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SUSPENSION AND LOOP OBJECTS IN THEORIES AND COHOMOLOGY

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Abstract. We introduce theories in which suspension, resp. loop objects, are defined and we describe the cohomology of such theories. Cohomology classes in degree 3 classify track theories.

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Since the work of Lawvere [16] it is well known that many algebraic categories (like the categories of monoids, groups, algebras, Lie algebras, etc.) are actually categories of models of a theory. In topology there are fundamental examples of theories as in (1) and (2) below in which also suspension objects ΣX or loop objects ΩX are defined. Such theories with suspension objects are also endowed with a *partial suspension* operator E and dually theories with loop objects are endowed with a *partial loop* operator L, compare for example Baues [5, 4, 2].

For example the homotopy category $\mathbf{S}$ of one point unions of spheres

$$S^{n_1} \vee ... \vee S^{n_k} \tag{1}$$

with $n_1, ..., n_k \geq 1$, $k \geq 0$, is a theory with suspension objects and partial suspension operator E. Moreover the homotopy category $\mathbf{K}(\mathbb{Z}/2)$ of products of Eilenberg–Mac Lane spaces

$$K^{n_1} \times ... \times K^{n_k} \tag{2}$$

with $n_1, ..., n_k \geq 0$, $k \geq 0$ and $K^n = K(\mathbb{Z}/2, n)$ is a theory with loop objects and partial loop operator L.

In this paper we introduce a complete system of axioms defining a theory with suspension objects and partial suspension operator termed a Σ-*theory*. The categorical dual of a Σ-theory is a theory with loop objects and partial loop operator termed Ω-theory.

The theory $\mathbf{S}$ given by spheres in (1) is a Σ-theory and the theory $\mathbf{K}(\mathbb{Z}/2)$ given by Eilenberg–Mac Lane spaces in (2) is an Ω-theory. These examples show the crucial significance of such theories. In [10] we show that the Ω-theory $\mathbf{K}(\mathbb{Z}/2)$ is completely determined by the Steenrod algebra.

We introduce the Σ-cohomology $H^*_\Sigma(\mathbf{T})$ of a Σ-theory $\mathbf{T}$ and dually the Ω-cohomology $H^*_\Omega(\mathbf{T})$ of an Ω-theory $\mathbf{T}$. The universal Toda brackets of Baues and Dreckmann [6] yield for the theories $\mathbf{S}$ and $\mathbf{K}(\mathbb{Z}/2)$ above the cohomology

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classes

$$\begin{aligned} \langle \mathbf{S} \rangle &\in H^3_\Sigma(\mathbf{S}), \\ \langle \mathbf{K}(\mathbb{Z}/2) \rangle &\in H^3_\Omega(\mathbf{K}(\mathbb{Z}/2)). \end{aligned} \tag{3}$$

The cohomology class $\langle \mathbf{S} \rangle$ determines all *secondary homotopy operations* in $\mathbf{S}$ like triple Toda brackets. The cohomology class $\langle \mathbf{K}(\mathbb{Z}/2) \rangle$ determines all *secondary cohomology operations* like triple Massey products or operations of Adams [1].

In [9] we study the concept of suspenders and loopers in groupoid enriched categories termed track categories. As a main result in this paper we show that a Σ-theory $\mathbf{T}$ together with an element $\alpha \in H^3_\Sigma(\mathbf{T})$ determines up to equivalence a track category which has the property of a "track theory with suspenders" and vice versa such track theories are classified by the Σ-cohomology H^3_Σ.

Dually an Ω-theory $\mathbf{T}$ together with an element $\alpha \in H^3_\Omega(\mathbf{T})$ determines up to equivalence a track category which has the properties of a "track theory with loopers" and vice versa such track theories are classified by the Ω-cohomology H^3_Ω.

For example the topological track category given by spaces in (1) and maps between such spaces (1-cells) and by homotopy classes of homotopies between such maps (2-cells) is a track theory with suspenders corresponding to the pair $(\mathbf{S}, \langle \mathbf{S} \rangle)$. Similarly the topological track category given by spaces in (2) is a track theory with loopers corresponding to the pair $(\mathbf{K}(\mathbb{Z}/2), \langle \mathbf{K}(\mathbb{Z}/2) \rangle)$.

1. Theories

A *theory* for us is a category $\mathbf{S}$ in which finite sums exist. We assume that a collection of generating objects X_i in $\mathbf{S}$ is given such that all objects of $\mathbf{S}$ are finite sums of objects from the collection. A *model* M of the theory $\mathbf{S}$ in a category $\mathbf{C}$ is a contravariant functor $M : \mathbf{S}^{\mathrm{op}} \to \mathbf{C}$ which carries sums to products. Hence a model can be thought of as a collection of objects M_i of $\mathbf{C}$ together with certain morphisms ("operations") $M_{i_1} \times ... \times M_{i_n} \to M_i$ which satisfy certain "identities", i. e. fit in certain commutative diagrams, according to the structure of $\mathbf{S}$. In most cases we consider models of $\mathbf{S}$ in the category $\mathbf{Set}$ of sets, so that a model is a collection of sets with algebraic structure.

1.1. **Example.** Let $\mathbf{S}$ be the full subcategory of the homotopy category $(\mathbf{Top}^*)_\simeq$ consisting of finite one point unions of spheres S^n with $n \geq 1$. Then $\mathbf{S}$ is a theory. *Homotopy groups* $\pi_n(X) = [S^n, X]$ of a space X together with homotopy operations given by maps in $\mathbf{S}$ yield a model of $\mathbf{S}$. Such models are also termed π-algebras in the work of Blanc [12], Stower [18], Dwyer–Kan–Stower [13], Baues–Goerss [7].

The categorical dual of a theory termed a *theory with products* is a category $\mathbf{P}$ with finite products denoted $A \times B$. A model of $\mathbf{P}$ is a covariant functor $\mathbf{P} \to \mathbf{C}$ which preserves products. Originally Lawvere used such theories with products. But the example of homotopy groups above shows that also theories with coproducts arise naturally.

1.2. **Example.** Let $\mathbf{K}(\mathbb{Z}/2)$ be the full subcategory of the homotopy category $(\mathbf{Top}^*)_{\simeq}$ consisting of finite products of Eilenberg–MacLane spaces $K(\mathbb{Z}/2, n)$ with $n \geq 1$. Then $\mathbf{K}$ is a theory with products. The *cohomology groups* $H^n(X; A) = [X, K(A, n)]$ of a space X together with cohomology operations given by maps in $\mathbf{K}(\mathbb{Z}/2)$ yield a model of $\mathbf{K}(\mathbb{Z}/2)$. For example, the cohomology of X as a module over the Steenrod algebra or the cohomology ring of X can be deduced from the model; see Gray [14].

Each object in $\mathbf{S}$ is a cogroup object and objects in $\mathbf{K}$ are group objects. Therefore $\mathbf{S}$ is a "theory of cogroups" as defined below and dually $\mathbf{K}$ is a "theory of groups".

1.3. **Definition.** A *theory of cogroups* is a theory $\mathbf{T}$ all of whose objects S are equipped with a cogroup structure $(0_S : S \to *, m_S : S \to S \vee S, n_S : S \to S)$ in a way which is compatible with sums, i. e.

$$0_{S\vee T} = \begin{pmatrix} 0_S \\ 0_T \end{pmatrix} : S \vee T \to *,$$

$$\begin{tikzcd}
 & S \vee S \vee T \vee T \arrow[dr, "\mathrm{id}_S \vee \binom{i_S}{i_T} \vee \mathrm{id}_T"] & \\
S \vee T \arrow[ur, "m_S \vee m_T"] \arrow[rr, "m_{S\vee T}"] & & S \vee T \vee S \vee T,
\end{tikzcd}$$

$$n_{S\vee T} = n_S \vee n_T : S \vee T \to S \vee T.$$

We shall deal more generally with the following theories of coactions which are fundamental in development of homotopy theory in [3]. Principal coactions are defined as in [9, Section 4].

1.4. **Definition.** A *theory of coactions* is a theory $\mathbf{T}$ all of whose objects X are assigned a cogroup S_X and a principal coaction with structure maps $a_X : X \to X \vee S_X$, $d_X : S_X \to X \vee X$. These structures are required to be compatible with sums, i. e. $S_{X\vee Y} = S_X \vee S_Y$, etc.

Clearly a theory of cogroups is a special case of a theory of coactions, with $S_X = X$, $a_X = m_X$, $d_X = (n_X \vee \mathrm{id}_X)m_X$, where m_X and n_X are the comultiplication and the coinverse respectively of the cogroup X.

1.5. **Definition.** Let S and T be cogroups in a category $\mathbf{C}$. We say that $f : S \to T$ is *linear* if f is compatible with the cogroup structure, i. e. $m_T f = (f \vee f)m_S$. More generally a map $f : S \to X \vee T$ is called *linear in* T if the diagram

$$\begin{tikzcd}
S \arrow[rr, "f"] \arrow[d, "m_S"] & & X \vee T \arrow[d, "\mathrm{id}_X \vee m_T"] \\
S \vee S \arrow[r, "f\vee f"'] & X \vee T \vee X \vee T \arrow[r, "\begin{pmatrix} i_X \\ i_{1,T} \\ i_X \\ i_{2,T} \end{pmatrix}"] & X \vee T \vee T
\end{tikzcd}$$

commutes.

We point out that in a theory of cogroups $\mathbf{T}$, morphisms in $\mathbf{T}$ are *not* required to be linear.

2. Theories with suspension

In this section we introduce and study "theories with suspension objects and partial suspension operator" termed Σ-theories. There are three different ways to characterize a Σ-theory. The categorical dual of a Σ-theory is termed a Ω-theory.

Let $\mathbf{T}$ be a theory of cogroups or more generally a theory of coactions.

For an object X of $\mathbf{T}$ we define the category $\mathbf{T}(X)$ as follows. Objects of $\mathbf{T}(X)$ are the objects of $\mathbf{T}$ having a cogroup structure. A morphism s from S to T in $\mathbf{T}(X)$ is given by a commutative diagram

$$
\begin{tikzcd}
 & X \arrow[dl, "i_X"'] \arrow[dr, "i_X"] & \\
X \vee S \arrow[rr, "s"] \arrow[dr, "\mathrm{id}_X \vee 0_S"'] & & X \vee T \arrow[dl, "\mathrm{id}_X \vee 0_T"] \\
 & X \vee * &
\end{tikzcd}
\tag{2.1}
$$

Note that each map $f : X \to Y$ induces a functor $\mathbf{T}(f) : \mathbf{T}(X) \to \mathbf{T}(Y)$ which is identity on objects and carries $a : X \vee S \to X \vee T$ to

$$\begin{pmatrix} i_Y \\ (f \vee T)ai_S \end{pmatrix} : Y \vee S \to Y \vee T.$$

In particular $\mathbf{T}(*)$ is the category whose objects are cogroups in $\mathbf{T}$ and whose morphisms are maps respecting counits. It is thus a theory of cogroups, and it is easy to see that the functor $\mathbf{T}(!_X) : \mathbf{T}(*) \to \mathbf{T}(X)$ induced as above preserves finite coproducts. Hence also $\mathbf{T}(X)$ is a theory of cogroups with the same structure on objects as $\mathbf{T}(*)$.

There is an alternative description of morphisms in $\mathbf{T}(X)$ as follows: assigning to $s : X \vee S \to X \vee T$ the map $si_S : S \to X \vee T$ gives an exact sequence of groups

$$0 \to \hom_{\mathbf{T}(X)}(S,T) \longrightarrow \hom_{\mathbf{T}}(S, X \vee T) \xrightarrow{\hom_{\mathbf{T}}(S, \mathrm{id}_X \vee 0_T)} \hom_{\mathbf{T}}(S, X \vee *),$$

where the group structures are induced by the cogroup structure of S. In other words, there are isomorphisms

$$\hom_{\mathbf{T}(X)}(S,T) \cong [S, X \vee T]_X \tag{2.2}$$

where by definition $[S, X \vee T]_X = \hom_{\mathrm{Pair}(\mathbf{T})}(0_S, \mathrm{id}_X \vee 0_T)$. This is the set of those maps $S \to X \vee T$ which are "trivial on X", i. e. those maps $g : S \to X \vee T$

which render the diagram

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "0_S"'] & X \vee T \arrow[d, "\mathrm{id}_X \vee 0_T"] \\ * \arrow[r, "!"] & X \vee * \end{tikzcd}$$

commutative. The inverse isomorphism $[S, X \vee T]_X \cong \hom_{\mathbf{T}(X)}(S, T)$ carries such a map g to

$$\begin{pmatrix} i_X \\ g \end{pmatrix} : X \vee S \to X \vee T.$$

It is easy to see that $s : X \vee S \to X \vee T$ is linear as a morphism of $\mathbf{T}(X)$ iff $si_S : S \to X \vee T$ is linear in T as a morphism in $\mathbf{T}$; see 1.5. The following definition of a Σ-theory is canonically derived from the notion of a Σ-representable track category as we shall see in our main result 4.2 below.

2.1. **Definition.** A *theory with suspensions* (or a Σ*-theory* for short) is a theory of coactions $\mathbf{T}$ together with a system of functors

$$E_X : \mathbf{T}(X) \to \mathbf{T}(X)$$

which preserve finite sums and carry each map to a linear map. These functors must be natural in the sense that for all maps $f : X \to Y$ the diagram of functors

$$\begin{tikzcd} \mathbf{T}(X) \arrow[r, "E_X"] \arrow[d, "\mathbf{T}(f)"] & \mathbf{T}(X) \arrow[d, "\mathbf{T}(f)"] \\ \mathbf{T}(Y) \arrow[r, "E_Y"] & \mathbf{T}(Y) \end{tikzcd}$$

commutes. In particular for $X = *$ one has the *suspension functor*

$$\Sigma = E_* : \mathbf{T}(*) \to \mathbf{T}(*).$$

Hence for each cogroup S in $\mathbf{T}$ the *suspension* ΣS is defined and is a cogroup in $\mathbf{T}$. Considering the above diagram for $f = !_X : * \to X$ one sees that all functors E_X coincide with Σ on objects. Moreover using the identification (2.2) above one may describe the effect of E_X on morphisms by maps which we will call *partial suspension* maps

$$E : [S, X \vee T]_X \to [\Sigma S, X \vee \Sigma T]_X \tag{2.3}$$

so that

$$E_X \begin{pmatrix} i_X \\ g \end{pmatrix} = \begin{pmatrix} i_X \\ Eg \end{pmatrix}.$$

By definition, for any map $f : S \to T$ in $\mathbf{T}(*)$, the map Σf is linear. More generally for any $g : S \to X \vee T$ with $(\mathrm{id}_X \vee 0_T)g = i_* 0_S : S \to X \vee *$, the map $Eg : \Sigma S \to X \vee \Sigma T$ is linear in ΣT. Considering the comultiplication $m_S : S \to S \vee S$ as a morphism in $\mathbf{T}(*)$ one sees that

$$\Sigma m_S : \Sigma S \to \Sigma(S \vee S) = \Sigma S \vee \Sigma S$$

is linear. This implies that the cogroup structure on ΣS is coabelian.

2.2. **Remark.** The categorical dual of a theory with suspensions (Σ-theory) is a *theory with loop objects* termed an Ω-*theory* for short. Hence an Ω-theory $\mathbf{T}$ is a category with finite products in which each object X is endowed with an action of a group object G, $X \times G \to X$, so that $\mathbf{T}$ is a theory of actions with properties dual to 1.4. Moreover $\mathbf{T}(X)$ is the category of diagrams

$$\begin{tikzcd} & X & \\ X \times G \arrow[ur, "p_X"] & & X \times H \arrow[ul, "p_X"'] \arrow[ll] \\ & X \times * \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{1}$$

dual to (2.1) and a system of functors

$$L_X : \mathbf{T}(X) \to \mathbf{T}(X) \tag{2}$$

is given with properties dual to 2.1. The *loop functor* is given by

$$\Omega = L_* : \mathbf{T}(*) \to \mathbf{T}(*) \tag{3}$$

which carries a group object G to the group objectΩG in $\mathbf{T}$. Moreover *partial loop operations*

$$L : [X \times H, G]_X \to [X \times \Omega H, \Omega G]_X \tag{4}$$

are defined dually to (2.3) by L_X in (2) above.

There are alternative ways to characterize a Σ-theory:

2.3. **Lemma.** *Equipping a theory of coactions $\mathbf{T}$ with the structure of a Σ-theory is equivalent to assigning a cogroup ΣS to each cogroup S in $\mathbf{T}$ and functions*

$$E : [S, X \vee T]_X \to [\Sigma S, X \vee \Sigma T]_X$$

to each object X and cogroups S, T in such a way that the following conditions are satisfied.

$$E\left(\begin{pmatrix} f \\ b \end{pmatrix} a\right) = \begin{pmatrix} f \\ Eb \end{pmatrix} Ea$$

for any $f : X \to Y$, $a \in [S, X \vee T]_X$, $b \in [T, Y \vee U]_Y$, and furthermore Ea is linear in ΣT. The functor Σ obtained from the particular case of this condition when $X = Y = *$, must preserve sums. And, for $i_S : S \to X \vee S$ one must have $Ei_S = i_{\Sigma S} : \Sigma S \to X \vee \Sigma S$.

Proof. We already saw how to derive partial suspension maps from the Σ-structure and the maps E_X from E. Conditions that they must obey are equivalent, to properties of E_X, namely, to preservation of composition, sums and identities, respectively.

Conversely, given the maps E, one defines E_X as above and then the required properties will correspond to properties of E as indicated. $\square$

For any Σ-theory $\mathbf{T}$ one has the functor

$$\Sigma_* : \mathbf{T} \to \mathbf{T}, \tag{2.4}$$

termed the *suspender* of $\mathbf{T}$, as follows. For an object X of $\mathbf{T}$ with the coaction structure (S_X, a_X, d_X) let $\Sigma_*(X) = X \vee \Sigma S_X$. Moreover define the *difference map* $\nabla(f) \in [S_X, Y \vee S_Y]_Y$ for any map $f : X \to Y$ by

$$\nabla(f) = (S_X \xrightarrow{d_X} X \vee X \xrightarrow{f \vee f} Y \vee Y \xrightarrow{\binom{i_Y}{a_Y}} Y \vee S_Y).$$

Then the partial suspension in $\mathbf{T}$ yields $E\nabla(f) \in [\Sigma S_X, Y \vee \Sigma S_Y]$ and one defines

$$\Sigma_*(f) = \begin{pmatrix} i_Y f \\ E\nabla(f) \end{pmatrix} : X \vee \Sigma S_X \to Y \vee \Sigma S_Y.$$

This indeed defines a functor Σ_* which also can be used to characterize the Σ-theory completely.

2.4. **Lemma.** *Let $\mathbf{T}$ be a theory of coactions. Then structures of a Σ-theory on $\mathbf{T}$ are in a one-to-one correspondence with sum-preserving endofunctors*

$$\Sigma_* : \mathbf{T} \to \mathbf{T}$$

equipped, for each coaction (X, S_X, a_X, d_X) in $\mathbf{T}$, with isomorphisms

$$\vartheta_X : \Sigma_*(X) \cong X \vee \Sigma S_X$$

for some cogroups ΣS_X depending only on the cogroups S_X. These isomorphisms must be natural in the sense that the inclusions

$$j_X = \vartheta_X^{-1} i_X : X \to \Sigma_*(X)$$

and the projections

$$p_X = \begin{pmatrix} \mathrm{id}_X \\ 0_{\Sigma S_X} \end{pmatrix} \vartheta_X : \Sigma_*(X) \to X$$

are natural in X. Moreover the composite maps

$$\Sigma S_X \xrightarrow{i_{\Sigma S_X}} X \vee \Sigma S_X \xrightarrow{\vartheta_X^{-1}} \Sigma_*(X) \xrightarrow{\Sigma_*(f)} \Sigma_*(Y) \xrightarrow{\vartheta_Y} Y \vee \Sigma S_Y$$

must be linear in ΣS_Y.

Proof. Given the functor Σ_* as above, for any cogroup S consider S as a coaction via $S_S = S$ coacting on itself via its own comultiplication; let $\Sigma S = \Sigma S_S$. These come equipped with maps $q_S : \Sigma_*(S) \to \Sigma S$ given by

$$q_S = (\ \Sigma_*(S) \xrightarrow{\vartheta_S} S \vee \Sigma S \xrightarrow{0_S \vee \mathrm{id}_{\Sigma S}} * \vee \Sigma S \cong \Sigma S\ ).$$

We call Σ the suspension and define the partial suspension Eg for $g \in [S, X \vee T]_X$ by

$$Eg = \Bigg(\Sigma S \xrightarrow{i_{\Sigma S}} S \vee \Sigma S \xrightarrow{\vartheta_S^{-1}} \Sigma_*(S) \xrightarrow{\Sigma_*(g)} \Sigma_*(X \vee T) \\ \cong \Sigma_*(X) \vee \Sigma_*(T) \xrightarrow{p_X \vee q_T} X \vee \Sigma T \Bigg).$$

It is clear that Eg is linear in ΣT and also other properties required of E and Σ are all satisfied. The converse procedure has been already described above. $\square$

3. Cohomology of Σ-theories and Ω-theories

Recall that a *natural system* D on a category $\mathbf{T}$ is a functor $D : F\mathbf{T} \to \mathbf{Ab}$ where $F\mathbf{T}$ is the category of factorizations of $\mathbf{T}$; see [8, 2.1]. Here D carries the object $f : A \to B$ in $F\mathbf{T}$ to an abelian group D_f and carries maps $\xleftarrow[g]{} \xleftarrow[f]{} \xleftarrow[h]{}$ to induced action maps $g_* : D_f \to D_{gf}$, $\xi \mapsto g\xi$, and $h^* : D_f \to D_{fh}$, $\xi \mapsto \xi h$.

For objects A, B in $\mathbf{T}$ let

$$[A, B] = \hom(A, B)$$

be the set of morphisms $A \to B$ in $\mathbf{T}$. If $\mathbf{T}$ is a Σ-theory we have for the coaction on each object X in $\mathbf{T}$ the structure maps

$$\begin{cases} a_X : X \to X \vee S_X \\ d_X : S_X \to X \vee X \end{cases}$$

as in 1.4.

3.1. **Proposition.** *Each Σ-theory $\mathbf{T}$ comes equipped with a canonical natural system $D^{\mathbf{T}}$ defined as follows: For a map $f : X \to Y$ let*

$$D_f^{\mathbf{T}} = [\Sigma S_X, Y],$$

with the abelian group structure induced by the abelian cogroup ΣS_X. Left action of a map $y : Y \to Y'$ on $a : \Sigma S_X \to Y$ is given by

$$y_*(a) = ya.$$

The right action of $x : X' \to X$ on a is given by the composite

$$\Sigma S_{X'} \xrightarrow{E\nabla(f)} X \vee \Sigma S_X \xrightarrow{\binom{f}{a}} Y,$$

where $\nabla(f)$ is the composite

$$\nabla(f) = (\ S_X \xrightarrow{d_X} X \vee X \xrightarrow{f \vee f} Y \vee Y \xrightarrow{\binom{i_Y}{a_Y}} Y \vee S_Y\ ).$$

Proof. It turns out that naturality of actions is evident if one uses the functor Σ_* in (2.4) above, whereas the fact that this is indeed the natural system of abelian groups and not just sets is clear if one uses the suspensions Σ. So let us first define the actions using the former and then transform them using the latter.

For any morphism $f : X \to Y$, put

$$D^{\mathbf{T}}_f = \{a : \Sigma_*(X) \to Y \mid aj_X = f\},$$

as in the diagram

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, "j_X"] \\
Y & \Sigma_*(X). \arrow[l, dashed, "a"']
\end{tikzcd}$$

Actions of this natural system are defined as follows: for $y : Y \to Y'$ let the induced action $D^{\mathbf{T}}_f \to D^{\mathbf{T}}_{yf}$ be given by $a \mapsto ya$, as in the diagram

$$\begin{tikzcd}
& & X \arrow[dl, "f"'] \arrow[d, "j_X"] \\
Y' & Y \arrow[l, "y"'] & \Sigma_*(X); \arrow[l, "a"'] \arrow[ll, bend left, "ya"]
\end{tikzcd}$$

whereas for $x : X' \to X$ let the action $D^{\mathbf{T}}_f \to D^{\mathbf{T}}_{fx}$ be given by $a \mapsto a\Sigma_*(x)$, as in the diagram

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, "j_X"] & X' \arrow[l, "x"'] \arrow[d, "j_{X'}"] \\
Y & \Sigma_*(X) \arrow[l, "a"'] & \Sigma_*(X'), \arrow[l, "\Sigma_*(x)"'] \arrow[ll, bend left, "x^*(a)"]
\end{tikzcd}$$

where the square commutes by 2.4.

Now let us use the isomorphisms

$$\vartheta_X : \Sigma_*(X) \cong X \vee \Sigma S_X$$

inducing bijections

$$D^{\mathbf{T}}_f \approx [\Sigma S_X, Y],$$

where the latter sets have the group structure induced by the canonical cogroup structure on ΣS_X. In fact as we have seen ΣS_X is an abelian cogroup (see 2.1), so the groups $D^{\mathbf{T}}_f$ are indeed abelian. Moreover the linearity conditions imply that the actions are indeed homomorphisms.

For convenience let us give explicit expressions for the actions in these new terms. Left actions are again given by composition: for $y : Y \to Y'$ the induced homomorphism is

$$[\Sigma S_X, Y] \to [\Sigma S_X, Y'], \text{ given by } a \mapsto ya$$

for any $a : \Sigma S_X \to Y$.

Right actions are more involved. For $x : X' \to X$ and $a : \Sigma S_X \to Y$ the map $x^*(a) : \Sigma S_{X'} \to Y$ is given by the lower horizontal composition in the diagram

$$\begin{tikzcd}
& \Sigma_*(X) \arrow[d, "\vartheta_X"', "\cong"] & \Sigma_*(X') \arrow[l, "\Sigma_*(f)"'] \arrow[d, "\vartheta_{X'}"', "\cong"] & \\
Y & X \vee \Sigma S_X \arrow[l, "\binom{f}{a}"'] & X' \vee \Sigma S_{X'} \arrow[l] & \Sigma S_{X'}. \arrow[l, "i_{\Sigma S_{X'}}"'] \arrow[lll, bend left, "x^*(a)"]
\end{tikzcd}$$

This yields the result by use of the definition of Σ_* in (2.4). $\square$

For each category $\mathbf{C}$ with natural system D the cohomology $H^n(\mathbf{C}; D)$ is defined as in [11]. This leads to the following notion of Σ-cohomology.

3.2. **Definition.** Let $\mathbf{T}$ be a Σ-theory and let $D^{\mathbf{T}}$ be the natural system determined by $\mathbf{T}$ as in 3.1. Then the Σ-*cohomology* of $\mathbf{T}$ is given by the abelian group $(n \geq 0)$

$$H^n_\Sigma(\mathbf{T}) = H^n(\mathbf{T}; D^{\mathbf{T}}).$$

The categorical dual of a Σ-theory is an Ω-theory $\mathbf{T}$ for which we have accordingly the Ω-*cohomology*

$$H^n_\Omega(\mathbf{T}) = H^n(\mathbf{T}; D^{\mathbf{T}}),$$

where $D^{\mathbf{T}}$ is the natural system associated to $\mathbf{T}$ with $D^{\mathbf{T}}_f = [A, \Omega B]$ for $f : A \to B$ in $\mathbf{T}$.

4. Track theories with suspensions.

A *track theory* $\mathscr{T}$ is a track category with sums $A \vee B$ in the weak sense so that for hom-groupoids in $\mathscr{T}$ the map

$$[\![A \vee B, X]\!] \xrightarrow{\sim} [\![A, X]\!] \times [\![B, X]\!]$$

is an equivalence of groupoids; see [8, §4]. Moreover $\mathscr{T}$ is Σ-*representable* if suspenders $(\Sigma_f, i_f, \upsilon_f)$ as defined in [9] exist for all maps $f : A \to B$ in $\mathscr{T}$.

4.1. **Definition.** A Σ-*track theory* is a track theory $\mathscr{T}$ which is Σ-representable and for which the homotopy category $\mathscr{T}_\simeq$ is a theory of coactions.

Recall that linear track extensions are defined as in [8, 2.2]. It is clear that a Σ-track theory $\mathscr{T}$ has abelian hom-groupoids so that $\mathscr{T}$ is a linear track extension of its homotopy category $\mathscr{T}_\simeq$ as in [8].

4.2. **Theorem.** *For any Σ-track theory $\mathscr{T}$ its homotopy category carries a canonical structure of a Σ-theory and $\mathscr{T}$ is a linear track extension of $\mathscr{T}_\simeq$ by the canonical natural system $D^{\mathscr{T}_\simeq}$ associated to the Σ-theory $\mathscr{T}_\simeq$. Conversely, any linear track extension of a Σ-theory by its canonical natural system is a Σ-track theory.*

Proof. Consider a track category $\mathscr{T}$ satisfying the hypothesis. To equip $\mathscr{T}_{\simeq}$ with a structure of a Σ-theory it suffices, according to 2.1, to define the functor Σ_*. We do this by declaring $\Sigma_*(X) = \Sigma_{\mathrm{id}_X}$ for all objects X and $\Sigma_*(f) = \Sigma_*(f, f)$ for all maps $f : X \to Y$, where (f, f) is considered as a morphism in $\mathrm{Pair}(\mathscr{T})$ from id_X to id_Y. By [9, 2.1] this indeed defines an endofunctor, equipped with natural maps $j_X = i_{\mathrm{id}_X} : X \to \Sigma_*(X)$. Σ_* preserves sums by [9, 2.4]. Moreover the isomorphisms ϑ are obtained, in view of [9, 2.2], by virtue of [9, 4.3]. The latter also implies that under these isomorphisms the maps j_X correspond to the coproduct inclusions $i_X : X \to X \vee \Sigma S_X$. To prove naturality of the projections

$$p_X = \begin{pmatrix} \mathrm{id}_X \\ 0_{\Sigma S_X} \end{pmatrix} \vartheta_X : \Sigma_*(X) \to X$$

it suffices to construct, for any map $f : X \to Y$, a track $f p_X \simeq p_Y \Sigma_*(f)$. By the universal property of the suspender Σ_{id_X}, such track exists iff there is a track in $\mathbf{G}_{\mathrm{id}_X}$ between $f p_X \upsilon_X \in \mathrm{Aut}(f p_X j_X)$ and $p_Y \Sigma_*(f) \upsilon_X \in \mathrm{Aut}(p_Y \Sigma_*(f) j_X)$. By [9, 2.1], one may take $\zeta_{(f,f)}$ for this purpose. Finally, linearity condition from 2.1 means commutativity in $\mathscr{T}_{\simeq}$ of the diagram

$$\begin{tikzcd}
\Sigma S_X \arrow[rr, "E"] \arrow[d, "\Sigma m_{S_X}"] & & Y \vee \Sigma S_Y \arrow[d, "\mathrm{id}_Y \vee \Sigma m_{S_Y}"] \\
\Sigma S_X \vee \Sigma S_X \arrow[r, "E \vee E"'] & Y \vee \Sigma S_Y \vee Y \vee \Sigma S_Y \arrow[r, "\begin{pmatrix} i_Y \\ i_{1,\Sigma S_Y} \\ i_Y \\ i_{2,\Sigma S_Y} \end{pmatrix}"] & Y \vee \Sigma S_Y \vee \Sigma S_Y,
\end{tikzcd}$$

for $f : X \to Y$, where E is the composite

$$\Sigma S_X \xrightarrow{i_{\Sigma S_X}} X \vee \Sigma S_X \xrightarrow{\vartheta_X^{-1}} \Sigma_*(X) \xrightarrow{\Sigma_*(f)} \Sigma_*(Y) \xrightarrow{\vartheta_Y} Y \vee \Sigma S_Y,$$

as in 2.1. In other words, for any object Z the induced diagram

$$\begin{tikzcd}
{[\Sigma S_X, Z]} & & {[Y, Z] \times [\Sigma S_Y, Z]} \arrow[ll, "E^*"'] \\
{[\Sigma S_X, Z]^2} \arrow[u, "+"] & {([Y, Z] \times [\Sigma S_Y, Z])^2} \arrow[l, "E^{*2}"'] & {[Y, Z] \times [\Sigma S_Y, Z]^2} \arrow[l, "(\mathrm{id}_{[Y,Z]} \times p_1, \mathrm{id}_{[Y,Z]} \times p_2)"'] \arrow[u, "\mathrm{id}_{[Y,Z]} \times +"']
\end{tikzcd}$$

must commute. Now using universal property of suspenders, the entries $[\Sigma ?, Z]$ may be replaced by the isomorphism classes of groupoids $\mathbf{G}_?(Z)$, thus obtaining

$$\begin{tikzcd}
\pi_0 \mathbf{G}_{0_{S_X}}(Z) & & {[Y, Z] \times \pi_0 \mathbf{G}_{0_{S_Y}}(Z)} \arrow[ll] \\
\pi_0 \mathbf{G}_{0_{S_X}}(Z)^2 \arrow[u] & {([Y, Z] \times \pi_0 \mathbf{G}_{0_{S_Y}}(Z))^2} \arrow[l] & {[Y, Z] \times \pi_0 \mathbf{G}_{0_{S_Y}}(Z)^2} \arrow[l] \arrow[u]
\end{tikzcd}$$

Calculating actions of maps in this diagram in the same way as in [9, 4.3], we see that commutativity of the diagram amounts to conjugatedness of the tracks

$$\mathrm{id}_{zf} \backslash (\mathrm{id}_z \cdot \omega_1) f + \mathrm{id}_{zf} \backslash (\mathrm{id}_z \cdot \omega_2) f \text{ and } \mathrm{id}_{zf} \backslash (\mathrm{id}_z \cdot (\omega_1 + \omega_2)) f$$

in $\mathrm{Aut}(!_Z 0_{S_X})$ for an element $(z, \omega_1, \omega_2) \in [Y, Z] \times \mathbf{G}_{0_{S_Y}}(Z)^2$ given by a map $z : Y \to Z$ and two tracks in $\mathrm{Aut}(!_Z 0_{S_Y})$, where $\cdot$ and $\backslash$ denote coaction and division maps, as in [9, 4.3]. But in fact these tracks are easily seen to be equal.

We now turn to the converse statement. Given a linear track category $\mathscr{T}$ with a structure of a Σ-theory on $\mathscr{T}_{\simeq}$, we know from [8, 4.6] that $\mathscr{T}$ has weak sums compatible with the sums in $\mathscr{T}_{\simeq}$. Moreover as $\mathscr{T}_{\simeq}$ is a theory of coactions, it is clear that each object of $\mathscr{T}$ has a structure of a weak principal coaction. Thus to have all suspenders it suffices by [9, 4.3] to construct suspenders of the maps $0_S : S \to *$, for weak cogroups S in $\mathscr{T}$. For $i_0 : * \to \Sigma_0$, take the suspension $\Sigma S = E_*(0_S : S \to *)$. Define the universal track

$$\upsilon_0 \in \mathrm{Aut}(\ S \xrightarrow{0_S} * \xrightarrow{!_{\Sigma S}} \Sigma S\ )$$

by $\upsilon_0 = \sigma_{!_{\Sigma S} 0_S}(\mathrm{id})$, utilizing the linear track extension action

$$D^{\mathscr{T}_{\simeq}}_{!_{\Sigma S}} = [\Sigma S, \Sigma S] \xrightarrow{\sigma_{0_S}} \mathrm{Aut}(!_{\Sigma S} 0_S)\ ,$$

where the equality follows from 3.1. To show that this indeed has the universal property of a suspender, we have to show the following:

- for any object Z and any track $\eta \in \mathrm{Aut}(!_Z 0_S)$ there exists a map $\Sigma_\eta : \Sigma S \to Z$ with $\eta = \Sigma_\eta \upsilon_0$;
- for any $h, h' : \Sigma S \to Z$ with $h\upsilon_0 = h'\upsilon_0$ there is a track $\delta : h \to h'$.

Using 3.1, the first condition becomes a tautology, whereas in the second one obtains $h = h'$, so trivially one can take for δ the identity track. $\square$

5. Classification of Σ-track categories

An *equivalence* between theories of coactions $F : \mathbf{T} \to \mathbf{T}'$ is an equivalence of the underlying categories which preserves sums and which carries the coaction structure of an object X in $\mathbf{T}$ to the coaction structure of the object $F(X)$ in $\mathbf{T}'$.

5.1. **Definition.** A *weak equivalence* between Σ-track theories is a weak equivalence of the underlying track categories as in [8, 1.5] which preserves weak sums and suspenders and for which the induced functor between homotopy categories is an equivalence between theories of coactions.

5.2. **Definition.** An *equivalence* (F, Φ) between Σ-theories is an equivalence $F : \mathbf{T} \to \mathbf{T}'$ between the underlying theories of coactions which is compatible with the suspension structure, that is for the diagram of functors

$$\begin{tikzcd} \mathbf{T}(X) \arrow[r, "F_X"] \arrow[d, "E_X"] & \mathbf{T}'(FX) \arrow[d, "E_{FX}"] \\ \mathbf{T}(X) \arrow[r, "F_X"] & \mathbf{T}'(FX) \end{tikzcd}$$

with F_X induced by F one has a natural isomorphism $\Phi_X : E_{FX}F_X \cong F_X E_X$.

5.3. **Definition.** A *classifying pair* is a pair $(\mathbf{T}, \tau)$ where $\mathbf{T}$ is a Σ-theory and τ is an element in the Σ-cohomology $H^3_\Sigma(\mathbf{T})$. Two such pairs $(\mathbf{T}, \tau)$ and $(\mathbf{T}', \tau')$ are *equivalent* if there exists an equivalence (F, Φ) with $F : \mathbf{T}' \to \mathbf{T}$ such that $F^*\tau = I_*\tau'$ in $H^3(\mathbf{T}'; F^*D^{\mathbf{T}})$. Here the natural isomorphism $I : D^{\mathbf{T}'} \cong F^*D^{\mathbf{T}}$ is given by Φ.

Using [8, 4.8] and 4.2 we obtain the following classification result.

5.4. **Classification.** *There is a* 1-1 *correspondence between weak equivalence classes of* Σ*-track theories and equivalence classes of classifying pairs.*

6. Examples

Since the track category associated to a Quillen model category is both Σ-representable and Ω-representable (see [9, 1.6]) we get the following result.

6.1. **Proposition.** *Let* $\mathbf{Q}_{\mathrm{cf}}$ *be the track category of cofibrant and fibrant objects in a Quillen model category and let* $\mathscr{T} \subset \mathbf{Q}_{\mathrm{cf}}$ *be a full track subcategory. If* $\mathscr{T}_\simeq$ *is a theory of coactions then* $\mathscr{T}$ *is a* Σ*-track theory and if dually* $\mathscr{T}_\simeq$ *is a theory of actions then* $\mathscr{T}$ *is an* Ω*-track theory. In particular* $\mathscr{T}_\simeq$ *is a* Σ*-theory and a* Ω*-theory respectively.*

A similar result is available for a cofibration category and a fibration category in the sense of Baues [4]. This, in particular, yields the following examples.

6.2. **Example.** Let $\mathscr{S}(k)$ be the track subcategory of $\mathbf{Top}^*$ consisting of one-point unions of spheres S^n with $n \geq k$. Then $\mathscr{S}(k)$ is a Σ-track theory and hence the homotopy category $\mathscr{S}(k)_\simeq$ is a Σ-theory, see 1.1. The partial suspension in $\mathscr{S}(k)_\simeq$, in fact, coincides with the partial suspension used in [5, 4, 2]. The partial suspension can be described totally by the ordinary suspension for homotopy groups of spheres and by Whitehead products. The Σ-track theory $\mathscr{S}(k)$ is classified by an element $\tau \in H^3_\Sigma(\mathscr{S}(k)_\simeq)$ in the Σ-cohomology. This is the universal Toda bracket $\langle \mathscr{S}(k) \rangle = \tau$ also considered in [6] and Baues [2].

6.3. **Example.** Let $\mathscr{K}(k)$ be the track subcategory of $\mathbf{Top}^*$ consisting of products of Eilenberg-MacLane spaces $K(A, n)$, $n \leq k$, and A a finitely generated abelian group. Then $\mathscr{K}(k)$ is an Ω-track theory and hence the homotopy category $\mathscr{K}(k)_\simeq$ is an Ω-theory. The partial loop operation in $\mathscr{K}(k)_\simeq$, in fact, coincides with the loop operation L used in [5, 4, 2]. Here L can be described totally by the ordinary loop operation (suspension) in cohomology and by cup products, see [4]. The Ω-track theory $\mathscr{K}(k)$ is classified by an element $\tau \in H^3_\Omega(\mathscr{K}(k)_\simeq)$ in the Ω-cohomology. This is the universal Toda bracket $\langle \mathscr{K}(k) \rangle = \tau$ also considered by Baues and Dreckmann [6] and Baues [2].

Recall that an additive category $\mathbf{A}$ is a category enriched in abelian groups which has sums. These sums are then also products (termed direct sums) denoted $A \oplus B$; see [17]. Each object is canonically an abelian group object by

$(1,1) : A \oplus A \to A$ and all maps in $\mathbf{A}$ are linear. A functor between additive categories is additive if it preserves biproducts.

6.4. **Lemma.** *An additive category $\mathbf{A}$ together with an additive functor $\sigma : \mathbf{A} \to \mathbf{A}$ has canonically the structure of both a Σ-theory and a Ω-theory.*

Proof. We simply define the suspender in 2.4 by $\Sigma_*(X) = X \oplus \sigma(X)$. Dually we define a looper by $\Omega_*(X) = X \oplus \sigma(X)$. One readily checks the properties in 2.4 and the dual properties respectively. $\square$

The Σ-cohomology of the Σ-theory $(\mathbf{A}, \sigma)$ in 6.4 is the cohomology

$$H^n_\Sigma(\mathbf{A}, \sigma) = H^n(\mathbf{A}; \mathrm{Hom}(\sigma, -)),$$

where $\mathrm{Hom}(\sigma, -)$ is the bifunctor $\mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Ab}$ which carries (X, Y) to $\mathrm{Hom}(\sigma X, Y)$. Moreover the Ω-cohomology of the Ω-theory $(\mathbf{A}, \sigma)$ is the cohomology

$$H^n_\Omega(\mathbf{A}, \sigma) = H^n(\mathbf{A}; \mathrm{Hom}(-, \sigma)),$$

where the bifunctor $\mathrm{Hom}(-, \sigma)$ carries (X, Y) to $\mathrm{Hom}(X, \sigma Y)$. Here Hom denotes the set of morphisms in $\mathbf{A}$ which is an abelian groups since $\mathbf{A}$ is an additive category. The cohomology groups above are special cases of those by Baues and Wirsching [2] which are in particular studied by Jibladze and Pirashvili [15].

Let R be a commutative ring with unit and let $\mathbf{Mod}_*(R)$ be the category of $\mathbb{Z}$-graded R-modules $X = (X_i)_{i \in \mathbb{Z}}$ with the graded tensor product $X \otimes Y$ given by

$$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes_R Y_j.$$

Here X is non-negatively graded if $X_i = 0$ for $i < 0$. A non-negatively graded R-module A which has the structure of a monoid in $(\mathbf{Mod}_*(R), \otimes)$ is termed a *graded algebra*. Let R be the graded R-module concentrated in degree 0 and given by the ring R. Then of course R is also a graded algebra; in fact, the initial object in the category of graded algebras. A map $\varepsilon : A \to R$ between graded algebras is termed an *augmentation* of A.

6.5. **Example.** Let A be a graded algebra and let $\mathbf{T}_A$ be the category of $\mathbb{Z}$-graded finitely generated free left A-modules. Objects are direct sums of objects $A[n]$ with $n \geq 0$, where $A[n]$ is the free left A-module generated by a single element $[n]$ in degree n. The elements of $A[n]$ are of the form $a \cdot [n]$ with $a \in A$ and degree $|a \cdot [n]| = |a| + n$. Morphisms $f : A[n] \to A[m]$ are determined by the element $\alpha \in A$ with $f[n] = \alpha \cdot [m]$ and we write $f = \alpha^*$ if $f(a \cdot [n]) = a \cdot \alpha \cdot [m]$. We define the *shift functor*

$$\sigma : \mathbf{T}_A \to \mathbf{T}_A$$

to be the unique additive functor which carries $A[n]$ to $A[n+1]$ and carries $f = \alpha^* : A[n] \to A[m]$ to $\alpha^* : A[n+1] \to A[m+1]$. Hence by 6.4 the category $\mathbf{T}_A$ together with σ is canonically both a Σ-theory and a Ω-theory.

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