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SUSPENSION AND LOOP OBJECTS AND REPRESENTABILITY OF TRACKS

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Abstract. In the general setting of groupoid enriched categories, notions of *suspender* and *looper* of a map are introduced, formalizing a generalization of the classical homotopy-theoretic notions of suspension and loop space. The formalism enables subtle analysis of these constructs. In particular, it is shown that the suspender of a principal coaction splits as a coproduct. This result leads to the notion of *theories with suspension* and to the cohomological classification of certain groupoid enriched categories.

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A category enriched in groupoids (termed a track category for short) is a special 2-category. A track category $\mathscr{T}$ consists of objects A, B,... and hom-groupoids $[\![A,B]\!]$ in which the objects are maps (1-arrows or 1-cells) and the morphisms are isomorphisms termed tracks (2-arrows or 2-cells). For each map $f : A \to B$ in $\mathscr{T}$ we have the group $\mathrm{Aut}(f)$ consisting of all tracks $\alpha : f \Rightarrow f$ in $\mathscr{T}$. This is an automorphism group in the hom-groupoid $[\![A,B]\!]$.

Our leading example is the track category $\mathbf{Top}^*$ consisting of spaces A, B,... with basepoint $*$, pointed maps $f : A \to B$ and tracks $\alpha : f \Rightarrow g$ which are homotopy classes (relative to the boundary) of homotopies $f \simeq g$; compare (1.3) in [5]. For the trivial map $0 : A \to * \to B$ in $\mathbf{Top}^*$ one has the well known isomorphism of groups

$$\mathrm{Aut}(A \to * \to B) = [\Sigma A, B]. \qquad (*)$$

Here the left-hand side is the group of automorphisms of $0 : A \to B$ in the track category $\mathbf{Top}^*$ and the right-hand side is the group of homotopy classes of maps $\Sigma A \to B$ where ΣA is the *suspension* of A. Dually we also have the canonical isomorphism

$$\mathrm{Aut}(A \to * \to B) = [A, \Omega B], \qquad (**)$$

where ΩB is the *loop space* of B. Via $(*)$ and $(**)$ certain tracks in the track category $\mathscr{T} = \mathbf{Top}^*$ are *represented* by morphisms in the homotopy category $\mathscr{T}_{\simeq}$ of $\mathscr{T}$. We study in this paper the categorical aspects of such a representability of tracks which we call Σ-representability in $(*)$ and Ω-representability in $(**)$. For this we introduce the notion of *suspender* generalizing the notion of

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suspension above by means of a universal property. The categorical dual of a suspender is a *looper* in a track category generalizing the notion of loop space. A track category $\mathscr{T}$ is Σ-representable, resp. Ω-representable, if suspenders, resp. loopers exist in $\mathscr{T}$. Of course the track category **Top*** of pointed spaces (more generally the track category associated to any Quillen model category) is both Σ-representable and Ω-representable.

We describe basic properties of suspenders and loopers. In particular we show that the suspender of a principal coaction splits as a coproduct. This is a crucial result which leads to the notion of *theories with suspension* and the cohomological classification of certain Σ-representable track categories in [6].

In topology a typical example of a suspender of a pointed space X is the space

$$\Sigma_* X = S^1 \times X / S^1 \times \{*\}.$$

Moreover a looper of X is given by the free loop space

$$\Omega_* X = (X^{S^1}, 0)$$

with the function space topology and basepoint given by the trivial loop 0. Splitting results $\Sigma_* X \simeq X \vee \Sigma X$ (resp. $\Omega_* X \simeq X \times \Omega X$) are well known in case X is a co-H-group (resp. H-group). For example Barcus and Barratt [5] or Rutter [12] use implicitly the splitting to obtain basic rules of homotopy theory. This paper and its sequel [6] specifies the categorical background of some of these rules. Suspenders and loopers are also responsible for the properties of *partial suspensions* and *partial loop operations* discussed in [6]; compare also [4, 3, 2].

The theory of Σ-representable track categories in this paper is also motivated by the approach of Gabriel and Zisman [7] who consider those properties of a track category $\mathscr{T}$ which imply existence of a Puppe sequence for mapping cones. The suspension ΣA plays a crucial rôle in this sequence. Enriching the results in [7] we show that the main categorical nature of a suspension in a track category is described by the notion of suspender which is the link between tracks in $\mathscr{T}$ and homotopy classes of maps in $\mathscr{T}$. Such a link, for example, is needed in results of Hardie, Kamps and Kieboom [9, 8] and Hardie, Marcum and Oda [10] who study homotopy-theoretic secondary operations like Toda brackets in 2-categories. This paper does not aim at combining the theory of exact sequences in homotopy theory as considered in [7] and the theory of suspenders since the notion of suspenders, resp. loopers, is quite sophisticated and new.

1. Σ-representable track categories

We first introduce the following notation. In a groupoid **G** the composition of morphisms

$$\xleftarrow{f} \xleftarrow{g}$$

is denoted by $f + g$. Accordingly also the composition of tracks

$$\xLeftarrow{\alpha} \; \xLeftarrow{\beta}$$

in a track category is denoted by $\alpha + \beta$. We also write $\alpha : a \simeq b$ for a track $\alpha : a \Rightarrow b$. We say that a groupoid $\mathbf{G}$ is *abelian* if all automorphism groups of objects in $\mathbf{G}$ are abelian. For $\beta : y \to y$ and $\varphi : x \to y$ in $\mathbf{G}$ we obtain the *conjugate*

$$\beta^{\varphi} = -\varphi + \beta + \varphi : x \to x$$

so that $(-)^{\varphi} : \mathrm{Aut}(y) \to \mathrm{Aut}(x)$ is a homomorphism. The *loop groupoid* $\mathbf{G}^{\circlearrowleft}$ of $\mathbf{G}$ is defined to have objects $\alpha : x \to x$, where x is an object of $\mathbf{G}$; a morphism from $\alpha : x \to x$ to $\beta : y \to y$ is a morphism $\varphi : x \to y$ in $\mathbf{G}$ such that $\alpha = -\varphi + \beta + \varphi$.

Track functors between track categories and track transformations between track functors are the enriched versions of functors and natural transformations enriched in the category $\mathfrak{Gpd}$ of groupoids. Here $\mathfrak{Gpd}$ is also an example of a track category with functors between groupoids as 1-arrows and natural transformations as 2-arrows. Each object C in a track category $\mathscr{T}$ yields the *representable track functor*

$$[\![C, -]\!] : \mathscr{T} \to \mathfrak{Gpd}$$

between track categories which carries $X \in \mathrm{Ob}(\mathscr{T})$ to the hom-groupoid $[\![C, X]\!]$ in $\mathscr{T}$.

In a track category $\mathscr{T}$, consider a map $f : A \to B$. For any object X, denote by $\mathbf{G}_f(X)$ the following groupoid: objects of $\mathbf{G}_f(X)$ are pairs (g, α), where $g : B \to X$ is a map and $\alpha : gf \Rightarrow gf$ is a track. A morphism from (g', α') to (g, α) is a track $\gamma : g' \Rightarrow g$ such that $\alpha' = \alpha^{\gamma f}$. Any map $h : X \to Y$ induces a functor $\mathbf{G}_f(h) : \mathbf{G}_f(X) \to \mathbf{G}_f(Y)$ sending (g, α) to $(hg, h\alpha)$ and γ to $h\gamma$. Moreover any track $\eta : h \Rightarrow h'$ induces a natural transformation $\mathbf{G}_f(\gamma) : \mathbf{G}_f(h) \to \mathbf{G}_f(h')$ with components $\eta g : hg \Rightarrow h'g$ for objects (g, α) of $\mathbf{G}_f(X)$. Thus we have defined a track functor

$$\mathbf{G}_f : \mathscr{T} \to \mathfrak{Gpd}.$$

Any object $(g : B \to C, \alpha : gf \Rightarrow gf)$ of $\mathbf{G}_f(C)$ gives rise to a track transformation

$$(g, \alpha)^* : [\![C, -]\!] \to \mathbf{G}_f$$

consisting of functors

$$[\![C, X]\!] \to \mathbf{G}_f(X)$$

which assign to $h : C \to X$ the pair $(hg, h\alpha)$ and to $\eta : h' \Rightarrow h$ the track ηg (this indeed defines a morphism in $\mathbf{G}_f$ as $\eta g f + h'\alpha = h\alpha + \eta g f$, i. e. $h'\alpha = (h\alpha)^{\eta g f}$).

1.1. **Definition.** For a map $f : A \to B$ in a track category $\mathscr{T}$, a *suspender* for f is a triple $(\Sigma_f, i_f, \upsilon_f)$ consisting of an object Σ_f, a map $i_f : B \to \Sigma_f$, and a track $\upsilon_f : i_f f \Rightarrow i_f f$ having the property that the induced track transformation

$$(i_f, \upsilon_f)^* : [\![\Sigma_f, -]\!] \to \mathbf{G}_f$$

induces a bijection of isomorphism classes of objects.

In other words, the following conditions must be satisfied:

(a) For any map $g : B \to C$ and any track $\eta : gf \Rightarrow gf$ there exists a map $\Sigma_\eta : \Sigma_f \to C$ and a track $\zeta_\eta : g \Rightarrow \Sigma_\eta i_f$ such that $\eta = (\Sigma_\eta \upsilon_f)^{\zeta_\eta f}$ (surjectivity);

(b) For any $h, h' : \Sigma_f \to C$ and any track $\gamma : h' i_f \Rightarrow h i_f$ with $h'\upsilon_f = (h\upsilon_f)^{\gamma f}$ one has $\gamma = \delta i_f$ for some track $\delta : h' \Rightarrow h$ (injectivity).

We point out that we do not assume for a suspender Σ_f that the map $(i_f, \upsilon_f)^*$ is an equivalence of groupoids since this, in fact, does not hold in the example of topological spaces. Hence topology forces us to think of a weaker universal property, namely that $(i_f, \upsilon_f)^*$ induces only a bijection of isomorphism classes of objects. A track category $\mathscr{T}$ is Σ-*representable* if each map f in $\mathscr{T}$ has a suspender $(\Sigma_f, i_f, \upsilon_f)$.

1.2. **Definition.** The dual notion of *looper* is obtained as a suspender in the opposite track category: a looper for $f : A \to B$ consists of a map $p_f : \Omega_f \to A$ and a track $\lambda_f : fp_f \Rightarrow fp_f$ satisfying conditions dual to the above ones for suspenders. A track category $\mathscr{T}$ is Ω-representable if each map f in $\mathscr{T}$ has a looper $(\Omega_f, p_f, \lambda_f)$.

Important particular cases are the suspenders and loopers for the identity map $1{=}\mathrm{id}_A : A \to A$ which will be denoted $\Sigma_*(A)$ and $\Omega_*(A)$ respectively; suspender for a map $0 : A \to *$ to the initial object will be called *suspension* of A and denoted $\Sigma_0(A)$, or simply $\Sigma(A)$ if the map 0 is uniquely determined by the context; and dually the looper for a map $0 : 1 \to A$ from the terminal object to A will be called *loop object* of A and denoted $\Omega_0(A)$ or $\Omega(A)$.

These examples are important in that sometimes suspenders or loopers of all maps can be constructed using solely Σ_* and Ω_* – indeed sometimes just using Σ_0 and Ω_0. See below.

We consider the following examples of Ω-representable and Σ-representable track categories.

1.3. **Example.** The track category $\mathfrak{Gpd}$ of groupoids is Ω-representable. In fact, for a functor $F : \mathbf{G} \to \mathbf{H}$ between groupoids the looper Ω_F is obtained by the pullback diagram

$$\begin{tikzcd}
\Omega_F \arrow[r] \arrow[d] & \mathbf{H}^{\circlearrowleft} \arrow[d] \\
\mathbf{G} \arrow[r, "F"] & \mathbf{H}.
\end{tikzcd}$$

Moreover the loop groupoid $\mathbf{H}^{\circlearrowleft}$ itself has the universal property for $\Omega_*(\mathbf{H})$.

1.4. **Example.** The track category $\mathbf{Top}^*$ of pointed topological spaces is Σ-representable. Let $IA = (A \times [0,1]) \,/\, (\{*\} \times [0,1])$ be the cylinder in $\mathbf{Top}^*$.

Then the suspender Σ_f of a map $f : A \to B$ is obtained by the pushout diagram

$$\begin{tikzcd} A \vee A \arrow[r, "(f,f)"] \arrow[d, "(i_0,i_1)"'] & B \arrow[d, "i_f"] \\ IA \arrow[r, "\upsilon_f"] & \Sigma_f. \end{tikzcd}$$

Here υ_f yields the track $\upsilon_f : i_f \Rightarrow i_f$ for the suspender Σ_f. Next let $PB = B^I$ be the space of maps $[0,1] \to B$ with the compact open topology. Then the looper Ω_f of f is obtained by the pullback diagram

$$\begin{tikzcd} \Omega_f \arrow[r, "\lambda_f"] \arrow[d, "p_f"'] & PB \arrow[d, "(q_0,q_1)"] \\ A \arrow[r, "(f,f)"] & B \times B. \end{tikzcd}$$

Here λ_f yields the track $\lambda_f : p_f \Rightarrow p_f$ for the looper Ω_f. In the next example we show that the properties 1.1 are satisfied for Σ_f and Ω_f respectively.

Let $\mathbf{C}$ be a cofibration category in the sense of Baues [3]. For each cofibrant object X in $\mathbf{C}$ we choose a cylinder

$$X \vee X \rightarrowtail IX \xrightarrow{\sim} X$$

which is a factorization of $(1,1) : X \vee X \to X$. For a fibrant object Y the homotopy classes relative to $X \vee X$ of maps $IX \to Y$ are the tracks in $\mathbf{C}$. Therefore the full subcategory $\mathbf{C}_{\mathrm{cf}}$ of cofibrant and fibrant objects in $\mathbf{C}$ is a track category; see [3, II §5].

1.5. **Lemma.** *For a cofibration category* $\mathbf{C}$ *the track category* $\mathbf{C}_{\mathrm{cf}}$ *is* Σ*-representable.*

Proof. For each cofibrant object X in $\mathbf{C}$ a fibrant model $j : X \overset{\sim}{\rightarrowtail} RX$ can be chosen. Now the suspender Σ_f of $f : A \to B$ in $\mathbf{C}_{\mathrm{cf}}$ is obtained by a fibrant model of the pushout Σ'_f in the following diagram

$$\begin{tikzcd} A \vee A \arrow[r, "(f,f)"] \arrow[d] \arrow[dr, phantom, "\text{push}"] & B \arrow[r, equal] \arrow[d, "i'_f"] & B \arrow[d, "i_f"] \\ IA \arrow[r, "\upsilon'_f"] & \Sigma'_f \arrow[r, tail, "\sim"] & \Sigma_f. \end{tikzcd}$$

The composite $\upsilon_f = j\upsilon'_f : IA \to \Sigma'_f \to \Sigma_f$ yields the track $\upsilon_f : i_f \Rightarrow i_f$. We now check that the properties (a) and (b) in 1.1 are satisfied. For a map $g : B \to C$ in $\mathbf{C}_{\mathrm{cf}}$ let $\eta : gf \simeq gf$ be a homotopy $\eta : IA \to C$. Then the pushout property of Σ'_f yields a map $g \cup \eta : \Sigma'_f \to C$ which admits an extension $\Sigma_\eta : \Sigma_f \to C$ so that $\Sigma_\eta i_f = g$ and $\eta = \Sigma_\eta \upsilon_f$. Hence we can actually choose the track ζ_η in

(a) to be the identity isomorphism of g. This proves (a). Now we check (b) as follows. Let $\gamma : h'i_f \simeq hi_f$ with

$$h'\upsilon_f = (h\upsilon_f)^{\gamma f} = -\gamma f + hf + \gamma f \tag{$*$}$$

as in (b). Here (*) is an equation of tracks. Now (*) implies that there is a map

$$\delta' : IIA \to C$$

with $\delta' i_0 = h\upsilon_f$, $\delta' i_1 = h'\upsilon_f$ and $\delta' Ii_0 = \delta' Ii_1 = \gamma If$. Here we choose the cylinder IB to be a fibrant object so that $If : IA \to IB$ is defined; see [3]. Now consider the following pushout diagram where $I(A \vee A) = IA \vee IA$.

$$\begin{tikzcd} I(A \vee A) \arrow[r, "{(If,If)}"] \arrow[d] \arrow[dr, phantom, "\text{push}"] & IB \arrow[r, "\gamma"] \arrow[d, "Ii'_f"] & C \\ IIA \arrow[r] & I\Sigma'_f . \end{tikzcd}$$

Here the pushout $I\Sigma'_f$ is actually a cylinder for Σ'_f and we define a cylinder $I\Sigma_f$ for Σ_f by the pushout diagram

$$\begin{tikzcd} \Sigma'_f \vee \Sigma'_f \arrow[r, tail, "j \vee j"] \arrow[d, "{(i_0,i_1)}"'] \arrow[dr, phantom, "\text{push}"] & \Sigma_f \vee \Sigma_f \arrow[d] \\ I\Sigma'_f \arrow[r, tail, "\sim"] & I\Sigma_f . \end{tikzcd}$$

Now the map $\delta' \cup \gamma : I\Sigma'_f \to C$ is defined with $(\delta' \cup \gamma) i_0 = h'j$ and $(\delta' \cup \gamma) i_1 = hj$. Hence a map $\delta = (\delta' \cup \gamma) \cup (h', h) : I\Sigma_f \to C$ is defined. The track defined by δ satisfies $\delta : h' \Rightarrow h$. Moreover $\delta i_f = \gamma$ since δi_f is represented by $(\delta' \cup \gamma)(Ii'_f) = \gamma$. □

For a model category $\mathbf{Q}$ as in [11] let $\mathbf{Q}_{\mathrm{c}}$ and $\mathbf{Q}_{\mathrm{f}}$ denote the full subcategory of cofibrant and fibrant objects, respectively. Then $\mathbf{Q}_{\mathrm{c}}$ is a cofibration category and $\mathbf{Q}_{\mathrm{f}}$ is a fibration category in the sense of Baues [3]. Here fibration category is the categorical dual of cofibration category. Therefore 1.5 above shows:

1.6. **Corollary.** *Let $\mathbf{Q}_{\mathrm{cf}}$ be the track category of cofibrant and fibrant objects in a Quillen model category. Then $\mathbf{Q}_{\mathrm{cf}}$ is Σ-representable and Ω-representable.*

The examples in 1.4 are also consequences of 1.5 since $\mathbf{Top}^*$ is a cofibration category and also a fibration category in which all objects are cofibrant and fibrant, compare [3]. Moreover using [3, Remark I.8.15] we see that also $\mathbf{Top}^*_0$ is a fibration category in which all objects are fibrant and cofibrant. Here we use the structure [3, I.3.3] and [3, I.4.6].

2. Functorial properties of suspenders

We consider functorial properties of suspenders. This implies a kind of uniqueness and compatibility with sums. For a category $\mathbf{T}$ the category $\mathrm{Pair}(\mathbf{T})$ is the usual category of pairs in $\mathbf{T}$. Objects of $\mathrm{Pair}(\mathbf{T})$ are morphisms $A \to B$ and morphisms from $(A \to B)$ to $(X \to Y)$ are commutative diagrams in $\mathbf{T}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y. \end{tikzcd}$$

2.1. **Lemma.** *For any commutative diagram of unbroken arrows*

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "q"] \arrow[d, "i_f"'] & Y \arrow[d, "i_g"] \\ \Sigma_f \arrow[r, dashed, "\Sigma_*(p,q)"'] \arrow[ur, Rightarrow, dashed, "\zeta_{(p,q)}"'] & \Sigma_g \end{tikzcd}$$

there exist a map $\Sigma_*(p,q)$ *and a track* $\zeta_{(p,q)}$ *as indicated, with*

$$\Sigma_*(p,q)\upsilon_f = (\upsilon_g p)^{\zeta_{(p,q)}f}. \tag{a}$$

Choosing such maps for each commutative square as above gives a functor

$$\Sigma_- : \mathrm{Pair}(\mathscr{T}) \to \mathrm{Pair}(\mathscr{T}_{\simeq})$$

carrying $f : A \to B$ *to* $[i_f] \in [B, \Sigma_f]$ *and the commutative square* $(p,q) : f \to g$ *as above to* $([q], [\Sigma_*(p,q)]) : [i_f] \to [i_g]$.

Proof. By the definition of suspenders, the track $\upsilon_g p$ considered as an automorphism of $i_g gp = i_g qf$ produces a map $\Sigma_{\upsilon_g p} : \Sigma_f \to \Sigma_g$ and a track $\zeta_{\upsilon_g p} : \Sigma_{\upsilon_g p} i_f \Rightarrow i_g q$ such that (a) holds. So one can define

$$\Sigma_*(p,q) = \Sigma_{\upsilon_g p}, \quad \zeta_{(p,q)} = \zeta_{\upsilon_g p}.$$

Then the injectivity condition for suspenders guarantees that there are tracks

$$\Sigma_*(\mathrm{id}_A, \mathrm{id}_B) \simeq \mathrm{id}_{\Sigma_f}$$

and

$$\Sigma_*(p,q)\Sigma_*(p',q') \simeq \Sigma_*(pp', qq')$$

for any two matching commutative squares. The lemma follows. $\square$

2.2. **Lemma.** *Let* $(\Sigma_f, i_f, \upsilon_f)$ *and* $(\Sigma'_f, i'_f, \upsilon'_f)$ *be two suspenders of a map* $f : A \to B$. *Then they are equivalent in* $\mathscr{T}$. *More precisely, there exist maps* $l : \Sigma_f \to \Sigma'_f$ *and* $l' : \Sigma'_f \to \Sigma_f$ *such that* $li_f = i'_f$, $l'i'_f = i_f$, $l\upsilon_f = \upsilon'_f$, *and* $l'\upsilon'_f = \upsilon_f$. *Moreover there exist tracks* $\lambda : l'l \simeq \mathrm{id}_{\Sigma_f}$, $\lambda' : ll' \simeq \mathrm{id}_{\Sigma'_f}$ *with* $\lambda i_f = \mathrm{id}_{i_f}$, $\lambda' i'_f = \mathrm{id}_{i'_f}$.

Proof. Existence of $l = \Sigma_{\upsilon'_f}$ and $l' = \Sigma_{\upsilon_f}$ satisfying the required identities is clear from the definition of suspenders. Then further by the uniqueness property of suspenders, for the identity track $\Sigma_{\upsilon'_f}\Sigma_{\upsilon_f} i_f = \Sigma_{\upsilon'_f} i'_f = i_f = \mathrm{id}_{\Sigma_f} i_f$ one has $\Sigma_{\upsilon'_f}\Sigma_{\upsilon_f}\upsilon_f = \Sigma_{\upsilon'_f}\upsilon'_f = \upsilon_f$, hence there is a track $\lambda : \Sigma_{\upsilon'_f}\Sigma_{\upsilon_f} \simeq \mathrm{id}_{\Sigma_f}$ with $\lambda i_f = \mathrm{id}_{i_f}$. In an exactly symmetric way one has λ' with required properties. $\square$

Also the converse is true:

2.3. **Lemma.** *Let* $(\Sigma_f, i_f, \upsilon_f)$ *be a suspender for the map* $f : A \to B$ *and let the maps* $l : \Sigma_f \to \Sigma$, $l' : \Sigma \to \Sigma_f$ *and tracks* $\lambda : l'l \simeq \mathrm{id}_{\Sigma_f}$, $\lambda' : ll' \simeq \mathrm{id}_{\Sigma}$ *realise a homotopy equivalence. Then* $(\Sigma, li_f, l\upsilon_f)$ *is another suspender for* f.

Proof. Consider the composite functor

$$[\![\Sigma, -]\!] \xrightarrow{[\![l,-]\!]} [\![\Sigma_f, -]\!] \xrightarrow{(i_f,\upsilon_f)^*} \mathbf{G}_f.$$

Clearly it coincides with $(li_f, l\upsilon_f)^*$. Moreover $(i_f, \upsilon_f)^*$ induces bijection on isomorphism classes by the universal property of suspenders, and so does $[\![l, -]\!]$ – in fact the latter is an equivalence, with inverse $[\![l', -]\!]$. Hence the lemma. $\square$

2.4. **Lemma.** *For any object* A, *a suspender for the map* $!_A : * \to A$ *from the (possibly weak) initial object to* A *is given by* $(A, \mathrm{id}_A, \mathrm{id}_{!_A})$. *Given suspenders* $(\Sigma_f, i_f, \upsilon_f)$ *and* $(\Sigma_{f'}, i_{f'}, \upsilon_{f'})$ *for the maps* $f : A \to B$ *and* $f' : A' \to B'$, *respectively,* $(\Sigma_f \vee \Sigma_{f'}, i_f \vee i_{f'}, \upsilon_f \vee \upsilon_{f'})$ *is a suspender for* $f \vee f' : A \vee A' \to B \vee B'$.

Proof. The first assertion follows easily from the fact that the functor $\mathbf{G}_{!_A}$ coincides with the covariant representable functor $[\![A, -]\!]$.

For the second, consider the functors

$$[\![\Sigma_f \vee \Sigma_{f'}, -]\!] \xrightarrow{\simeq} [\![\Sigma_f, -]\!] \times [\![\Sigma_{f'}, -]\!] \xrightarrow{(i_f,\upsilon_f)^* \times (i_{f'},\upsilon_{f'})^*} \mathbf{G}_f \times \mathbf{G}_{f'} \longrightarrow \mathbf{G}_{f \vee f'},$$

where the rightmost functor is the one assigning to $((g, \alpha), (g', \alpha'))$ with $g : B \to C$, $\alpha : gf \simeq gf$, $g' : B' \to C'$, $\alpha' : g'f' \simeq g'f'$ the pair $(\binom{g}{g'}, \binom{\alpha}{\alpha'})$, where $\binom{g}{g'} : B \vee B' \to C$ and $\binom{\alpha}{\alpha'} : \binom{gf}{g'f'} \simeq \binom{gf}{g'f'} \simeq \binom{g}{g'}(f \vee f')$ are obtained from the equivalences $[\![B \vee B', C]\!] \simeq [\![B, C]\!] \times [\![B', C]\!]$. It is clear how to define this functor on morphisms. One sees directly that this functor induces bijection on isomorphism classes of objects; hence so does the composite, which is easily seen to coincide with $(i_f \vee i_{f'}, \upsilon_f \vee \upsilon_{f'})^*$. The lemma follows. $\square$

3. Suspensions

Let $*$ be the initial object of a track category $\mathscr{T}$ in the strong sense so that the hom-groupoid $[\![*, X]\!]$ is the trivial groupoid for any X. Then the suspender $\Sigma_0 A$ of a map $0 : A \to *$ is termed a *suspension* (associated to 0) of A.

3.1. **Proposition.** *For any map* $0 : A \to *$ *to the initial object, the corresponding suspension* $\Sigma_0(A)$ *is canonically equipped with a cogroup structure in the homotopy category* $\mathscr{T}_{\simeq}$. *Moreover for any* $a : A' \to A$ *the induced map* (*see* 2.1) $\Sigma_*(f, \mathrm{id}_*) : \Sigma_{0a}(A) \to \Sigma_0(A)$ *respects this cogroup structure.*

Proof. Recall that the initial object is understood in the strong sense, so that $[\![*, X]\!]$ is a trivial groupoid for any X. It then follows that the groupoid $\mathbf{G}_0(X)$ has as many objects as there are tracks $\alpha : !_X 0 \simeq !_X 0$, and only identity morphisms. In other words, it is the discrete groupoid on the set $\mathrm{Aut}(!_X 0)$. Let us equip this set with a group structure coming from the obvious one on $\mathrm{Aut}(!_X 0)$. Then moreover the functor $\mathbf{G}_0(X) \to \mathbf{G}_0(Y)$ induced by a map $f : X \to Y$ is given on objects by $\alpha \mapsto f\alpha$, hence is a homomorphism of groups. One so obtains a lifting of the functor $\mathbf{G}_0$ to groups. But by the universal property of the suspender, this functor coincides with $[\Sigma_0, -]$. So considered as an object of $\mathscr{T}_{\simeq}$, the suspension Σ_0 has the property that its covariant representable functor lifts to the category of groups. It then follows by the standard categorical argument that this object has a cogroup structure in $\mathscr{T}_{\simeq}$. Explicitly, the cozero of this cogroup is Σ_{id_0}, i. e. the map $\Sigma_0 \to *$ induced by the pair $(\mathrm{id}_* : * \to *, \mathrm{id}_0 : 0\mathrm{id}_* = 0 \simeq 0 = 0\mathrm{id}_*)$. The coaddition map $+ : \Sigma_0 \to \Sigma_0 \vee \Sigma_0$ is induced by the pair $(!_{\Sigma_0 \vee \Sigma_0} : * \to \Sigma_0 \vee \Sigma_0, i_1 \upsilon_0 + i_2 \upsilon_0)$, where $\upsilon_0 \in \mathrm{Aut}(!_{\Sigma_0} 0)$ is the universal track and $i_1, i_2 : \Sigma_0 \to \Sigma_0 \vee \Sigma_0$ are the coproduct inclusions. The inverse map $\Sigma_0 \to \Sigma_0$ is induced by $(!_{\Sigma_0} 0, -\upsilon_0)$.

Now given any $a : A' \to A$, it obviously respects counit. To show that it respects coaddition, one must find a track $+\Sigma_*(a, \mathrm{id}_*) \simeq (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*))+$. According to the uniqueness property of the suspender Σ_{0a}, for this it is enough to find a track $\alpha : +\Sigma_*(a, \mathrm{id}_*) i_{0a} \simeq (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) + i_{0a}$ satisfying $+\Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} = ((\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) + \upsilon_{0a})^{\alpha 0a}$. There is a unique choice for such α – namely the identity track, as * is initial in the strong sense. Then $+\Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} = +\upsilon_0 a = (i_1 \upsilon_0 + i_2 \upsilon_0) a = i_1 \upsilon_0 a + i_2 \upsilon_0 a = i_1 \Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} + i_2 \Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} = ((\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) i_1 \upsilon_{0a} + (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) i_2 \upsilon_{0a} = (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*))(i_1 \upsilon_{0a} + i_2 \upsilon_{0a}) = (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) + \upsilon_{0a}$ as required. □

3.2. **Corollary.** *Suppose that an object* A *has a co-*H*-structure, i. e. a coaddition* $a : A \to A \vee A$ *with a two-sided cozero* $0 : A \to *$ *in* $\mathscr{T}_{\simeq}$. *Then the above canonical cogroup structure on* Σ_0 (*see* 3.1) *is coabelian.*

Proof. By 2.1 and 2.4, there are maps $0' = \Sigma_*(0, \mathrm{id}_*) : \Sigma_0 \to *$ and $+' = \Sigma_*(a, \mathrm{id}_*) : \Sigma_0 \to \Sigma_0 \vee \Sigma_0$ which equip Σ_0 with a co-H-structure in $\mathscr{T}_{\simeq}$. On the other hand it has a canonical cogroup structure $(\Sigma_{\mathrm{id}_0}, +, -)$ in $\mathscr{T}_{\simeq}$ by 3.1.

But in fact Σ_{id_0} and $\Sigma_*(0, \mathrm{id}_*)$ coincide in $\mathscr{T}_\simeq$, so it follows that these cogroup structures have the same cozero.

Moreover the fact that $+'$ respects the cogroup structure means commutativity of

$$\begin{tikzcd}
\Sigma_0 \arrow[r, "+"] \arrow[d, "+'"'] & \Sigma_0 \vee \Sigma_0 \arrow[d, "+' \vee +'"] \\
\Sigma_0 \vee \Sigma_0 \arrow[r, "+_2"] & \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0
\end{tikzcd}$$

in $\mathscr{T}_\simeq$, where $+_2$ is the coaddition for the canonical cogroup structure on $\Sigma_0 \vee \Sigma_0$ considered as $\Sigma_{\binom{0}{0}}$. It is clear that this cogroup structure coincides with the coproduct of cogroup structures on Σ_0. In general, for two cogroups X and Y the coaddition on their coproduct is given by

$$X \vee Y \xrightarrow{+_X \vee +_Y} X \vee X \vee Y \vee Y \xrightarrow{X \vee \binom{i_X}{i_Y} \vee Y} X \vee Y \vee X \vee Y,$$

so that $+_2$ is given by the composite

$$\Sigma_0 \vee \Sigma_0 \xrightarrow{+\vee+} \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \xrightarrow{(23)} \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0,$$

where (23) denotes the map permuting second and third summands.

Composing the above diagram with

$$\mathrm{id}_{\Sigma_0} \vee \begin{pmatrix} 0' \\ 0' \end{pmatrix} \vee \mathrm{id}_{\Sigma_0} : \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \to \Sigma_0 \vee * \vee * \vee \Sigma_0 \cong \Sigma_0 \vee \Sigma_0$$

then gives that there is a track $+' \simeq +$, whereas composing it with

$$\begin{pmatrix} 0' \vee \mathrm{id}_{\Sigma_0} \\ \mathrm{id}_{\Sigma_0} \vee 0' \end{pmatrix} : \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \to * \vee \Sigma_0 \vee \Sigma_0 \vee * \cong \Sigma_0 \vee \Sigma_0$$

gives that there is a track $+' \simeq (12)+$. This means that $+$ is coabelian in $\mathscr{T}_\simeq$. $\square$

4. Suspenders of coactions

We show that the suspender Σ_f of a map $f : A \to B$ splits as a coproduct if A has the structure of a principal coaction in the homotopy category. Recall that a *theory* **C** is a category with finite sums $A \vee B$.

4.1. **Definition.** A *principal coaction* in a theory **C** consists of an object A together with a cogroup object S in **C** and a right coaction

$$a : A \to A \vee S$$

on A such that the map

$$(i_A, q) : A \vee A \to A \vee S$$

is an isomorphism in **C**. The cogroup structure of S is given by maps $\mu : S \to S \vee S$, $\nu : S \to S$ and $e : S \to *$. The inverse of (i_A, a) yields the map $d : S \to A \vee A$.

A principal coaction is *trivial* if A is isomorphic to S in such a way that μ corresponds to the cogroup structure of S. It is well known that a principal coaction (A, a) is trivial if and only if there exists a map $A \to *$ in $\mathbf{C}$, where $*$ is the initial object of $\mathbf{C}$.

4.2. **Remark.** A *principal action* in a category with finite products consists of an object T together with an internal group G in this category and a right action $a : T \times G \to T$ of G on T such that the map $(p_T, a) : T \times G \to T \times T$ is an isomorphism (p_T being the product projection). Of course a principal action is the categorical dual of a principal coaction.

For our purposes we need a weaker notion which we call principal *quasi* action or *quasi torsor*. It consists of objects T, G and morphisms $T \times G \to T$, $d : T \times T \to G$, $1 \to G$, denoted via $(x, g) \mapsto x \cdot g$, $(x, y) \mapsto x\backslash y$, and e respectively, for $x, y :? \to T$, $g :? \to G$, such that the following identities hold:

- $x\backslash x = e$;
- $x \cdot (x\backslash y) = y$.

(Note that the above conditions imply also $x \cdot e = x$.)

Clearly, any principal action is a particular case of this, as one can define d to be the composite map

$$T \times T \xrightarrow{(p_T,a)^{-1}} T \times G \xrightarrow{p_G} G\ .$$

The categorical dual of a principal quasi action is a *principal quasi coaction* which generalizes the principal coaction in 4.1.

Recall that a *track theory* is a track category with coproducts $A \vee B$ in the weak sense (see [5]) so that for all X one has the equivalence of hom-groupoids

$$[\![A \vee B, X]\!] \xrightarrow{\sim} [\![A, X]\!] \times [\![B, X]\!].$$

Let ω be an inverse of this equivalence.

4.3. **Theorem.** *Let $\mathscr{T}$ be a track theory and let $f : A \to B$ be a map in $\mathscr{T}$. Assume A has the structure of a principal (quasi)coaction in the homotopy category $\mathscr{T}_{\simeq}$ represented by a map $a : A \to A \vee S$ in $\mathscr{T}$, where S is a cogroup in $\mathscr{T}_{\simeq}$. Let $\Sigma S = \Sigma_e S$ be a suspension of S in $\mathscr{T}$ associated to a map $e : S \to *$ in $\mathscr{T}$ representing the counit of S. Then there is a suspender of f with*

$$\Sigma_f = B \vee \Sigma S$$

and $i_f = i_B : B \to B \vee \Sigma S$ the coproduct inclusion and $\upsilon_f : i_B f \Rightarrow i_B f$ a certain canonically defined track.

The theorem shows that existence of certain suspensions in a track category implies existence of a wider class of suspenders. Moreover by 2.2 we get the following corollary.

4.4. **Corollary.** *Let $\mathscr{T}$ be a Σ-representable track theory and let $f : A \to B$ be a map in $\mathscr{T}$ where A admits the structure of a principal (quasi)coaction $A \to A \vee S$ in $\mathscr{T}_{\simeq}$. Then there exists a homotopy equivalence $\Sigma_f \simeq B \vee \Sigma S$ where ΣS is a suspension associated to a map $S \to *$ representing the counit of S.*

Proof of 4.3. To simplify exposition, let us introduce the following notation. The given principal coaction gives rise, for each object X, to functors

$$[\![A, X]\!] \times [\![S, X]\!] \xrightarrow{\omega} [\![A \vee S, X]\!] \xrightarrow{[\![a,X]\!]} [\![A, X]\!]$$

and

$$[\![A, X]\!] \times [\![A, X]\!] \xrightarrow{\omega} [\![A \vee A, X]\!] \xrightarrow{[\![d,X]\!]} [\![S, X]\!],$$

whose actions on both objects and morphisms will be denoted by

$$(x, s) \mapsto a \cdot s, \quad (x, y) \mapsto x \backslash y,$$

respectively. The principal coaction structure in $\mathscr{T}_{\simeq}$ implies existence of tracks $\varkappa$, λ which for any $x, y : A \to X$ induce tracks

$$x\varkappa : e \Rightarrow x \backslash x,$$

$$\binom{x}{y}\lambda : x \cdot (x \backslash y) \Rightarrow y.$$

Let us define another track ι by

$$\begin{tikzcd}
& x \cdot (x \backslash x) \arrow[dr, Rightarrow, "x\binom{\mathrm{id}_A}{\mathrm{id}_A}\lambda"] & \\
x \cdot e \arrow[ur, Rightarrow, "\mathrm{id}_x \cdot \varkappa"] \arrow[rr, Rightarrow, "x\iota"] & & x.
\end{tikzcd}$$

We now turn to the construction of the universal track υ_f. It is the composite track in the diagram

$$\begin{tikzcd}
& B \vee * \arrow[dl, bend right, "\mathrm{id}_B \vee !_{\Sigma S}"'] & B \arrow[l, "\cong"'] & & \\
B \vee \Sigma S \arrow[r, phantom, "\Downarrow \mathrm{id}_{\mathrm{id}_B} \vee \upsilon_e"] & {} & B \vee S \arrow[ul, "\mathrm{id}_B \vee e"'] \arrow[dl, "\mathrm{id}_B \vee e"] & A \vee S \arrow[l, "f \vee \mathrm{id}_S"'] \arrow[ul, "\binom{f}{!_B e}"'] \arrow[dl, "\binom{f}{!_B e}"] \arrow[u, phantom, "\Downarrow -f\iota"] \arrow[d, phantom, "\Downarrow f\iota"] & A \arrow[l, "a"'] \arrow[ull, bend right, "f"'] \arrow[dll, bend left, "f"] \\
& B \vee * \arrow[ul, bend left, "\mathrm{id}_B \vee !_{\Sigma S}"] & B \arrow[l, "\cong"'] & {} &
\end{tikzcd},$$

where the two parallelograms commute. More formally, $\upsilon_f = (\upsilon a)^{-i_B f \iota}$, where the track

$$\upsilon = (\mathrm{id}_{\mathrm{id}_B} \vee \upsilon_e)(f \vee \mathrm{id}_S) \in \mathrm{Aut}((\mathrm{id}_B \vee (!_{\Sigma S} e))(f \vee \mathrm{id}_S))$$

is considered as an automorphism of the map

$$
\begin{aligned}
i_B\begin{pmatrix} f \\ !_B e \end{pmatrix} &= (\mathrm{id}_B \vee !_{\Sigma S})(f \vee e) \\
&= (\mathrm{id}_B \vee !_{\Sigma S})(\mathrm{id}_B \vee e)(f \vee \mathrm{id}_S) \\
&= (\mathrm{id}_B \vee (!_{\Sigma S} e))(f \vee \mathrm{id}_S).
\end{aligned}
$$

To show that υ_f is indeed universal, we must show that, for each object X, the functor

$$\llbracket B, X \rrbracket \times \llbracket \Sigma S, X \rrbracket \cong \llbracket B \vee \Sigma S, X \rrbracket \xrightarrow{(i_B, \upsilon_f)^*} \mathbf{G}_f(X)$$

induces bijection on isomorphism classes of objects. Now υ_f is chosen in such a way that given $x : B \to X$ and a track $\varepsilon \in \mathrm{Aut}(!_X e)$ with the corresponding map $\Sigma_\varepsilon : \Sigma S \to X$, one has

$$(i_B, \upsilon_f)^*(x, \Sigma_\varepsilon) = (\mathrm{id}_{xf} \cdot \varepsilon)^{-xf\iota} : xf \simeq xf.$$

Taking into account the universal property of ΣS, we may replace isomorphism classes of $\llbracket \Sigma S, X \rrbracket$ by those of $\mathbf{G}_e(X)$. We thus must show

- For any $x : B \to X$ and any track $\alpha \in \mathrm{Aut}(xf)$ there is a track $\varepsilon \in \mathrm{Aut}(!_X e)$ such that $\mathrm{id}_{xf} \cdot \varepsilon = \alpha^{xf\iota}$;
- For any $x, x' : B \to X$, any $\varepsilon, \varepsilon' \in \mathrm{Aut}(!_X e)$ and any $\chi : x \simeq x'$ with $\mathrm{id}_{x'f} \cdot \varepsilon' = (\mathrm{id}_{xf} \cdot \varepsilon)^{\chi f}$ there is a track $\eta : \Sigma_{\varepsilon'} \to \Sigma_\varepsilon$.

For the first, define, for $\alpha \in \mathrm{Aut}(xf)$, the track $\varepsilon = (\mathrm{id}_{xf} \backslash \alpha)^{xf\varkappa}$. Then because of our special choice of ι the required identity will be satisfied.

For the second, note that if $\chi : x \simeq x'$ satisfies the hypothesis, then in the diagram

$$
\begin{tikzcd}
e \arrow[rrr, equal] \arrow[ddd, Rightarrow, "\varepsilon"'] & & & e \arrow[ddd, Rightarrow, "\varepsilon'"] \\
& xf\backslash(xf\cdot e) \arrow[ul, Rightarrow, "\binom{xf}{e}\lambda"'] \arrow[r, Rightarrow, "\chi f\backslash(\chi f\cdot e)"] \arrow[d, Rightarrow, "xf\backslash(xf\cdot\varepsilon)"] & x'f\backslash(x'f\cdot e) \arrow[ur, Rightarrow, "\binom{x'f}{e}\lambda"] \arrow[d, Rightarrow, "x'f\backslash(x'f\cdot\varepsilon')"] & \\
& xf\backslash(xf\cdot e) \arrow[dl, Rightarrow, "\binom{xf}{e}\lambda"] \arrow[r, Rightarrow, "\chi f\backslash(\chi f\cdot e)"] & x'f\backslash(x'f\cdot e) \arrow[dr, Rightarrow, "\binom{x'f}{e}\lambda"'] & \\
e \arrow[rrr, equal] & & & e
\end{tikzcd}
$$

all inner squares commute, hence the outer square commutes too, i. e. actually $\varepsilon = \varepsilon'$. $\square$

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