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ON THE RATIONALITY OF CERTAIN STRATA OF THE LANGE STRATIFICATION OF STABLE VECTOR BUNDLES ON CURVES

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Abstract. Let X be a smooth projective curve of genus $g \geq 2$ and $S(r,d)$ the moduli scheme of all rank r stable vector bundles of degree d on X. Fix an integer k with $0 < k < r$. H. Lange introduced a natural stratification of $S(r,d)$ using the degree of a rank k subbundle of any $E \in S(r,d)$ with maximal degree. Every non-dense stratum, say $W(k, r-k, a, d-a)$, has in a natural way a fiber structure $h : W(k, r-k, a, d-a) \to \mathrm{Pic}^a(X) \times \mathrm{Pic}^b(X)$ with h dominant. Here we study the rationality or the unirationality of the generic fiber of h.

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1. Introduction

Let X be a smooth complete algebraic curve of genus $g \geq 2$ defined over an algebraically closed base field K with $\mathrm{char}(K) = 0$. Fix integers r, d with $r \geq 1$ and $L \in \mathrm{Pic}(X)$. Let $S_L(r,d)$ be the moduli scheme of stable rank r vector bundles on X with determinant L and $S(r,d)$ the moduli scheme of all stable rank r vector bundles on X with degree d. It is well-known ([14]) that $S(r,d)$ (resp. $S_L(r,d)$) is smooth, irreducible, of dimension $(r^2-1)(g-1)+g$ (resp. $(r^2-1)(g-1)$) and that $S_L(r,d)$ is unirational. The variety $S_L(r,d)$ is a fine moduli scheme if and only if $(r,d) = 1$. P. E. Newstead ([11]) proved in many cases that $S_L(r,d)$ is rational. For other cases, see [1]. By [5] $S_L(r,d)$ is rational if $(r,d) = 1$. In [6] H. Lange introduced the following stratification (called the Lange stratification) of the moduli scheme $S(r,d)$, $r \geq 2$, depending on the choice of an integer k with $0 < k < r$. For any rank r vector bundle E set $s_k(E) := k(\deg(E)) - r(\deg(A))$, where A is a rank k subsheaf of E with maximal degree. By [9] we have $s_k(E) \leq gk(r-k)$. If E is stable, then $s_k(E) > 0$. By [4], sect. 4, (see [8], Remark 3.14) for any L and a general $E \in S_L(r,d)$ we have $s_k(E) = k(r-k)(g-1)+e$, where e is the unique integer with $(r-1)(g-1) \leq e \leq (r-1)g$ and $e + k(r-k)(g-1) \equiv kd \bmod (r)$. For any integer a set $V(k, r-k, a, d-a) := \{E \in S(r,d) : s_k(E) = kd - ra\}$. This gives a stratification of $S(r,d)$ which will be called the Lange stratification of $S(r,d)$. Here we study the rationality or the unirationality of smaller strata

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of this stratification. Hence (setting $b = d - a$) we fix integers r, k, a, b with $0 < k < r$ and $a/r < b/(r-k) < a/r + g - 1$. By [12], Th. 0.1, there is a non-empty open irreducible subset $W(k, r-a, a, b)$ of $V(k, r-k, a, b)$ such that every $E \in W(k, r-k, a, b)$ fits in an exact sequence

$$0 \to H \to E \to Q \to 0 \tag{1}$$

with H computing $s_k(E)$ (i.e. with $\mathrm{rank}(H) = k$, $\deg(H) = a$, $\mathrm{rank}(Q) = r-k$ and $\deg(Q) = b$), H and Q stable and such that H is the only rank k subsheaf of E computing $s_k(E)$. This means that (up to a scalar) E fits in a unique extension (1). Furthermore, varying E in $W(k, r-k, a, a, b)$, the pairs (H, Q) obtained in this way cover a Zariski dense constructible subset of $S(k, a) \times S(r-k, b)$. Conversely, the generic extension of the generic element of $S(r-k, b)$ by the generic element of $S(k, a)$ is the generic element of $W(k, r-k, a, b)$. Hence there is a rational dominant map $W(k, r-k, a, b) \to \mathrm{Pic}^a(X) \times \mathrm{Pic}^b(X) \cong \mathrm{Alb}(X) \times \mathrm{Alb}(X)$ sending E into $(\det(H), \det(Q))$. For any $L \in \mathrm{Pic}^a(X)$ and any $M \in \mathrm{Pic}^b(X)$ set $W(k, r-k, a, b, L, M) := \{E \in W(k, r-k, a, b) : E$ fits in a unique exact sequence (1) and $L \cong \det(H)$ and $M \cong \det(Q)\ \}$. We are interested in the rationality or unirationality of the strata $W(k, r-k, a, b, L, M)$. In this paper we prove the following results.

Theorem 1. *Fix integers r, k, a, b with $0 < k < r$ and $a/r < b/(r-k) < a/r + g - 1$. Then for a general pair $(L, M) \in Pic^a(X) \times Pic^b(X)$ the variety $W(k, r-k, a, b, L, M)$ is unirational.*

Theorem 2. *Fix integers r, k, a, b with $0 < k < r$, $a/r < b/(r-k) < a/r + g - 1$, $(k, a) = 1$ and $(r-k, b) = 1$. Then for a general pair $(L, M) \in Pic^a(X) \times Pic^b(X)$ the variety $W(k, r-k, a, b, L, M)$ is rational.*

Proofs of Theorems 1 and 2

Lemma 1. *Fix integers u, v, a and b with $u > 0$ and $v > 0$ and take a general pair $(L, M) \in Pic^a(X) \times Pic^b(X)$. Then for a general pair $(A, B) \in S_L(u, a) \times S_M(v, b)$ we have $h^0(X, Hom(A, B)) = \max\{0, bu - av + uv(1-g)\}$ and $h^1(X, Hom(A, B)) = \max\{0, -bu + av + uv(g-1)\}$.*

Proof. Without the restrictions $\det(A) \cong L$ and $\det(B) \cong M$, this is a result of A. Hirschowitz (see [2], sect. 4, or [13], Th. 1.2, for a published proof). By semicontinuity and the openness of stability we obtain the result for a general pair (L, M). $\square$

Lemma 2. *Fix integers u, v, a and b with $u > 0$, $v > 0$ and $a/u < b/v$ and take a general pair $(L, M) \in Pic^a(X) \times Pic^b(X)$. Then for a general pair $(A, B) \in S_L(u, a) \times S_M(v, b)$ the general extension of B by A is stable.*

Proof. Without the restrictions $\det(A) \cong L$ and $\det(B) \cong M$, this is proved in [13] during the proof of [13], Theorems 0.1 and 0.2. By the openness of stability we obtain the result for a general pair (L, M). $\square$

Lemma 3. *Fix integers u, v, a and b with $u > 0$, $v > 0$ and $a/u < b/v < a/u + g - 1$. Take a general pair $(L, M) \in Pic^a(X) \times Pic^b(X)$. Then for a general pair $(A, B) \in S_L(u, a) \times S_M(v, b)$ the general extension, E, of B by A is stable, $s_u(E) = ub - va$ and A is the only rank u subbundle of E computing $s_u(E)$.*

Proof. Without the restrictions $\det(A) \cong L$ and $\det(B) \cong M$, this is [13], Th. 0.1. By the openness of stability and the semicontinuity of the Lange invariant s_u we obtain the result for a general pair (L, M). $\square$

Now we can prove Theorems 2 and 1.

Proof of Theorem 2. The variety $S_L(k, a) \times S_M(r - k, b)$ is rational by [5], Th. 1.2. Since $(k, a) = (r - k, b) = 1$, both $S_L(k, a)$ and $S_M(r - k, b)$ are fine moduli spaces and hence there is a universal family, U, of pairs (A, B) of vector bundles on $S_L(k, a) \times S_M(r - k, b)$. For every $(A, B) \in S_L(k, a) \times S_M(r - k, b)$ we have $h^0(X, Hom(A, B)) = 0$ because $\mu(B) = b/(r - k) > a/k = \mu(A)$ and both A and B are stable. Thus $h^1(X, Hom(A, B)) = kb - (r - k)a + k(r - k)(g - 1)$ (Riemann–Roch), i.e. $h^1(X, Hom(A, B))$ does not depend from the choice of the pair $(A, B) \in S_L(k, a) \times S_M(r - k, b)$ but only from the integers k, r, a and b. Thus the vector spaces $H^1(X, Hom(A, B))$, $(A, B) \in S_L(k, a) \times S_M(r - k, b)$, fit together to form a vector bundle EXT on $S_L(k, a) \times S_M(r - k, b)$: the relative Ext-functor considered in [7]; here we need the existence of U (i.e. the conditions $(k, a) = (r - k, b) = 1$) for the construction of EXT. Since EXT is a vector bundle over an irreducible rational variety, the total space of EXT is an irreducible rational variety. By [13], Th. 0.1, a non-empty open subset V of EXT corresponds to elements of $W(k, r - k, a, b, L, M)$ and conversely a general element of $W(k, r - k, a, b, L, M)$ corresponds to a general element of EXT. Hence there is a rational dominant map, f, from EXT into $W(k, r - k, a, b, L, M)$. As explained in the introduction, the uniqueness part in [13], Th. 0.1, means that the rational map f induces a generically bijective map from the projective bundle $P(EXT))$ onto $W(k, r - k, a, b, L; M)$. Since $P(EXT))$ is rational and $\text{char}(K) = 0$, we conclude. $\square$

Proof of Theorem 1. Fix integers x, y with $x > 0$, $P \in X$ and $R \in \text{Pic}^y(X)$. Since $S_R(x, y) \cong S_{R(uxP)}(x, y + ux)$ for every integer u, we will assume y very large, say $y > x(2g - 1)$. By the very construction of $S_R(x, y)$, $y > x(2g - 1)$, using Geometric Invariant Theory, there is a smooth variety $U_R(x, y)$ with a PGL(N)-action, $N = y + x(1 - g)$, without any fixed point and a morphism $f_{x,y} : U_R(x, y) \to S_R(x, y)$ which make $S_R(x, y)$ the GIT-quotient of $U_R(x, y)$ and such that on $U_R(x, y) \times X$ there exists a total family of vector bundles on X with R as determinant. We repeat the proof of Theorem 2 using $U_L(k, a) \times U_M(r - k, b)$ instead of $S_L(k, a) \times S_M(r - k, b)$. Since on $U_L(k, a) \times U_M(r - k, b)$ there is a family of pairs of stable vector bundles, we may take a global EXT which is a vector bundle over $U_L(k, a) \times U_M(r - k, b)$ and hence it is irreducible and rational. By [13], Th. 0.1, there are a non-empty open subset V of EXT and a dominant morphism $f : V \to W(k, r-k, a, b, L, M)$. Thus $W(k, r-k, a, b, L, M)$ is unirational. $\square$

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