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ON THE ξ-EXPONENTIALLY ASYMPTOTIC STABILITY OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Necessary and sufficient conditions and effective sufficient conditions are established for the so-called ξ-exponentially asymptotic stability of the linear system

$$dx(t) = dA(t) \cdot x(t) + df(t),$$

where $A : [0, +\infty[\to \mathbb{R}^{n\times n}$ and $f : [0, +\infty[\to \mathbb{R}^n$ are respectively matrix- and vector-functions with bounded variation components, on every closed interval from $[0, +\infty[$ and $\xi : [0, +\infty[\to [0, +\infty[$ is a nondecreasing function such that $\lim\limits_{t\to+\infty} \xi(t) = +\infty$.

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Let the components of matrix-functions $A : [0, +\infty[\to \mathbb{R}^{n\times n}$ and vector-functions $f : [0, +\infty[\to \mathbb{R}^n$ have bounded total variations on every closed segment from $[0, +\infty[$.

In this paper, sufficient (necessary and sufficient) conditions are given for the so-called ξ-exponentially asymptotic stability in the Lyapunov sense for the linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t). \tag{1}$$

The theory of generalized ordinary differential equations enables one to investigate ordinary differential, difference and impulsive equations from the unified standpoint. Quite a number of questions of this theory have been studied sufficiently well ([1]–[3], [5], [6], [8], [10], [11]).

The stability theory has been investigated thoroughly for ordinary differential equations (see [4], [7] and the references therein). As to the questions of stability for impulsive equations and for generalized ordinary differential equations they are studied, e.g., in [3], [9], [10] (see also the references therein).

The following notation and definitions will be used in the paper:

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ and $]a, b[$ (a, $b \in \mathbb{R}$) are, respectively, closed and open intervals.

$\operatorname{Re} z$ is the real part of the complex number z.

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$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|,\quad |X|=(|x_{ij}|)_{i,j=1}^{n,m},$$

$$\mathbb{R}_{+}^{n\times m}=\Big\{X=(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\Big\}.$$

The components of the matrix-function X are also denoted by $[x]_{ij}$ $(i=1,\dots,n;\ j=1,\dots,m)$.

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X. I_n is the identity $n\times n$-matrix; $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1,\dots,\lambda_n$.

$r(H)$ is the spectral radius of the matrix $H\in\mathbb{R}^{n\times n}$.

$\bigvee\limits_{0}^{+\infty}(X)=\sup\limits_{b\in\mathbb{R}_+}\bigvee\limits_{0}^{b}(X)$, where $\bigvee\limits_{0}^{b}(X)$ is the sum of total variations on $[0,b]$ of the components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$ of the matrix-function $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(0)=0$ and $v(x_{ij})(t)=\bigvee\limits_{0}^{t}(x_{ij})$ for $0<t<+\infty$ $(i=1,\dots,n;\ j=1,\dots,m)$.

$X(t-)$ and $X(t+)$ are the left and the right limit of the matrix-function $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ at the point t; $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ of bounded total variation on every closed segment from $\mathbb{R}_+$.

$L_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ such that their components are measurable and integrable functions in the Lebesgue sense on every closed segment from $\mathbb{R}_+$.

$\widetilde{C}_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ such that their components are absolutely continuous functions on every closed segment from $\mathbb{R}_+$.

$s_0:BV_{loc}(\mathbb{R}_+,\mathbb{R})\to BV_{loc}(\mathbb{R}_+,\mathbb{R})$ is the operator defined by

$$s_0(x)(t)\equiv x(t)-\sum_{0<\tau\leq t}d_1x(\tau)-\sum_{0\leq\tau<t}d_2x(\tau).$$

If $g:\mathbb{R}_+\to\mathbb{R}$ is a nondecreasing function $x:\mathbb{R}_+\to\mathbb{R}$ and $0\leq s<t<+\infty$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,dg_1(\tau)-\int_{]s,t[}x(\tau)\,dg_2(\tau)$$
$$+\sum_{s<\tau\leq t}x(\tau)\,d_1g(\tau)-\sum_{s\leq\tau<t}x(\tau)\,d_2g(\tau),$$

where $g_j:\mathbb{R}_+\to\mathbb{R}$ $(j=1,2)$ are continuous nondecreasing functions such that $g_1(t)-g_2(t)\equiv s_0(g)(t)$, and $\int_{]s,t[}x(\tau)\,dg_j(\tau)$ is the Lebesgue–Stiltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function g_j $(j=1,2)$ (if $s=t$, then $\int\limits_s^t x(\tau)\,dg(\tau)=0$).

A matrix-function is said to be nondecreasing if each of its components is nondecreasing.

If $G=(g_{ik})_{i,k=1}^{\ell,n}:\mathbb{R}_+\to\mathbb{R}^{\ell\times n}$ is a nondecreasing matrix-function, $X=(x_{ik})_{i,k=1}^{n,m}:\mathbb{R}_+\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\Big(\sum_{k=1}^n\int\limits_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{\ell,m}\ \text{ for }\ 0\le s\le t<+\infty,$$

$$S_0(G)(t)\equiv\Big(s_0(g_{ik})(t)\Big)_{i,k=1}^{\ell,n}.$$

If $G_j:\mathbb{R}_+\to\mathbb{R}^{\ell\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G(t)\equiv G_1(t)-G_2(t)$ and $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\int\limits_s^t dG_1(\tau)\cdot X(\tau)-\int\limits_s^t dG_2(\tau)\cdot X(\tau)\ \text{ for }\ 0\le s\le t<+\infty.$$

$\mathcal{A}$ and $\mathcal{B}:BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})\times BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})\to BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ are the operators defined, respectively, by

$$\begin{aligned}\mathcal{A}(X,Y)(t)=Y(t)&+\sum_{0<\tau\le t}d_1X(\tau)\cdot\Big(I_n-d_1X(\tau)\Big)^{-1}d_1Y(\tau)\\&-\sum_{0\le\tau<t}d_2X(\tau)\cdot\Big(I_n+d_2X(\tau)\Big)^{-1}d_2Y(\tau)\ \text{ for }\ t\in\mathbb{R}_+\end{aligned}$$

and

$$\mathcal{B}(X,Y)(t)=X(t)Y(t)-X(0)Y(0)-\int\limits_0^t dX(\tau)\cdot Y(\tau)\ \text{ for }\ t\in\mathbb{R}_+.$$

$\mathcal{L}:BV_{loc}^2(\mathbb{R}_+,\mathbb{R}^{n\times n})\to BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ is an operator given by

$$\mathcal{L}(X,Y)(t)=\int\limits_0^t d\Big(X(\tau)+\mathcal{B}(X,Y)(\tau)\Big)\cdot X^{-1}(\tau)\ \text{ for }\ t\in\mathbb{R}_+.$$

We will use the following properties of these operators (see [2]):

$$\mathcal{B}\Big(X,\mathcal{B}(Y,Z)\Big)(t)\equiv\mathcal{B}(XY,Z)(t),$$

$$\mathcal{B}\Big(X,\int\limits_0^{\cdot} dY(s)\cdot Z(s)\Big)(t)\equiv\int\limits_0^t d\mathcal{B}(X,Y)(s)\cdot Z(s).$$

Under a solution of the system (1) we understand a vector-function $x \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \ \ (0 \le s \le t < +\infty).$$

Note that the linear system of ordinary differential equations

$$\frac{dx}{dt} = P(t)x + q(t) \ \ (t \in \mathbb{R}_+), \tag{2}$$

where $P \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $q \in L_{loc}(\mathbb{R}_+, \mathbb{R}^n)$, can be rewritten in form (1) if we set

$$A(t) \equiv \int_0^t P(\tau)\,d\tau, \quad f(t) \equiv \int_0^t q(\tau)\,d\tau.$$

We assume that $A \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $f \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^n)$, $A(0) = O_{n\times n}$ and

$$\det\Big(I_n + (-1)^j\, d_j A(t)\Big) \ne 0 \ \text{ for } \ t \in \mathbb{R}_+ \ \ (j = 1, 2).$$

These conditions guarantee the unique solvability of the Cauchy problem for system (1) (see [11]).

Definition 1. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty. \tag{3}$$

Then the solution x_0 of system (1) is called ξ-exponentially asymptotic stable if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of system (1), satisfying the inequality $\|x(t_0) - x_0(t_0)\| < \delta$ for some $t_0 \in \mathbb{R}_+$, admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp\Big(-\eta\Big(\xi(t) - \xi(t_0)\Big)\Big) \ \text{ for } \ t \ge t_0.$$

Stability, uniform stability, asymptotic stability and exponentially asymptotic stability are defined just in the same way as for systems of ordinary differential equations, i.e., when $A(t) \equiv \operatorname{diag}(t, \dots, t)$ (see, e.g., [4] or [7]). Note that the exponentially asymptotic stability is a particular case of the ξ-exponentially asymptotic stablility if we assume $\xi(t) \equiv t$.

Definition 2. System (1) is called stable in this or another sense if every solution of this system is stable in the same sense.

We will use the following propositions.

Proposition 1. *System* (1) *is ξ-exponentially asymptotically stable* (*uniformly stable*) *if and only if its corresponding homogeneous system*

$$dx(t) = dA(t) \cdot x(t) \tag{1_0}$$

is ξ-exponentially asymptotically stable (*uniformly stable*).

Proposition 2. *System* (1_0) *is* ξ*-exponentially asymptotically stable* (*uniformly stable*) *if and only if its zero solution is* ξ*-exponentially asymptotically stable* (*uniformly stable*).

Proposition 3. *System* (1_0) *is* ξ*-exponentially asymptotically stable* (*uniformly stable*) *if and only if there exist positive numbers* ρ *and* η *such that*

$$\|U(t,s)\| \leq \rho \exp\Big(-\eta\big(\xi(t)-\xi(s)\big)\Big) \ \ for \ \ t \geq s \geq 0$$
$$\Big(\|U(t,s)\| \leq \rho \ \ for \ \ t \geq s \geq 0\Big),$$

where U *is the Cauchy matrix of system* (1_0).

The proofs of these propositions are analogous to those for ordinary differential equations.

Therefore, the ξ-exponentially asymptotic stability (uniform stability) is not the property of a solution of system (1). It is the common property of all solutions and a vector-function f does not influence on this property. Hence the ξ-exponentially asymptotic stability (uniform stability) is the property of the matrix-function A and the following definition is natural.

Definition 3. The matrix-function A is called ξ-exponentially asymptotically stable (uniformly stable) if the system (1_0) is ξ-exponentially asymptotically stable (uniformly stable).

Theorem 1. *Let the matrix-function* $A_0 \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ *be* ξ*-exponentially asymptotically stable,*

$$\det\Big(I_n + (-1)^j\, d_j A_0(t)\Big) \neq 0 \ \ for \ \ t \in \mathbb{R}_+ \ \ (j=1,2) \tag{4}$$

and

$$\lim_{t\to+\infty} \bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(A_0, A-A_0) = 0, \tag{5}$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing function satisfying condition* (3),

$$\nu(\xi)(t) = \sup\Big\{\tau \geq t : \ \ \xi(\tau) \leq \xi(t+) + 1\Big\}.$$

Then the matrix-function A *is* ξ*-exponentially asymptotically stable as well.*

To prove the theorem we will use the following lemma.

Lemma 1. *Let the matrix-function* $A_0 \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ *satisfy condition* (4). *Let, moreover, the following conditions hold:*

(a) *the Cauchy matrix* U_0 *of the system*

$$dx(t) = dA_0(t)\cdot x(t) \tag{6}$$

satisfies the inequality

$$|U_0(t,t_0)| \leq \Omega \exp\Big(-\xi(t)+\xi(t_0)\Big) \ \ (t \geq t_0) \tag{7}$$

for some $t_0 \in \mathbb{R}_+$, *where* $\Omega \in \mathbb{R}_+^{n\times n}$, *and* ξ *is a function from* $BV_{loc}(\mathbb{R}_+,\mathbb{R})$ *satisfying* (3)*;*

(b) *there exists a matrix* $H \in \mathbb{R}_+^{n\times n}$ *such that*

$$r(H) < 1 \tag{8}$$

and

$$\int_{t_0}^{t} \exp\Big(\xi(t)-\xi(\tau)\Big)|U_0(t,\tau)|\,dV\Big(\mathcal{A}(A_0,A-A_0)\Big)(\tau) < H \ \ \textit{for}\ \ t\geq t_0. \tag{9}$$

Then an arbitrary solution x *of system* (1) *admits an estimate*

$$|x(t)| \leq R|x(t_0)|\exp\Big(-\xi(t)+\xi(t_0)\Big) \ \ \textit{for}\ \ t\geq t_0, \tag{10}$$

where $R=(I_n-H)^{-1}\Omega$.

Proof. Let $A=(a_{ik})_{i,k=1}^n$, $A_0=(a_{0ik})_{i,k=1}^n$, $U_0=(u_{0ik})_{i,k=1}^n$, $H=(h_{ik})_{i,k=1}^n$, and $x=(x_i)_{i=1}^n$ be an arbitrary solution of system (1_0).

According to the variation of constants formula and properties of the Cauchy matrix U_0 (see [11]) we have

$$\begin{aligned}
x(t) &= U_0(t,t_0)x(t_0) + \int_{t_0}^{t} U_0(t,s)\,d\Big(A(s)-A_0(s)\Big)\cdot x(s)\\
&\quad - \sum_{t_0<s\leq t} d_1U_0(t,s)\cdot d_1\Big(A(s)-A_0(s)\Big)\cdot x(s)\\
&\quad + \sum_{t_0\leq s<t} d_2U_0(t,s)\cdot d_2\Big(A(s)-A_0(s)\Big)\cdot x(s)\\
&= U_0(t,t_0)x(t_0) + \int_{t_0}^{t} U_0(t,s)\,d\Big(A(s)-A_0(s)\Big)\cdot x(s)\\
&\quad + \sum_{t_0<s\leq t} U_0(t,s)\,d_1A(s)\cdot\Big(I_n-d_1A_0(s)\Big)^{-1}d_1\Big(A(s)-A_0(s)\Big)\cdot x(s)\\
&\quad - \sum_{t_0\leq s<t} U_0(t,s)\,d_2A(s)\cdot\Big(I_n+d_2A_0(s)\Big)^{-1}d_2\Big(A(s)-A_0(s)\Big)\cdot x(s).
\end{aligned}$$

Therefore

$$x(t) = U_0(t,t_0)x(t_0) + \int_{t_0}^{t} U(t,\tau)\,d\mathcal{A}(A_0,A-A_0)(\tau)\cdot x(\tau) \ \ \text{for}\ \ t\geq t_0. \tag{11}$$

Let

$$\begin{aligned}
y_k(t) &= \max\Big\{\exp\Big(\xi(\tau)-\xi(t_0)\Big)\cdot|x_k(\tau)|:\ \ t_0\leq\tau\leq t\Big\},\\
y(t) &= \Big(y_k(t)\Big)_{k=1}^n.
\end{aligned}$$

Then

$$\left|\sum_{j,k=1}^{n}\int_{t_0}^{t} u_{0ij}(t,\tau)x_k(\tau)\,d(b_{jk})(\tau)\right| \le \sum_{j,k=1}^{n}\int_{t_0}^{t} |u_{0ij}(t,\tau)|\,|x_k(\tau)|\,dv(b_{jk})(\tau)$$

$$\le \sum_{k,j=1}^{n}\int_{t_0}^{t} \exp\Big(-\xi(\tau)+\xi(t_0)\Big)|u_{0ij}(t,\tau)|\,dv(b_{jk})(\tau)\cdot y_k(t)$$

$$\text{for } t\ge t_0, \quad (i=1,\dots,n),$$

where $b_{jk}(t)\equiv \mathcal{A}(a_{0jk},a_{jk}-a_{0jk})(t)$ $(j,k=1,\dots,n)$. From this and (11) we have

$$\exp\Big(\xi(t)-\xi(t_0)\Big)\cdot|x_i(t)| \le \sum_{k=1}^{n}\exp\Big(\xi(t)-\xi(t_0)\Big)|u_{0ik}(t,t_0)|\,|x_k(t_0)|$$

$$+\sum_{k,j=1}^{n}\int_{t_0}^{t} \exp\Big(\xi(t)-\xi(\tau)\Big)|u_{0ij}(t,\tau)|\,dv(b_{jk})(\tau)\cdot y_k(t)$$

$$\text{for } t\ge t_0, \quad (i=1,\dots,n).$$

By this, (7) and (9) we obtain

$$y(t)\le \Omega|x(t_0)|+Hy(t) \ \text{ for } \ t\ge t_0.$$

Hence

$$(I_n-H)y(t)\le \Omega|x(t_0)| \ \text{ for } \ t\ge t_0. \tag{12}$$

On the other hand, by (8) the matrix I_n-H is nonsingular and the matrix $(I_n-H)^{-1}$ is nonnegative since H is a nonnegative matrix. From this, (12) and the definition of y we have

$$y(t)\le (I_n-H)^{-1}\Omega|x(t_0)| \ \text{ for } \ t\ge t_0$$

and

$$|x(t)|\le (I_n-H)^{-1}\Omega|x(t_0)|\exp\Big(-\xi(t)+\xi(t_0)\Big) \ \text{ for } \ t\ge t_0.$$

Therefore estimate (10) is proved. □

Proof of Theorem 1. By the ξ-exponentially asymptotic stability of the matrix-function A_0 and Proposition 3 there exist positive numbers η and ρ_0 such that the Cauchy matrix U_0 of system (6) satisfies the estimate

$$|U_0(t,\tau)|\le R_0\exp\Big(-2\eta\Big(\xi(t)-\xi(\tau)\Big)\Big) \ \text{ for } \ t\ge\tau\ge 0, \tag{13}$$

where R_0 is an $n\times n$ matrix whose every component equals ρ_0.

Let

$$\varepsilon=(4n\rho_0)^{-1}\Big(\exp(\eta)-1\Big)\exp(-2\eta). \tag{14}$$

By (5) there exists $t^* \in \mathbb{R}_+$ such that

$$\bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(A_0, A - A_0) < \varepsilon \ \text{ for } \ t \geq t^*. \tag{15}$$

On the other hand, by (13) we have

$$\int_{t_0}^{t} \exp\Big(\eta\big(\xi(t) - \xi(\tau)\big)\Big)|U_0(t,\tau)|\, dV(B)(\tau) \leq \mathcal{J}(t) \ \ (t \geq t_0) \tag{16}$$

for every $t_0 \geq 0$, where $B(t) \equiv \mathcal{A}(A_0, A - A_0)(t)$ and

$$\mathcal{J}(t) \equiv R_0 \int_{t_0}^{t} \exp\Big(-\eta\big(\xi(t) - \xi(\tau)\big)\Big)\, dV(B)(\tau).$$

Let $k(t)$ be the integer part of $\xi(t) - \xi(t_0)$ for every $t \geq t_0$,

$$T_i = \Big\{\tau \geq t_0 : \ \ \xi(t_0) + i \leq \xi(\tau) < \xi(t_0) + i + 1\Big\} \ \ (i = 0, \dots, k(t)),$$

where $k_i = k(t_i)$ $(i = 0, \dots, k(t))$, the points $t_0, t_1, \dots, t_{k(t)}$ are defined by

$$t_0 = \sup T_0, \quad t_i = \begin{cases} t_{i-1} & \text{if } \ T_i = \varnothing \\ \sup T_i & \text{if } \ T_i \neq \varnothing \end{cases} \quad (i = 1, \dots, k(t)).$$

Let us show that

$$t_i \leq \nu(\xi)(t_{i-1}) \ \ (i = 1, \dots, k(t)). \tag{17}$$

If $T_i = \varnothing$, then (17) is evident.

Let now $T_i \neq \varnothing$. It is sufficient to show that

$$T_i \subset Q_i, \tag{18}$$

where

$$Q_i = \Big\{\tau : \ \xi(\tau) < \xi(t_{i-1}+) + 1\Big\}.$$

It is easy to verify that

$$\xi(t_{i-1}+) \geq \xi(t_0) + i. \tag{19}$$

Indeed, otherwise there exists $\delta > 0$ such that

$$\xi(t_{i-1} + s) < \xi(t_0) + i \ \text{ for } \ 0 \leq s \leq \delta.$$

On the other hand, by the definition of t_{i-1} we have

$$\xi(t_0) + i - 1 \leq \xi(t_{i-1}-)$$

and therefore

$$\xi(t_0) + i - 1 \leq \xi(t_{i-1} + s) < \xi(t_0) + i \ \text{ for } \ 0 \leq s \leq \delta.$$

But this contradicts the definition of t_{i-1}.

Let $\tau \in T_i$. Then from (19) and the inequality $\xi(\tau) < \xi(t_0) + i + 1$ it follows that $\xi(\tau) < \xi(t_{i-1}+) + 1$, $\tau_i \in Q_i$. Hence (17) is proved.

Let $t_0 \geq t^*$. Then according to (15) and (17) we get

$$\mathcal{J}(t) \leq R_0 \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big) \sum_{i=1}^{1+k(t)} \int_{t_{i-1}}^{t_i} \exp\Big(\eta\big(\xi(\tau) - \xi(t_0)\big)\Big)\, dV(B)(\tau)$$

$$= R_0 \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big)\Bigg(\sum_{i=1,\, i=1+k_i}^{1+k(t)} \int_{t_{i-1}}^{t_i} \exp\Big(\eta\big(\xi(\tau) - \xi(t_0)\big)\Big)\, dV(B)(\tau)$$

$$+ \sum_{i=1,\, i\neq 1+k_i}^{1+k(t)} \int_{t_{i-1}}^{t_i} \exp\Big(\eta\big(\xi(\tau) - \xi(t_0)\big)\Big)\, dV(B)(\tau)\Bigg)$$

$$\leq R_0 \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big)\Bigg(\sum_{i=1,\, i=1+k_i}^{1+k(t)} \exp(\eta\, i)\Big[V(B)(t_i) - V(B)(t_{i-1})\Big]$$

$$+ \sum_{i=1,\, i\neq 1+k_i}^{1+k(t)} \exp(\eta\, i)\Big[V(B)(t_i) - V(B)(t_{i-1})\Big]$$

$$+ \sum_{i=1,\, i\neq 1+k_i}^{1+k(t)} \exp\Big((1+k_i)\eta\Big)\, d_1B(t_i)\Bigg)$$

$$\leq \varepsilon R_0 \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big)\Bigg(\sum_{i=1}^{1+k(t)} \exp(\eta\, i) + \sum_{i=1,\, i\neq 1+k_i}^{1+k(t)} \exp\Big((1+k_i)\eta\Big)\Bigg)$$

$$\leq 2\varepsilon R_0 \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big) \sum_{i=1}^{1+k(t)} \exp(\eta\, i)$$

$$= 2\varepsilon R_0 \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big) \exp(\eta)\Big(\exp\Big((1+k(t))\eta\Big) - 1\Big)\Big(\exp(\eta) - 1\Big)^{-1}$$

$$\leq 2\varepsilon R_0 \exp\Big(-\eta k(t)\Big) \exp\Big((2+k(t))\eta\Big)\Big(\exp(\eta) - 1\Big)$$

$$= 2\varepsilon R_0 \exp(2\eta)\Big(\exp(\eta) - 1\Big)^{-1}.$$

From (14), (16) and (20) it follows that inequality (9) holds for $t_0 \geq t^*$, where $H \in \mathbb{R}^{n\times n}$ is the matrix whose every component equals $\frac{1}{2n}$. On the other hand, it can be easily shown that

$$r(H) < \frac{1}{2}\,.$$

Consequently, by Lemma 1 an arbitrary solution x of the system (1_0) admits an estimate

$$\|x(t)\| \leq \rho \exp\Big(-\eta\big(\xi(t) - \xi(t_0)\big)\Big) \quad \text{for} \quad t \geq t_0 \geq t^*,$$

where $\rho > 0$ is a constant independent of t_0. $\square$

Note that a similar theorem is proved in [7] for the case of ordinary differential equations.

Corollary 1. *Let the components* a_{ik} $(i,k=1,\dots,n)$ *of the matrix-function* A *satisfy the conditions*

$$1+(-1)^j\,d_j a_{ii}(t)\neq 0 \ \textit{ for } \ t\in\mathbb{R}_+ \ \ (i=1,\dots,n;\ \ j=1,2), \tag{20}$$

$$\lim_{t\to+\infty}\bigvee_{t}^{\nu(\xi)(t)}\mathcal{A}(a_{ii},a_{ik})=0 \ \ (i,k=1,\dots,n), \tag{21}$$

and

$$a_{ii}(t)-a_{ii}(\tau)\leq-\eta\Big(\xi(t)-\xi(\tau)\Big) \ \textit{ for } \ t\geq\tau\geq 0 \ \ (i=1,\dots,n), \tag{22}$$

where $\eta>0$, $\xi:\mathbb{R}_+\to\mathbb{R}_+$ *is a nondecreasing function satisfying condition* (3), *and* $\nu(\xi):\mathbb{R}_+\to\mathbb{R}_+$ *is the function defined as in Theorem* 1. *Then the matrix-function* A *is* ξ*-exponentially asymptotically stable.*

Proof. Corollary 1 follows from Theorem 1 if we assume that

$$A_0(t)\equiv\operatorname{diag}\Big(a_{11}(t),\dots,a_{nn}(t)\Big).$$

Indeed, by the definition of the operator $\mathcal{A}$ we have

$$\Big[\mathcal{A}(A_0,A-A_0)(t)\Big]_{ik}=a_{ik}(t)+\sum_{0<\tau\leq t}\frac{d_1a_{ii}(\tau)}{1-d_1a_{ii}(\tau)}\,d_1a_{ik}(\tau)$$
$$-\sum_{0\leq\tau<t}\frac{d_2a_{ii}(\tau)}{1+d_2a_{ii}(\tau)}\,d_2a_{ik}(\tau)=\mathcal{A}(a_{ii},a_{ik})(t)$$
$$\text{for } \ t\in\mathbb{R}_+ \ \ (i\neq k;\ \ i,k=1,\dots,n)$$

and

$$\Big[\mathcal{A}(A_0,A-A_0)(t)\Big]_{ii}=0 \ \text{ for } \ t\in\mathbb{R}_+ \ \ (i=1,\dots,n).$$

Therefore, by (21) and (22) the matrix-function A is ξ-exponentially asymptotically stable. □

Corollary 2. *Let the matrix-function* $P\in L_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ *be* ξ*-exponentially asymptotically stable and*

$$\lim_{t\to+\infty}\bigvee_{t}^{\xi(t)+1}(A-A_0)=0 \ \textit{ for } \ t\in\mathbb{R}_+,$$

where $A_0(t)\equiv\int\limits_0^t P(\tau)\,d\tau$, $\xi:\mathbb{R}_+\to\mathbb{R}_+$ *is a continuous nondecreasing function satisfying condition* (3). *Then the matrix-function* A *is* ξ*-exponentially asymptotically stable as well.*

Proof. Corollary 2 immediately follows from Theorem 1 if we observe that

$$\mathcal{A}(A_0, A - A_0)(t) = A(t) - A_0(t) \ \ (t \in \mathbb{R}_+)$$

in this case and, moreover,

$$\nu(\xi)(t) = \xi(t) + 1 \ \ (t \in \mathbb{R}_+)$$

because ξ is a nondecreasing continuous function. $\square$

Theorem 2. *The matrix-function A is ξ-exponentially asymptotically stable if and only if there exist a positive number η and a nonsingular matrix-function $H \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ such that*

$$\sup\Big\{\|H^{-1}(t)H(s)\| : \ \ t \geq s \geq 0\Big\} < +\infty \tag{23}$$

and

$$\bigvee_0^{+\infty} B_\eta(H, A) < +\infty, \tag{24}$$

where

$$B_\eta(H, A)(t) \equiv \int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\Big[\exp\Big(\eta\xi(\tau)\Big)H(\tau)$$
$$+ \exp\Big(\eta\xi(\tau)\Big)H(\tau)A(\tau) - \int_0^\tau d\Big(\exp\Big(\eta\xi(s)\Big)H(s)\Big)\cdot A(s)\Big]. \tag{25}$$

Proof. Let U and U^* be the Cauchy matrices of systems (1_0) and

$$dy(t) = dA^*(t)\cdot y(t),$$

respectively, where $A^*(t) = \mathcal{L}(\exp(\eta\xi(\cdot))H, A)(t)$. Then by the definition of the operator $\mathcal{L}$ and by the equality

$$U(t, s) = \exp\Big(-\eta\Big(\xi(t) - \xi(s)\Big)\Big)H^{-1}(t)U^*(t, s)H(s) \ \ \text{for} \ \ t,\ s \in \mathbb{R}_+$$

we obtain that

$$\exp\Big(\eta\Big(\xi(\tau) - \xi(s)\Big)\Big)U(t, s) = H^{-1}(t)H(s)$$
$$+H^{-1}(t)\int_s^t \exp\Big(\eta\Big(\xi(\tau) - \xi(s)\Big)\Big)\, dB_\eta(H, A)(\tau)\cdot U(\tau, s) \ \ \text{for} \ \ t,\ s \in \mathbb{R}_+.$$

Hence

$$W(t, s) = H^{-1}(t)H(s) + H^{-1}(t)d_1 B_\eta(H, A)(t)\cdot W(t, s)$$
$$+ H^{-1}(t)\int_s^t dG(\tau)\cdot W(\tau, s) \ \ \text{for} \ \ t,\ s \in \mathbb{R}_+, \tag{26}$$

where

$$W(t,s)=\exp\Big(\eta\big(\xi(t)-\xi(s)\big)\Big)U(t,s),\quad G(t)=B_\eta(H,A)(t-).$$

On the other hand by (23), (24) and by the equalities

$$\det\Big(I_n+(-1)^j\,d_jA^*(t)\Big)=\exp\Big((-1)^jn\eta\,d_j\xi(t)\Big)\det\Big(H(t)+(-1)^j\,d_jH(t)\Big)$$
$$\times\det\Big(I_n+(-1)^j\,d_jA(t)\Big)\det\Big(H^{-1}(t)\Big)\ \text{ for }\ t\in\mathbb{R}_+\ \ (j=1,2)$$

and

$$I_n+(-1)^jH^{-1}(t)\,d_jB_\eta(H,A)(t)$$
$$=H^{-1}(t)\Big(I_n+(-1)^j\,d_jA^*(t)\Big)H(t)\ \text{ for }\ t\in\mathbb{R}_+\ \ (j=1,2)$$

there exists a positive number r_0 such that

$$\det\Big(I_n+(-1)^jH^{-1}(t)\,d_jB_\eta(H,A)(t)\Big)\neq0\ \text{ for }\ t\in\mathbb{R}_+\ \ (j=1,2)\tag{27}$$

and

$$\Big\|\Big(I_n+(-1)^jH^{-1}(t)\,d_jB_\eta(H,A)(t)\Big)^{-1}\Big\|<r_0\ \text{ for }\ t\in\mathbb{R}_+\ \ (j=1,2).\tag{28}$$

From (26), by (23), (27) and (28) we get

$$\|W(t,s)\|\leq r_0\bigg(\rho+\rho_1\int_s^t\|W(\tau,s)\|\,d\|V(G)(\tau)\|\bigg)\ \text{ for }\ t\geq s\geq0,$$

where

$$\rho=\sup\Big\{\|H^{-1}(t)H(s)\|:\ t\geq s\Big\},\quad \rho_1=\rho\|H^{-1}(0)\|.$$

Hence, according to the Gronwall inequality ([11])

$$\|W(t,s)\|\leq M<+\infty\ \text{ for }\ t\geq s\geq0,$$

where

$$M=r_0\exp\Big(r_0\rho_1\bigvee_0^{+\infty}B_\eta(H,A)\Big).$$

Therefore

$$\|U(t,s)\|\leq M\exp\Big(-\eta\big(\xi(t)-\xi(s)\big)\Big)\ \text{ for }\ t\geq s\geq0,$$

i.e., the matrix-function A is ξ-exponentially asymptotically stable.

Let us show the necessity. Let the matrix-function A is ξ-exponentially asymptotically stable. Then there exist positive numbers η and ρ such that

$$\|Z(t)Z^{-1}(s)\|\leq\rho\exp\Big(-\eta\big(\xi(t)-\xi(s)\big)\Big)\ \text{ for }\ t\geq s\geq0,\tag{29}$$

where Z ($Z(0)=I_n$) is the fundamental matrix of system (1_0).

Let

$$H(t)\equiv\exp\Big(-\eta\xi(t)\Big)Z^{-1}(t).$$

Then according to (25), (29) and the equality

$$Z^{-1}(t) = I_n - Z^{-1}(t)A(t) + \int_0^t dZ^{-1}(\tau) \cdot A(\tau) \ \text{ for } \ t \in \mathbb{R}_+ \tag{30}$$

(see [11]) we have

$$\|H^{-1}(t)H(s)\| = \|Z(t)Z^{-1}(s)\| \exp\Big(\eta\Big(\xi(t) - \xi(s)\Big)\Big) \le \rho \ \text{ for } \ t \ge s \ge 0$$

and

$$B_\eta(H, A)(t) = B_\eta\Big(\exp(-\eta\xi)Z^{-1}, A\Big)(t) = 0 \ \text{ for } \ t \in \mathbb{R}_+.$$

Therefore conditions (23) and (24) are fulfilled. □

Remark 1. If in Theorem 2 the function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, then condition (24) can be rewritten as

$$\left\| \int_0^{+\infty} dV\Big(\mathcal{I}(H, A) + \eta \operatorname{diag}(\xi, \dots, \xi)\Big)(t) \cdot |H(t)| \right\| < +\infty.$$

Corollary 3. *Let the matrix-function $Q \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ be uniformly stable and*

$$\det\Big(I_n + (-1)^j \, d_j Q(t)\Big) \ne 0 \ \textit{ for } \ t \in \mathbb{R}_+ \ \ (j = 1, 2). \tag{31}$$

Let, moreover, there exist a positive number η such that

$$\left\| \int_0^{+\infty} |Z^{-1}(t)| \, dV\Big(G_\eta(\xi, Q, A)\Big)(t) \right\| < +\infty \tag{32}$$

where Z ($Z(0) = I_n$) is the fundamental matrix of the system

$$dz(t) = dQ(t) \cdot z(t), \tag{33}$$

and

$$\begin{aligned} G_\eta(\xi, Q, A)(t) &\equiv \mathcal{A}(Q, A - Q)(t) + \eta s_0(\xi)(t) \cdot I_n \\ &+ \sum_{0<\tau\le t} \exp\Big(-\eta\xi(\tau)\Big) \, d_1 \exp\Big(\eta\xi(\tau)\Big) \cdot \Big(I_n - d_1 Q(\tau)\Big)^{-1} \Big(I_n - d_1 A(\tau)\Big) \\ &+ \sum_{0\le\tau<t} \exp\Big(-\eta\xi(\tau)\Big) \, d_2 \exp\Big(\eta\xi(\tau)\Big) \cdot \Big(I_n + d_2 Q(\tau)\Big)^{-1} \Big(I_n + d_2 A(\tau)\Big). \end{aligned} \tag{34}$$

Then the matrix-function A is ξ-exponentially asymptotically stable.

Proof. Let $B_\eta(H, A)$ be the matrix-function defined by (25), where $H(t) \equiv Z^{-1}(t)$. Using the formula of integration by parts ([11]), the properties of the operator $\mathcal{B}$ given above and equality (30), we conclude that

$$B_\eta(H, A)(t)$$

$$= \int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\Big(\exp(\eta\xi(\tau))Z^{-1}(\tau) + \mathcal{B}\Big(\exp(\eta\xi)Z^{-1}, A\Big)(\tau)\Big)$$

$$= \int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\Big(\exp(\eta\xi(\tau))Z^{-1}(\tau)\Big)$$

$$+ \int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\mathcal{B}\Big(\exp(\eta\xi)I_n, \mathcal{B}\Big(\exp(\eta\xi)Z^{-1}, A\Big)(\tau)\Big)$$

$$= \int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\Big(\exp(\eta\xi(\tau))Z^{-1}(\tau)\Big)$$

$$+ \int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\mathcal{B}\Big(\exp(\eta\xi)I_n, \mathcal{B}(Z^{-1}, A)\Big)(\tau) \ \text{ for } \ t \in \mathbb{R}_+; \tag{35}$$

$$\int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\Big(\exp(\eta\xi(\tau))Z^{-1}(\tau)\Big)$$

$$= \int_0^t Z^{-1}(\tau)\, d\Big(\eta s_0(\xi)(\tau)I_n - \mathcal{A}(Q,Q)(\tau)\Big)$$

$$+ \sum_{0<s\le\tau} \exp\Big(-\eta\xi(s)\Big)\, d_1\exp\Big(\eta\xi(s)\Big)\cdot\Big(I_n - d_1Q(s)\Big)^{-1}$$

$$+ \sum_{0\le s<\tau} \exp\Big(-\eta\xi(s)\Big)\, d_2\exp\Big(\eta\xi(s)\Big)\cdot\Big(I_n + d_2Q(s)\Big)^{-1} \ \text{ for } \ t \in \mathbb{R}_+; \tag{36}$$

$$\mathcal{B}(Z^{-1}, A)(t) \equiv \int_0^t Z^{-1}(\tau)\, dA(\tau) - \sum_{0<\tau\le t} d_1Z^{-1}(\tau)\cdot d_1A(\tau)$$

$$+ \sum_{0\le\tau<t} d_2Z^{-1}(\tau)\cdot d_2A(\tau) = \int_0^t Z^{-1}(\tau)\, d\mathcal{A}(Q, A-Q)(\tau) \ \text{ for } \ t \in \mathbb{R}_+, \tag{37}$$

$$\mathcal{B}\Big(\exp(\eta\xi)I_n, \mathcal{B}(Z^{-1}, A)\Big)(t)$$

$$= \int_0^t Z^{-1}(\tau)\, d\mathcal{B}\Big(\exp(\eta\xi)I_n, \mathcal{A}(Q,A)\Big)(\tau) \ \text{ for } \ t \in \mathbb{R}_+ \tag{38}$$

and

$$\int_0^t \exp\Big(-\eta\xi(\tau)\Big)\, d\mathcal{B}\Big(\exp(\eta\xi)I_n, \mathcal{A}(Q,A)\Big)(\tau)$$

$$= \mathcal{A}(Q,A)(t) - \sum_{0<\tau\le t} \exp\Big(-\eta\xi(\tau)\Big)\, d_1\exp\Big(\eta\xi(\tau)\Big)\cdot\Big(I_n - d_1Q(\tau)\Big)^{-1} d_1A(\tau)$$

$$+\sum_{0\le\tau<t}\exp\Big(-\eta\xi(\tau)\Big)\,d_2\exp\Big(\eta\xi(\tau)\Big)\cdot\Big(I_n+d_2Q(\tau)\Big)^{-1}d_2A(\tau) \tag{39}$$
$$\text{for } t\in\mathbb{R}_+.$$

From (35), by (36)–(39) we get

$$B_\eta(H,A)(t)=\int_0^t\exp\Big(-\eta\xi(\tau)\Big)\,d\Big(\exp(\eta\xi(\tau))\cdot Z^{-1}(\tau)\Big)$$
$$+\int_0^t Z^{-1}(\tau)\,d\bigg(\int_0^\tau\exp\Big(-\eta\xi(s)\Big)\,d\mathcal{B}\Big(\exp(\eta\xi)I_n,\mathcal{A}(Q,A)\Big)(s)\bigg)$$
$$=\int_0^t Z^{-1}(\tau)\,dG_\eta(\xi,Q,A)(\tau)\ \text{ for }\ t\in\mathbb{R}_+$$

and

$$\bigvee_0^{+\infty}B_\eta(H,A)\le\bigg\|\int_0^{+\infty}|Z^{-1}(t)|\,dV\Big(G_\eta(\xi,Q,A)\Big)(t)\bigg\|.$$

Therefore from (32) and the fact that the matrix-function Q is ξ-exponentially asymptotically stable, it follows that the conditions of Theorem 2 are fulfilled. □

Remark 2. In Corollary 3 if the function $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is continuous, then

$$G_\eta(\xi,Q,A)(t)=\mathcal{A}(Q,A-Q)(t)+\eta\xi(t)I_n\ \text{ for }\ t\in\mathbb{R}_+.$$

Corollary 4. *Let the matrix-function $Q\in BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times n})$, satisfying condition* (31)*, be ξ-exponentially asymptotically stable and*

$$\bigvee_0^{+\infty}\mathcal{B}(Z^{-1},A-Q)<+\infty, \tag{40}$$

where Z $(Z(0)=I_n)$ is the fundamental matrix of system (33)*. Then the matrix-function A is ξ-exponentially asymptotically stable as well.*

Proof. Since Q is ξ-exponentially asymptotically stable there exists a positive number η such that the estimate (29) holds.

Let now $B_\eta(H,A)$ be the matrix-function defined by (25), where

$$H(t)\equiv\exp\Big(-\eta\xi(t)\Big)Z^{-1}(t).$$

Using equality (30) for the matrix-function Q we conclude that

$$Z^{-1}(t)=I_n+\mathcal{B}(Z^{-1},-Q)(t)\ \text{ for }\ t\in\mathbb{R}_+$$

and

$$B_\eta(H,A)(t)=\int_0^t\exp\Big(-\eta\xi(\tau)\Big)\,d\mathcal{B}(Z^{-1},A-Q)(\tau)\ \text{ for }\ t\in\mathbb{R}_+.$$

By this and (40), condition (24) holds. Therefore, the conditions of Theorem 2 are fulfilled. □

Remark 3. By the equality

$$\mathcal{B}(Z^{-1}, A - Q)(t) = \int_0^t Z^{-1}(\tau)\, d\Big(A(\tau) - Q(\tau)\Big) \ \text{ for } \ t \in \mathbb{R}_+$$

the condition

$$\Big\| \int_0^{+\infty} |Z^{-1}(t)|\, dV\Big(\mathcal{A}(Q, A - Q)\Big)(t) \Big\| < +\infty$$

guarantees the fulfilment of condition (40) in Corollary 4. On the other hand,

$$\lim_{\eta \to 0+} G_\eta(\xi, Q, A)(t) = \mathcal{A}(Q, A - Q)(t) \ \text{ for } \ t \in \mathbb{R}_+,$$

where $G_\eta(\xi, Q, A)(t)$ is defined by (34). Consequently, Corollary 3 is true in the limit case ($\eta = 0$), too, if we require the ξ-exponentially asymptotic stability of Q instead of the uniform stability.

Corollary 5. *Let $Q \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ be a continuous matrix-function satisfying the Lappo-Danilevskiĭ condition*

$$\int_0^t Q(\tau)\, dQ(\tau) = \int_0^t dQ(\tau) \cdot Q(\tau) \ \ \textit{for} \ \ t \in \mathbb{R}_+.$$

Let, moreover, there exist a nonnegative number η such that

$$\Big\| \int_0^{+\infty} \Big|\exp(-Q(t))\Big|\, dV(A - Q + \eta\xi I_n)(t) \Big\| < +\infty,$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying condition (3). *Then:*

(a) *the uniform stability of the matrix-function Q guarantees the ξ-exponentially asymptotic stability of the matrix-function A for $\eta > 0$;*

(b) *the ξ-exponentially asymptotic stability of Q guarantees the ξ-exponentially asymptotic stability of A for $\eta = 0$.*

Proof. The corollary follows immediatelly from Corollaries 3 and 4 and Remark 3 if we note that

$$Z(t) = \exp(Q(t)) \ \text{ for } \ t \in \mathbb{R}_+$$

and in this case

$$G_\eta(\xi, Q, A)(t) = A(t) - Q(t) + \eta\xi(t) I_n \ \text{ for } \ t \in \mathbb{R}_+. \qquad \square$$

Corollary 6. *Let there exist a nonnegative number η such that the components a_{ik} $(i,k=1,\dots,n)$ of the matrix-function A satisfy conditions* (20),

$$s_0(a_{ii})(t)-s_0(a_{ii})(\tau)-\sum_{\tau<s\le t}\ln\Big|1-d_1a_{ii}(s)\Big|+\sum_{\tau\le s<t}\ln\Big|1+d_2a_{ii}(s)\Big|$$
$$\le-\eta\Big(s_0(\xi)(t)-s_0(\xi)(\tau)\Big)-\mu\Big(\xi(t)-\xi(\tau)\Big)\ \text{for}\ t\ge\tau\ge0 \quad (i=1,\dots,n), \tag{41}$$

$$(-1)^j\sum_{0\le t<+\infty}|z_i^{-1}(t)|\Big(\exp\Big((-1)^j\,d_j\xi(\tau)\Big)-1\Big)<+\infty \quad (j=1,2;\ \ i=1,\dots,n) \tag{42}$$

and

$$\int_0^{+\infty}|z_i^{-1}(t)|\,dv(g_{ik})(t)<+\infty\ \ (i\ne k;\ \ i,k=1,\dots,n), \tag{43}$$

where $\mu=0$ if $\eta>0$, $\mu>0$ if $\eta=0$,

$$z_i(t)\equiv\exp\Big(s_0(a_{ii})(t)+\eta s_0(\xi)(t)\Big)\times\prod_{0<\tau\le t}\Big(1-d_1a_{ii}(\tau)\Big)^{-1}\prod_{0\le\tau<t}\Big(1+d_2a_{ii}(\tau)\Big)\ \ (i=1,\dots,n),$$
$$g_{ik}(t)\equiv s_0(a_{ik})(t)+\sum_{0<\tau\le t}\exp\Big(-\eta\,d_1\xi(\tau)\Big)\,d_1a_{ik}(\tau)\cdot\Big(1-d_1a_{ii}(\tau)\Big)^{-1}$$
$$+\sum_{0\le\tau<t}\exp\Big(\eta\,d_2\xi(\tau)\Big)\,d_2a_{ik}(\tau)\cdot\Big(1+d_2a_{ii}(\tau)\Big)^{-1} \quad (i\ne k;\ \ i,k=1,\dots,n),$$

and $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing function satisfying condition (3). *Then the matrix-function A is ξ-exponentially asymptotically stable.*

Proof. For $\eta>0$ the corollary follows from Corollary 3 if we assume that

$$Q(t)\equiv\operatorname{diag}\Big(a_{11}(t)+\eta s_0(\xi)(t),\dots,a_{nn}(t)+\eta s_0(\xi)(t)\Big).$$

Indeed, by the definition of the operator $\mathcal{A}$ we have

$$\Big[\mathcal{A}(Q,A-Q)(t)\Big]_{ik}=a_{ik}(t)+\sum_{0<\tau\le t}d_1a_{ii}(\tau)\cdot\Big(1-d_1a_{ii}(\tau)\Big)^{-1}d_1a_{ik}(\tau)$$
$$-\sum_{0\le\tau<t}d_2a_{ii}(\tau)\cdot\Big(1+d_2a_{ii}(\tau)\Big)^{-1}d_2a_{ik}(\tau)\ \ \text{for}\ \ t\in\mathbb{R}_+\ \ (i\ne k;\ \ i,k=1,\dots,n),$$
$$\Big[\mathcal{A}(Q,A-Q)(t)\Big]_{ii}=-\eta s_0(\xi)(t)\ \ \text{for}\ \ t\in\mathbb{R}_+\ \ (i=1,\dots,n).$$

From these relations, using (34) we obtain

$$\Big[G_\eta(\xi,Q,A)(t)\Big]_{ik}=a_{ik}(t)+\sum_{0<\tau\le t}d_1a_{ii}(\tau)\cdot\Big(1-d_1a_{ii}(\tau)\Big)^{-1}d_1a_{ik}(\tau)$$

$$- \sum_{0\le\tau<t} d_2a_{ii}(\tau)\cdot\Big(1+d_2a_{ii}(\tau)\Big)^{-1} d_2a_{ik}(\tau)$$

$$- \sum_{0<\tau\le t} d_1a_{ik}(\tau)\cdot\Big(1-d_1a_{ii}(\tau)\Big)^{-1}\Big(1-\exp\Big(-\eta d_1\xi(\tau)\Big)\Big)$$

$$- \sum_{0\le\tau<t} d_2a_{ik}(\tau)\cdot\Big(1+d_2a_{ii}(\tau)\Big)^{-1}\Big(1-\exp\Big(\eta d_2\xi(\tau)\Big)\Big)$$

$$= g_{ik}(t) \ \text{ for } \ t\in\mathbb{R}_+ \ \ (i\ne k;\ \ i,k=1,\dots,n)$$

and

$$\Big[G_\eta(\xi,Q,A)(t)\Big]_{ii} = \sum_{0<\tau\le t}\Big(1-\exp\Big(-\eta d_1\xi(\tau)\Big)\Big)$$

$$+ \sum_{0\le\tau<t}\Big(1-\exp\Big(\eta d_2\xi(\tau)\Big)\Big) \ \text{ for } \ t\in\mathbb{R}_+ \ \ (i=1,\dots,n).$$

On the other hand, the matrix-function $Z(t) = \operatorname{diag}(z_1(t),\dots,z_n(t))$ is the fundamental matrix of the system (33), satisfying the condition $Z(0) = I_n$. Therefore, by (41)–(43) the conditions of Corollary 3 are valid. For $\eta = 0$ the corollary follows from Corollary 4 and Remark 3. □

Remark 4. If, in addition to (20), the components a_{ik} $(i,k=1,\dots,n)$ of the matrix-function A satisfy the condition

$$(-1)^j\, d_ja_{ik}(t)\cdot\Big(1+(-1)^j\, d_ja_{ii}(t)\Big)^{-1} \ge 0 \ \text{ for } \ t\in\mathbb{R}_+$$
$$(i\ne k;\ \ i,k=1,\dots,n;\ \ j=1,2),$$

then we can assume without loss of generality that $\eta > 0$ and $\mu = 0$ in Corollary 6.

Corollary 7. *Let there exist a nonnegative number η such that*

$$\int\limits_0^{+\infty} t^{n_\ell-1}\exp(-t\operatorname{Re}\lambda_\ell)\,dv(b_{ik})(t) < +\infty \ \ (\ell=1,\dots,m;\ \ i,k=1,\dots,n),$$

where $b_{ik}(t)\equiv a_{ik}(t)-p_{ik}t+\eta\xi(t)$ $(i,k=1,\dots,n)$, $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing function satisfying condition (3), *and $\lambda_1,\dots,\lambda_m$ ($\lambda_i\ne\lambda_j$ for $i\ne j$) are the characteristic values of the matrix $P=(p_{ik})_{i,k=1}^n$ with multiplicities $n_1,\dots,n_m$, respectively. Then:*

(a) *if $\eta>0$, $\operatorname{Re}\lambda_\ell\le 0$ $(\ell=1,\dots,m)$, and, in addition, $n_\ell=1$ for $\operatorname{Re}\lambda_\ell=0$, then A is ξ-exponentially asymptotically stable;*

(b) *if $\eta=0$ and $\operatorname{Re}\lambda_\ell<0$ $(\ell=1,\dots,m)$, then A is exponentially asymptotically stable;*

(c) *if $\eta=0$ and P is ξ-exponentially asymptotically stable, then A is ξ-exponentially asymptotically stable as well.*

Proof. The corollary immediatelly follows from Corollary 5 if we assume $Q(t)\equiv Pt$ and derive the matrix-function $\exp(-Pt)$ by the standard way using the Jordan canonical form of the matrix P. □

Consider now system (2), where $P \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $q \in L_{loc}(\mathbb{R}_+, \mathbb{R}^n)$. Theorem 2 and Corollary 6 have the following form for this system.

Theorem 2′. *Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be an absolutely continuous nondecreasing function satisfying condition (3). Then the matrix-function $P \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ is ξ-exponentially asymptotically stable if and only if there exist a positive number η and nonsingular matrix-function $H \in \widetilde{C}_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ such that*

$$\sup\Big\{\|H^{-1}(t)H(s)\| : \quad t \ge s \ge 0\Big\} < +\infty$$

and

$$\int\limits_0^{+\infty} \Big\|H'(t) + H(t)\Big(P(t) + \eta\xi'(t)I_n\Big)\Big\|\,dt < +\infty.$$

Corollary 6′. *Let there exist a positive number η such that the components $p_{ik} \in L_{loc}(\mathbb{R}_+, \mathbb{R})$ $(i, k = 1, \dots, n)$ of the matrix-function P satisfy the conditions*

$$p_{ii}(t) \le -\eta \ \ \textit{for} \ \ t \ge t^* \quad (i = 1, \dots, n)$$

and

$$\int\limits_{t^*}^{+\infty} \exp\bigg(-\int\limits_{t_*}^{t}\Big(p_{ii}(\tau) + \eta\Big)\,d\tau\bigg)\,|p_{ik}(t)|\,dt < +\infty \ \ (i \ne k; \ \ i, k = 1, \dots, n)$$

for some $t^* \in \mathbb{R}_+$. Then the matrix-function P is exponentially asymptotically stable.

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