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# QUASICONFORMAL DEFORMATIONS OF HOLOMORPHIC FUNCTIONS

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*Dedicated to the memory of N. I. Muskhelishvili*
*on the occasion of his 110th birthday*

**Abstract**. This paper deals with holomorphic functions from Bergman spaces $B^p$ in the disk and provides the existence of deformations (variations) which do not increase the norm of functions and preserve some other prescribed properties. Admissible variations are constructed (for even integer $p \geq 2$) using special quasiconformal maps of the complex plane (in line with a new approach to variational problems for holomorphic functions in Banach spaces).

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## 1. Introduction

Quasiconformal maps have now become one of the basic tools in many fields of mathematics and its applications. This paper promotes the development of a new approach to solving variational problems in Banach spaces of holomorphic functions, which recently has been outlined in [5], [6].

The methods adopted in those problems rely mainly on the integral representations of holomorphic functions by means of certain measures, which reduces the problems to the investigation of these measures. This method often encounters great difficulties. Our approach is completely different. It is based on the special kind of quasiconformal deformations of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which ensure a controlled change of the norm of a given function under variation and also preserve some other prescribed properties of the function.

This approach was established in [5], [6] mainly for the Hardy spaces $H^p$ in the disk with an even integer $p$ and applied to some well-known conjectures concerning nonvanishing functions. The specific features of these spaces are used there in an essential way. The present paper deals with more general Bergman spaces $B^p$ of holomorphic functions. Its purpose is to show the existence of deformations (variations) which do not increase the norm of a function and

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preserve some other prescribed properties, for example, vary independent of a suitable number of the Taylor coefficients of this function. Such requirements are, of course, rather rigid.

Such problems naturally arise, for example, while studying nonvanishing holomorphic functions. These are closely connected with nonbranched holomorphic coverings.

To preserve nonvanishing, we construct quasiconformal deformations of the complex plane which fix the origin. This produces additional rigidity of variations, and we show how it can be resolved in the case of Bergman functions.

Recall that the Bergman space $B^p$, $1 < p < \infty$, in the unit disk $\Delta = \{z = x + iy \in \mathbb{C} : |z| < 1\}$ consists of holomorphic functions $f$ with the finite norm $\|f\|_p = \Big(\iint\limits_{\Delta} |f(z)|^p dxdy\Big)^{1/p}$. Let $|\mathbf{x}|$ denote the Euclidean norm in $\mathbb{R}^n$.

## 2. Main Existence Theorem

The next theorem gives a complete answer to the question on the existence of admissible deformations with controlled distortion of the norm and coefficients of the Bergman functions for even integers $p$.

**Theorem 1.** *Let $m$ and $n$ be two fixed positive integers $(m, n \geq 1)$. For every function $f_0(z) = c_0^0 + \sum\limits_{k=j}^{\infty} c_k^0 z^k \in B^{2m} \cap H^\infty$ with $c_0^0 \neq 0$ and $c_j^0 \neq 0$ $(1 \leq j < n)$, which is not a polynomial of degree $n_1 \leq n$, there exists a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and any points $\mathbf{d} = (d_1, d_{j+1}, \dots, d_n) \in \mathbb{C}^{n-j+1}$ and $a \in \mathbb{R}$ satisfying the inequalities $|\mathbf{d}| \leq \varepsilon$, $|a| \leq \varepsilon$, there is a quasiconformal automorphism $h$ of $\hat{\mathbb{C}}$, which is conformal at least in the disk*

$$\Delta_0' := \{w : \ |w| < \sup_{\Delta} |f_0(z)| + 1\}$$

*and satisfies*

(a) $h(0) = 0$;

(b) $h'(0) = 1 + d_1$;

*rom(c)* $h^{(k+1)}(0) = k!\ d_k,\ k = j+1, \dots, n$;

(d) $\|(h \circ F_0)'(z)\|_{2m}^{2m} - \|F_0'\|_{2m}^{2m} = \|h'(F_0) f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = a$,

*where $F_0(z) = \int\limits_0^z f_0(t)dt$.*

*The map $h$ can be chosen to have the Beltrami coefficient $\mu_h = \partial_{\bar{w}} h / \partial_w h$ with $\|\mu_h\|_\infty \leq M_0 \varepsilon$. The quantities $\varepsilon_0$, $M_0$ as well as the bound of the remainder in* (c) *depend only on $f_0$, $m$ and $n$.*

*Proof.* We associate with the functions $f(z) = \sum\limits_0^\infty c_k z^k \in B^p$ the complex Banach space $\tilde{B}^p$ of their primitive functions

$$F(z) = \int\limits_0^z f(t)dt = \sum_0^\infty \frac{c_k}{k+1} z^{k+1}, \tag{1}$$

letting

$$\|F\|_p := \|F\|_{\tilde{B}^p} = \|f\|_{B^p}.$$

Note that $\sup_\Delta |F(z)| \le \sup_\Delta |f(z)|$, so if $f \in B^p \cap H^\infty$, then $F \in \tilde{B}^p \cap H^\infty$. In the sequel, we use $p = 2m$.

Now define for the annulus

$$E = \{w : \ R < |w| < R+1\},$$

with a fixed $R \ge \sup_\Delta |f_0(z)| + 1$, the integral operators

$$T\rho = -\frac{1}{\pi} \iint\limits_E \frac{\rho(\zeta) d\xi d\eta}{\zeta - w}, \quad \Pi\rho = \partial_w T = -\frac{1}{\pi} \iint\limits_E \frac{\rho(\zeta) d\xi d\eta}{(\zeta - w)^2},$$

assuming $\rho \in L_q(E), \ q \ge 2$, and regarding the second integral as a principal Cauchy value.

Let us seek the required quasiconformal automorphism $h = h^\mu$ of $\hat{\mathbb{C}}$ in the form

$$h(w) = w - \frac{1}{\pi} \iint\limits_E \frac{\rho(\zeta) d\xi d\eta}{\zeta - w} = w + T\rho(w), \tag{2}$$

with the Beltrami coefficient $\mu = \mu_h$ supported in $E$, i.e., with $\|\mu\|_\infty < \kappa < 1$ and $\mu(w) = 0$ on $\mathbb{C} \setminus E$. Substituting (2) into the Beltrami equation $\partial_{\bar{w}} h = \mu \partial_w h$ (with $\partial_w = (1/2)\ (\partial_u - i\partial_v),\ \partial_{\bar{w}} = (1/2)\ (\partial_u + i\partial_v),\ w = u + iv$), one obtains

$$\rho = \mu + \mu\Pi\mu + \mu\Pi(\mu\Pi\mu) + \cdots .$$

As is well-known, this series is convergent in $L_q(E)$ for some $q > 2$; thus the distributional derivatives $\partial_{\bar{w}} h = \rho$ and $\partial_w h = 1 + \Pi\rho$ belong to $L_q^{\text{loc}}(\mathbb{C})$. This implies the estimates

$$\|\rho\|_{L_q(\Delta_r)} \le M_1(\kappa, r, q)\|\mu\|_{L_\infty(\mathbb{C})}, \quad \|\Pi\rho\|_{L_q(\Delta_r)} \le M_1(\kappa, r, q)\|\mu\|_{L_\infty(\mathbb{C})};$$
$$\|h\|_{C(\Delta_r)} \le M_1(\kappa, r, q)\|\mu\|_\infty$$

in the disks $\Delta_r = \{w \in \mathbb{C} : \ |w| < r\}$ $(r < \infty)$ and the smoothness of quasiconformal maps as the functions of parameters: if $\mu(z; t)$ is a $C^1$-smooth $L_\infty(\mathbb{C})$-function of a real (respectively complex) parameter $t$, then $\partial_w h^{\mu(\cdot,t)}$ and $\partial_{\bar{w}} h^{\mu(\cdot,t)}$ are smoothly $\mathbb{R}$-differentiable (respectively, $\mathbb{C}$-differentiable) $L_p$ functions of $t$; hence, the map $t \mapsto h^{\mu(\cdot,t)}(z)$ is $C^1$ smooth as an element of $C(\overline{\Delta}_r)$. For the proof of these results, which will be exploited below, see, e.g., [8], Ch.2; [1]; [4], Ch. 2.

Setting

$$\langle \nu, \varphi \rangle = -\pi^{-1} \iint\limits_E \nu(\zeta)\varphi(\zeta) d\xi d\eta, \quad (\nu \in L_\infty(E), \quad \varphi \in L_1(E)),$$

one can rewrite (2) for $|w| < R$ in the form

$$h(w) = w + T\mu(w) + \omega(w) = w + \sum_{k=0}^{\infty} \langle \mu, \varphi_k \rangle w^k + \omega(w) \tag{3}$$

with

$$\varphi_k(\zeta) = 1/\zeta^{k+1}, k = 0, 1, \dots\ ; \quad \|\omega\|_{C(\Delta_r)} \le M_2(\kappa, r)\|\mu\|_\infty^2 \ \ (r < \infty).$$

Comparing (3) for $w = F_0(z)$ with the above-given conditions (a), (b) and (c), one finds that the following relations are satisfied by the desired Beltrami coefficient $\mu$:

$$\begin{aligned} \langle \mu, \varphi_0 \rangle + O(\|\mu\|_\infty^2) &= 0, \\ \langle \mu, \varphi_1 \rangle + O(\|\mu\|_\infty^2) &= d_1, \\ \langle \mu, \varphi_{k+1} \rangle + O(\|\mu\|_\infty^2) &= d_k, \quad k = j, \dots, n. \end{aligned} \tag{4}$$

On the other hand, the condition (d), together with (3), gives the equality

$$\begin{aligned} \|(h \circ F_0)'(z)\|_{2m}^{2m} &= \Big\|(F_0 + T\rho \circ F_0)'(z)\Big\|_{2m}^{2m} \\ &= \iint\limits_{\Delta} |f_0(z) + (T\mu \circ F_0)'(z)|^{2m} dxdy + O(\|\mu\|_\infty^2) \\ &= \iint\limits_{\Delta} |f_0(z) - (\Pi\mu \circ F_0)'(z) f_0(z)|^{2m} dxdy + O(\|\mu\|_\infty^2) \\ &= \iint\limits_{\Delta} [|f_0(z)|^2 - 2\,\mathrm{Re}[\overline{f_0(z)}\Pi\mu \circ F_0(z) f_0(z)] \\ &+ |\Pi\mu \circ F_0(z)|^2 |f_0(z)|^2\Big]^m dxdy + O(\|\mu\|_\infty^2) = \|f_0\|_{2m}^{2m} \\ &+ \frac{2m}{\pi}\,\mathrm{Re}\Big[\iint\limits_{E} \mu(\zeta) d\xi d\eta \iint\limits_{\Delta} \frac{|f_0(z)|^{2m+1}}{(\zeta - F_0(z))^2} dxdy\Big] + O_m(\|\mu\|_\infty^2). \end{aligned}$$

Define

$$\phi(\zeta) = -2m \iint\limits_{\Delta} \frac{|f_0(z)|^{2m+1}}{(\zeta - F_0(z))^2} dxdy, \tag{5}$$

then the previous equality takes the form

$$\|(h \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \mathrm{Re}\langle \mu, \phi \rangle + \ O_m(\|\mu\|_\infty^2). \tag{6}$$

The function $\phi$ is holomorphic in a domain $D \subset \hat{\mathbb{C}}$ containing the disk $\Delta_0$, and $\phi|E$ belongs to the span $A_2^0$ of the system $\{\varphi_k\}_0^\infty$ in $L_2(E)$. We also have

$$\phi(\zeta) \not\equiv 0$$

in $D$ because for large $|\zeta|$ integral (5) expands to $\phi(\zeta) = \sum\limits_{k=2}^{\infty} b_k \zeta^{-k}$ with

$$b_2 = 2m\pi |c_0^0|^{2m+1} > 0.$$

We need a stronger fact. □

**Lemma 1.** *Under the assumptions of Theorem* 1, *the function* $\phi$ *does not reduce to a linear combination of the fractions* $\varphi_1, \dots, \varphi_l$, $l \le n$, *i.e., it cannot be of the form*

$$\phi(\zeta) = \sum_{k=1}^{l} b_k \zeta^{-k}, \quad l \le n. \tag{7}$$

*Proof.* We modify the arguments exploited in [5]. They rely on the real analyticity of the $L_p$-norm.

Assuming the contrary that equality (7) holds, one obtains by (6) that any $\mu \in L_\infty(E)$ with $\|\mu\| \le \varepsilon$, $\varepsilon \to 0$ and such that $\langle \mu, \phi_k \rangle = 0$, $k = 0, 1, \dots, l$, satisfies the relation

$$\|(h \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = O(\varepsilon^2). \tag{8}$$

Fix an integer $s > n$ such that $c_s^0 \ne 0$, and consider in $L_1(E)$ the span $\mathcal{A}_{l,s}$ of the functions $\varphi_k | E$, with $0 \le k \le l$ and $k > s$. Then

$$\inf\{\|\varphi_s - \chi\|_{L_1(E)} : \ \chi \in \mathcal{A}_{l,s}\} = d > 0.$$

By the Hahn–Banach theorem, there exists $\mu_0 \in L_\infty(E)$ such that

$$\langle \mu_0, \chi \rangle = 0, \quad \chi \in \mathcal{A}_{l,s}; \quad \langle \mu_0, \varphi_s \rangle = 1, \quad \|\mu_0\|_\infty = d.$$

By (2) and (3), the quasiconformal homeomorphism $h^{\alpha\mu_0}$ with $|\alpha| = \varepsilon$ now assumes the form

$$h^{\alpha\mu_0}(w) = w + \alpha w^s + O(\varepsilon^2). \tag{9}$$

After replacing the element $\varphi_s(w)$ by $\varphi_{s,\beta}(w) = 1/(w-\beta)^s$ with small $|\beta|$, we obtain the system

$$\varphi_k(w), \quad k = 0, 1, 2, \dots, \quad k \ne s; \quad \varphi_{s,\beta}(\zeta) = 1/(w-\beta)^s,$$

which also constitutes a basis in $A_2^0$, and pass to the corresponding map

$$h_\beta^{\alpha\mu_0}(w) = w + \alpha(w-\beta)^s + O(\varepsilon^2) \tag{10}$$

with the same remainder as in (9). Then

$$h_\beta^{\alpha\mu_0} \circ F_0(z) := \sum_1^\infty C_k^{\alpha,\beta} z^k = \sum_1^{s-2} \frac{c_{k-1}^0}{k} z^k + \left[\frac{c_{s-2}^0}{s-1} - s\alpha\beta(c_0^0)^{s-1} + \cdots\right] z^{s-1} + \left[\frac{c_{s-1}^0}{s} + \alpha(c_0^0)^s + \cdots\right] z^s + \cdots + O(\varepsilon_1^2),$$

where

$$\varepsilon_1 = \sqrt{|\alpha|^2 + |\beta|^2}.$$

Accordingly,

$$\begin{aligned}\left(h_\beta^{\alpha\mu_0} \circ F_0\right)'(z) = \sum_1^{s-2} c_{k-1}^0 z^{k-1} + [c_{s-2}^0 - s(s-1)\alpha\beta(c_0^0)^{s-1} + \cdots] z^{s-2} \\ + [c_{s-1}^0 + s\alpha(c_0^0)^s + \cdots] z^{s-1} + \cdots + O(\varepsilon_1^2).\end{aligned} \tag{11}$$

In the disk $\Delta$, let

$$f_0(z)^m = \sum_1^\infty c_{k,m}^0 z^k, \quad [(h_\beta^{\alpha\mu_0} \circ F_0)'(z)]^m = \sum_1^\infty c_{k,m}^{\alpha,\beta} z^k.$$

Applying Parseval's equality in $L_2(\Delta)$, one obtains

$$\begin{aligned} \|(h_\beta^{\alpha\mu_0} \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} &= \|[(h_\beta^{\alpha\mu_0} \circ F_0)']^m\|_2^2 - \|f_0^m\|_2^2 \\ &= \sum_{k=0}^\infty r_k^2 \Big(|c_{k,m}^{\alpha,\beta}|^2 - |c_{k,m}^0|^2\Big) \end{aligned} \tag{12}$$

with $r_k^2 = \pi/(k+1)$. The right-hand side of (12) is a nonconstant real analytic function of $\alpha$ and $\beta$. We have

$$c_{k,m}^{\alpha,\beta} = c_{k,m}^0 + \alpha g_{k,m}(\beta) = c_{k,m}^0 + O_k(\alpha) + O_k(|\alpha\beta|),$$

where $O_k(\varepsilon)/\varepsilon \asymp d$ with the bounds depending only on $\|f_0\|_\infty$. It follows from (11) that the linear term $C\alpha\beta$ in (12) cannot vanish identically for all small $|\alpha\beta|$. Therefore

$$\|(h^{\alpha\mu_0} \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = O(\varepsilon_1), \quad O(\varepsilon_1)/\varepsilon_1 \asymp d,$$

which contradicts (8) for suitable choices of $\varepsilon = |\alpha| \to 0$ and $\beta \to 0$. $\square$

Denote the scalar product in $A_2^0$ by

$$\langle \bar{\chi}, \varphi \rangle_E = \iint_E \overline{\chi(z)} \varphi(z) dx dy.$$

By Lemma 1, the power series of $\phi$ in the disk $\Delta_R^*$ must contain the powers $\zeta^{-k-1}$ with $k > n$. Thus the remainder

$$\psi(\zeta) = \phi(\zeta) - \sum_0^n b_k \zeta^{-k-1} = \Big(\sum_0^{j-1} + \sum_s^\infty\Big) b_k \zeta^{-k-1}, \quad s \ge n+1, \tag{13}$$

does not vanish identically in $\Delta_R^*$. It also satisfies

$$\langle \bar{\psi}, \varphi_k \rangle_E = 0, \quad k = j+1, \dots, n.$$

Further, for any homeomorphism $h = h^\mu$ satisfying $h(0) = 0$, with $\mu \in L_\infty(\mathbb{C})$ vanishing on $\mathbb{C} \setminus E$, we have by (3) the expansion

$$h \circ F_0(z) = (1 + \langle \mu, \varphi_1 \rangle) F_0(z) + \sum_{k=2}^\infty \langle \mu, \varphi_k \rangle F_0(z)^k + O(\|\mu\|_\infty^2);$$

hence,

$$\begin{aligned} (h \circ F_0)'(z) = (1 + \langle \mu, \varphi_1 \rangle) f_0(z) + \sum_{k=2}^\infty k \langle \mu, \varphi_k \rangle F_0(z)^{k-1} f_0(z) \\ + O(\|\mu\|_\infty^2) =: \sum_{k=1}^\infty c_k^* z^k, \quad \|\mu\|_\infty \to 0. \end{aligned}$$

This yields, in particular, that $c_0^* = c_0^0 h'(0)$ and since

$$kF_0(z)^{k-1} f_0(z) = k(c_0^0 z + \frac{c_j^0}{j+1} z^{j+1} + \cdots)^{k-1}(c_0^0 + c_j^0 z^j + \cdots),$$

one concludes that

$$c_j^* = (1 + \langle \mu, \varphi_1 \rangle) c_1^0 + O(\|\mu\|_\infty^2)$$

for $j = 1$, and

$$c_j^* = (1 + \langle \mu, \varphi_1 \rangle) c_j^0 + (j+1)\langle \mu, \varphi_{j+1} \rangle (c_0^0)^{j+1} + O(\|\mu\|_\infty^2)$$

when $j > 1$, provided $\mu$ is chosen so that $\langle \mu, \varphi_k \rangle = 0, \quad k = 0, 1, \ldots, j$, and

$$\langle \mu, \varphi_{j+1} \rangle = -\pi^{-1} \iint\limits_E \mu(\zeta) \zeta^{-j-2} d\xi d\eta \neq 0.$$

Then

$$|c_j^*|^2 - |c_j^0|^2 = 2(j+1) \operatorname{Re}\{d_{j+1} c_j^0 (\bar{c}_0^0)^{j+1}\} + O(\|\mu\|_\infty^2).$$

Put here $\mu = t\nu$ with $\|\mu\|_\infty = 1$ and $t \to 0$ ($t \in \mathbb{C}$). Combining with (6) and (7), this results in

$$\begin{aligned}\lim_{t \to 0} \frac{|c_j^*|^2 - |c_j^0|^2}{t} &= 2(j+1) \operatorname{Re}\{c_j^0 (\bar{c}_0^0)^{j+1} \langle \nu, \varphi_{j+1} \rangle\} \\ &= 2(j+1) \operatorname{Re}\{c_j^0 (\bar{c}_0^0)^{j+1} b_{j+1} \langle \nu, \psi \rangle \neq 0,\end{aligned}$$

which shows that

$$b_{j+1} \neq 0. \tag{14}$$

We now choose the desired Beltrami coefficient $\mu$ of the form

$$\mu = \xi_0 \bar{\varphi}_0 + \xi_1 \bar{\varphi}_1 + \sum_{k=j}^{n} \xi_k \bar{\varphi}_{k+1} + \tau \bar{\psi}, \quad \mu | \mathbb{C} \setminus E = 0, \tag{15}$$

with the unknown constants

$$\xi_0, \xi_1, \xi_j, \ldots, \xi_n, \tau$$

to be determined from equalities (4) and (6). Substituting expression (15) into (4) and taking into account the mutual orthogonality of $\varphi_k$ on $E$, one obtains for determining $\xi_k$ and $\tau$ the nonlinear equations

$$k! d_k = \xi_k \sigma_{k+1}^2 + O(\|\mu\|_\infty^2), \quad k = 0, 1, j+1, \ldots, n, \tag{16}$$

with $d_0 = 0$ and $\sigma_{k+1} = \|\varphi_{k+1}\|_{A_2^0}$ . A comparison of (15) and (6) gives the equation

$$\begin{aligned}&\|(h \circ F_0)'(z)\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} \\ &= \operatorname{Re} \left\langle \xi_0 \bar{\varphi}_0 + \xi_1 \bar{\varphi}_1 + \sum_{k=j}^{n} \xi_k \bar{\varphi}_{k+1} + \tau \bar{\psi}, \phi \right\rangle_E + O(\|\mu\|_\infty^2),\end{aligned} \tag{17}$$

hence the only remaining equation is a relation for $\operatorname{Re}\xi_j, \operatorname{Im}\xi_j, \operatorname{Re}\tau, \operatorname{Im}\tau$. Thus we add to (17) three real equations to distinguish a unique solution of the above system. First of all we will seek $\xi_j$ satisfying

$$\xi_j b_{j+1}\sigma_{j+1}^2 = -\xi_0 b_0 \sigma_0^2 - \xi_1 b_1 \sigma_1^2 - \sum_{k=j+1}^{n} \xi_k b_{k+1} r_{k+1}^2 \tag{18}$$

(which annihilates the main part of the increment of the magnitude $\sum_0^n |c_k|^2$ after deformation). Then (17) takes the form

$$\|(h \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \operatorname{Re}\langle \tau\bar{\psi}, \phi\rangle_E + O(\|\mu\|^2),$$

and, letting $\tau$ be real, we obtain

$$\|(h \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \tau\varkappa + O(\|\mu\|_\infty^2). \tag{19}$$

Separating the real and imaginary parts in (16), (18) and adding (19), we obtain $2(n-j)+5$ real equalities, which define a nonlinear $C^1$-smooth (in fact, real analytic) map

$$\mathbf{y} = W(\mathbf{x}) = W'(\mathbf{0})\mathbf{x} + O(|\mathbf{x}|^2)$$

of the points

$$\mathbf{x} = (\operatorname{Re}\xi_0, \operatorname{Im}\xi_0, \operatorname{Re}\xi_1, \operatorname{Im}\xi_1, \operatorname{Re}\xi_j, \operatorname{Im}\xi_j + 1, \dots, \operatorname{Re}\xi_n, \operatorname{Im}\xi_n, \tau)$$

in a small neighborhood $U_0$ of the origin in $\mathbb{R}^{2(n-j)+5}$, taking the values

$$\begin{aligned}\mathbf{y} =\Big(&\operatorname{Re} d_0, \operatorname{Im} d_0, \operatorname{Re} d_1, \operatorname{Im} d_1, \operatorname{Re} d_{j+1}, \operatorname{Im} d_{j+1}, \dots, \operatorname{Re} d_n, \operatorname{Im} d_n,\\ &\|(h \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m}\Big)\end{aligned}$$

also near the origin of $\mathbb{R}^{2(n-j)+5}$. Its linearization

$$\mathbf{y} = W'(\mathbf{0})\mathbf{x}$$

defines a linear map $\mathbb{R}^{2(n-j)+5} \to \mathbb{R}^{2(n-j)+5}$ whose Jacobian differs from the product $r_0^2 r_1^2 \prod_j^n r_{k+1}^2$ by a nonzero constant factor.

Therefore, $\mathbf{x} \mapsto W'(\mathbf{0})\mathbf{x}$ is a linear isomorphism of $\mathbb{R}^{2(n-j)+5}$ onto itself, and one can apply to $W$ the inverse mapping theorem. It implies the existence of a Beltrami coefficient $\mu$ of the form (15), for which we have the assertions of Theorem 1.

The relations (14) and (18) show a special role played by the first nonzero coefficient $c_j^0$ of $f_0$, which cannot be replaced in general by some $c_k^0$ with $j < k < n$.

## 3. Holomorphic Dependence on Coefficient Parameters $d_{j+1}, \dots, d_n$

**3.1.** As a consequence of the proof of Theorem 1 we obtain

**Theorem 2.** *For a fixed small $a \in \mathbb{R}$, the Beltrami coefficient $\mu$ determined by* (15) *can be chosen to be a complex $L_\infty$-holomorphic function of the parameters $d_1, d_{j+1}, \dots, d_n$.*

*Proof.* This follows from the uniqueness of solutions of the system (16), (18), (19). Let $|\mathbf{d}| < \varepsilon$ and $|a| < \varepsilon$ satisfy the assumptions of Theorem 1 and define the desired $\mu$ by (15). Fix a such $a$ and a $\tau = \tau_0 > 0$ found in the proof of Theorem 1 and solve for this

$$\mu = \xi_0 \bar{\varphi}_0 + \xi_1 \bar{\varphi}_1 + \sum_{k=j}^{n} \xi_k \bar{\varphi}_{k+1} + \tau_0 \bar{\psi}$$

the complex system (16), (18) separately. We already know that it has a unique solution $\xi_0, \xi_{j+1}, \dots, \xi_n$, and by the inverse mapping theorem these $\xi_k$, hence $\mu$, depend on the given $\mathbf{d}$ holomorphically.

In particular, the value $d_j = h^{(j)}(0)/j!$ also moves holomorphically, simultaneously with the independent parameters $d_{j+1}, \dots, d_n$, and one can use by variations the openness of holomorphic maps. □

**3.2.** If we need to construct a quasiconformal homeomorphism $h$ satisfying

$$\|h'(F_0) f_0\|_{2m}^{2m} = \|f_0\|_{2m}^{2m},$$

i.e., for $a = 0$, then Lemma 1 again allows us to seek a Beltrami coefficient $\mu$ of the form (15) with unknown constants $\xi_0, \xi_{j+1}, \dots, \xi_n, \tau$ to be determined from equations (4) and

$$\langle \mu, \phi \rangle = 0.$$

In this case, the existence of a desired $\mu$ of such form is again ensured, for small $\varepsilon$, by the inverse mapping theorem, and moreover, this theorem provides also $L_\infty$ holomorphy of $\tau$ and $\mu$ in all parameters $d_j, d_{j+1}, \dots, d_n$, which move independently.

**3.3. A modification of Theorem 1.** Omitting the assertion (a) of Theorem 1 (i.e., that $h(0) = 0$), one can drop the requirement for the original function $f_0$ to be bounded in $\Delta$.

Indeed, for $f \in B^p$ with $p > 2$, the Hölder inequality yields

$$\|f\|_2 \leq \pi^{(p-2)/p} \|f\|_p;$$

and hence the image domain $F(\Delta)$, where $f$ is the primitive function (1) for $f$, has a finite area (counting with multiplicity). Therefore this domain must have the exterior points.

Now, conjugating the desired quasiconformal homeomorphism $h$ with a linear fractional transformation $\gamma$ of $\hat{\mathbb{C}}$, one reduces the proof to the case, where the primitive function $F_0(z) = \int\limits_0^z f_0(t)dt$ is bounded in the unit disk, and this is, in fact, exactly what was used in the proof of Theorem 1. As a result, the homeomorphism $h$ obtained is conformal in some domain containing $F_0(\Delta)$.

## 4. Infinitesimal Deformations

The following infinitesimal version of Theorem 1 admits a simpler proof.

**Theorem 3.** *Given a function* $f_0(z) = \sum\limits_{k=1}^{\infty} c_k^0 z^k \in B^{2m} \cap H^\infty$, *distinct from a polynomial of degree* $n_1 \le n$, *then for sufficiently small* $\varepsilon_0 > 0$ *and* $\varepsilon \in (0, \varepsilon_0)$ *we have that for every point* $\mathbf{d} = (d_2, d_3, \dots, d_n) \in \mathbb{C}^{n-1}$ *and* $a \in \mathbb{R}$ *so that* $|\mathbf{d}| \le \varepsilon$, $|a| \le \varepsilon$, *there is a quasiconformal homeomorphism* $h$ *of* $\hat{\mathbb{C}}$, *which is conformal in the disk* $D_0'$ *and satisfies:*

(a) $h(0) = 0,\ h'(0) = 1$;

(b) $h^{(k)}(0) = k!\ d_k + O(\varepsilon^2),\ k = 2, 3, \dots, n$;

(c) $\|h'(F_0) f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = a + O(\varepsilon^2)$, *where again* $F_0(z) = \int\limits_0^z f_0(t)dt$.

*The Beltrami coefficient of* $h$ *is estimated similar to Theorem* 1.

*Proof.* By assumption, $f_0(z)$ contains at least one nonzero term $c_s^0 z^s$ with $s > n$. Split the space $A_2^0$ into the span

$$\mathcal{A}_l = \langle \varphi_0, \varphi_1, \dots, \varphi_l \rangle$$

and its orthogonal complement

$$\mathcal{A}_l^\perp = \{g \in A_2 :\ \langle g, \varphi \rangle_E = 0,\ \varphi \in \mathcal{A}_l\}$$

for $0 < l \le n$ and observe that $\mathcal{A}_l^\perp$ contains all convergent series $g(\zeta) = \sum\limits_{k=s}^{\infty} g_k \zeta^{-k-1}$ in the disk $\Delta_R^*$, with $g_s \ne 0$. Any such $g$ determines the map

$$h_{\varepsilon,s}[g](w) := w - \varepsilon (T\bar{g})^{(s-1)}(w) = w - \frac{\varepsilon(s-1)!}{\pi} \iint\limits_E \frac{\overline{g(\zeta)} d\xi d\eta}{(\zeta - w)^s} \tag{20}$$

if $\varepsilon \in \mathbb{C}$ is chosen close to 0. Similarly to (3), the restriction of $T\bar{g}$ onto the disk $\Delta_R = \{|w| < R\}$ assumes the form

$$T\bar{g}(w) = -\frac{1}{\pi} \iint\limits_E \frac{\overline{g(\zeta)} d\xi d\eta}{\zeta - w} = -\frac{1}{\pi} \sum_{k=s}^{\infty} \bar{g}_k \sigma_k^2 w^k \tag{21}$$

and thus in (20)

$$(T\bar{g})^{(s-1)}(w) = -\frac{1}{\pi} \sum_{k=s}^{\infty} \bar{g}_k \sigma_k^2\ k(k-1)\cdots(k-s+2)\ w^{k-s+1}.$$

It follows from (20) that

$$\|(h_{\varepsilon,s}[g] \circ F_0)'(z)\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \operatorname{Re}\langle \bar{g}, \phi_s \rangle_E + O(\varepsilon^2), \quad \varepsilon \to 0, \tag{22}$$

with

$$\phi_s(\zeta) = -\frac{2ms!}{\pi} \iint\limits_{\Delta} \frac{|f_0(z)|^{2m+1}}{(\zeta - F_0(z))^{s+1}} dxdy. \tag{23}$$

We first verify that as in Lemma 1, $\phi_s$ cannot be a linear combination only of $\varphi_0, \dots, \varphi_l$ for some $l \le n$, i.e., the equality

$$\phi_s(\zeta) = \sum_0^l \frac{b_k}{\zeta^{k+1}}, \quad l \le n, \tag{24}$$

cannot take place. Assume that (24) holds, then the relation (22) must reduce to

$$\|(h_{\varepsilon,s}[g]'(F_0)f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = O(\varepsilon^2), \quad \varepsilon \to 0. \tag{25}$$

Define

$$\begin{aligned} I(f_0; g) &= \iint\limits_{\Delta} |f_0(z)|^{2m} \Big[(T\bar{g})^{(s-1)} \circ F_0\Big]'(z) dxdy \\ &= \iint\limits_{\Delta} |f_0(z)|^{2m+1} (T\bar{g})^{(s)} \circ F_0(z) dxdy. \end{aligned}$$

Using (21), this integral is evaluated as follows:

$$\begin{aligned} I(f_0; g) &= \int_0^1 \int_0^{2\pi} |f_0(re^{i\theta})|^{2m} \Big[(T\bar{g})^{(s-1)} \circ F_0\Big]'(re^{i\theta}) r dr d\theta \\ &= -\frac{1}{\pi} \int_0^1 r dr \int\limits_{|z|=r} f_0(z)^m \overline{f_0\Big(\frac{r^2}{\bar{z}}\Big)^m} \times \\ &\times \sum_{k=s}^{\infty} \bar{g}_k \sigma_k^2 \, k(k-1)\cdots(k-s+2)(k-s+1) F_0(z)^{k-s} f_0(z) d\theta \\ &= -\frac{1}{\pi i} \int_0^1 r dr \int\limits_{|z|=r} \Big(\sum_{k=0}^{\infty} c_k^0 z^k\Big)^m \Big(\sum_{k=0}^{\infty} \frac{\bar{c}_k^0 r^{2k}}{z^k}\Big)^m \times \\ &\times \sum_{k=s}^{\infty} \bar{g}_k \sigma_k^2 \, k \cdots (k-s+1) \, \Big(\sum_{k=1}^{\infty} \frac{c_{k-1}^0}{k} z^k\Big)^{k-s} \sum_{k=0}^{\infty} c_k^0 z^k \, \frac{dz}{z} \\ &= \int_0^1 \Big[-2\bar{g}_s \sigma_s^2 s! \, \Big(\sum_{k=0}^{\infty} |c_k^0|^2 r^{2k}\Big)^m + \sum_{k=s+1}^{\infty} \bar{g}_k B_k(r)\Big] r dr. \end{aligned}$$

This shows that by suitable choice of the numbers $g_s, g_{s+1}, \dots$, the corresponding function $g \in \mathcal{A}_l^{\perp}$ satisfies $I(f_0; g) > 0$, and then

$$\frac{\|h_{\varepsilon,s}[g] \circ F_0\|_{2m}^{2m} - \|F_0\|_{2m}^{2m}}{\varepsilon} = \frac{2\,\mathrm{Re}\Big\{\varepsilon I(F_0; g)\Big\}}{\varepsilon} + O(\varepsilon) = O(1), \quad \varepsilon \to 0.$$

This shows that equalities (24) and (25) cannot occur for sufficiently small $|\varepsilon|$ and thus the function

$$\psi_s(\zeta) := \phi_s(\zeta) - \sum_0^n \varphi_k(\zeta) = \sum_{k=n'}^{\infty} b_k \zeta^{-k-1}, \quad n' \ge n+1,$$

does not vanish identically in $\Delta_R^*$.

We now construct a linear functional $L$ on the space $A_2$ whose kernel is the space $\mathcal{A}_{n,s}^{\perp}$, the orthogonal complement of

$$\mathcal{A}_{n,s} = \langle \varphi_0, \varphi_1, \dots, \varphi_n, \psi_s \rangle.$$

Fix a complex number $d_1$ with $0 < |d_1| < \varepsilon$, and first define a linear functional $L_1$ on $A_2$ satisfying

$$L_1(\varphi_0) = 0, L_1(\varphi_k) = d_k,\ k = 1, 2, \dots, n,$$

with given $d_2, \dots, d_n$ and

$$L_1(\varphi) = 0, \quad \varphi \in \mathcal{A}_n^{\perp} = \langle \varphi_0, \varphi_2, \dots, \varphi_n \rangle^{\perp}.$$

Noting that all $\varphi \in A_2$ are of the form $\varphi = \sum\limits_0^\infty \eta_k r_k^{-1} \varphi_k$ and by Parseval's equality $\|\varphi\|_{A_2}^2 = \sum\limits_0^\infty |\eta_k|^2$, the norm of $L_1$ can be estimated (using also Schwarz's inequality) by

$$\begin{aligned} \|L_1\| &= \sup_{\|\varphi\|_{A_2}=1} |L_1(\varphi)| \\ &= \sup_{\sum\limits_0^\infty |\eta_k|^2 = 1} \Big| \sum_{k=1}^n \eta_k d_k / r_k \Big| \le \sup_{\sum\limits_0^\infty |\eta_k|^2 = 1} \sum_1^n |d_k|^2 / r_k^2 \sum_0^n |\eta_k^2| < M_2(n)\varepsilon. \end{aligned}$$

There exists a function $g_1 \in A_2$ representing $L_1$ so that $L_1(\varphi) = \langle \bar{g}_1, \varphi \rangle_E$ and $\|g_1\|_2 = \|L_1\| = O(\varepsilon)$, which defines a *polynomial* map $h_1 :\ D_R \to \mathbb{C}$ by

$$h_1(w) = w - \frac{1}{\pi} \iint_E \frac{\overline{g_1(\zeta)} d\xi d\eta}{\zeta - w} = w - \frac{1}{\pi} \sum_0^\infty \langle \bar{g}_1, \varphi_k \rangle_E w^k = \sum_1^n d_k w^k.$$

Let

$$h_1 \circ F_0(z) = \sum_1^\infty \frac{c_{k-1,1}}{k} z^k,$$

and put

$$a_1 = 2|c_1^0|^2 \operatorname{Re} d_1 + \sum_{k=2}^\infty \frac{|c_{k-1,1}|^2 - |c_{k-1}^0|^2}{k^2}. \tag{26}$$

Now define another linear functional $L_2$ on $A_2$, setting

$$L_2(\psi_s) = a - a_1$$

and $L_2(\varphi) = 0$ for $\varphi \perp \psi_s$ and extending by the Hahn–Banach theorem from the space $\langle \psi_s \rangle \bigoplus \langle \psi_s \rangle^{\perp}$, where $\langle \psi_s \rangle$ denotes the span $\{t\psi_s : \ t \in \mathbb{C}\}$ of $\psi_s$ onto $A_2$. Its norm is estimated by

$$\|L_2\| \le M_3(1/\delta)\varepsilon, \quad \delta = \mathrm{dist}(\psi_s, \langle \psi_s \rangle^{\perp}) > 0.$$

The corresponding function $g_2 \in A_2$ for which

$$L_2(\varphi) = \langle \bar{g}_2, \varphi \rangle_E, \quad \|g_2\|_2 = \|L_2\| = O(\varepsilon),$$

determines a holomorphic map $D_R \to \mathbb{C}$ by

$$\begin{aligned} h_2(w) &= -\frac{(s-1)!}{\pi} \iint\limits_E \frac{\overline{g_2(\zeta)} d\xi d\eta}{(\zeta - w)^s} \\ &= -\frac{1}{\pi} \sum_{k=s}^{\infty} \langle \bar{g}_2, \varphi_k \rangle_E \ k(k-1) \cdots (k-s+2) \ w^{k-s+1}. \end{aligned}$$

Define finally the map $h = h_1 + h_2 : \ \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ corresponding to $L = L_1 + L_2$. This map is holomorphic in the disk $\Delta_R$, and it follows from above that

$$h(0) = 0, \quad h'(0) = 1 + O(\varepsilon), \quad h^{(k)}(0) = k! d_k + O(\varepsilon^2), \quad k = 2, 3, \ldots, n,$$

with given $d_k$, and

$$\|(h \circ F_0)'\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = a + O(\varepsilon^2).$$

Restricting $h$ to a smaller disk $\overline{\Delta}_{R_1} = \{|w| \le R_1 < R\}$, we get

$$\|h'(w) - 1\|_{\overline{\Delta}_{R_1}} \le M_4(R_1)\varepsilon, \quad \|h^{(j)}(w)\|_{\overline{\Delta}_{R_1}} \le M_4(R_1)\varepsilon, \quad j = 2, 3, \ldots, n. \tag{27}$$

If $\varepsilon$ is sufficiently small, $h|\overline{\Delta}_{R_1}$ is univalent in this disk and admits quasiconformal extensions across the circle $\{|w| = R_1\}$ onto the whole complex plane. One can use, for example, its Ahlfors–Weill extension [2] with the harmonic Beltrami coefficient

$$\mu_h(w) = -\frac{1}{2} \overline{S_h\Big(\frac{R_1^2}{w}\Big)} \ \frac{(|w|^2 - R_1^2)^2}{\overline{w}^4}, \quad |w| > R_1,$$

where

$$S_h = (h''/h')' - (h''/h')^2/2$$

denotes the Schwarzian derivative of $h$ on $\overline{\Delta}_{R_1}$. It follows from (27) that $\|\mu_h\|_{\infty} = O(\varepsilon)$ as $\varepsilon \to 0$.

The quasiconformal homeomorphism $h$ constructed above satisfies all the assertions of the theorem except $h'(0) = 1$. To establish the latter property, observe that the parameter $d_1$ has been chosen arbitrarily in the above proof, and take now those values of $d_1$ for which equality (26) is reduced to

$$a_1 = 0.$$

One obtains a linear equation for the $\varepsilon$-linear term of $\operatorname{Re} d_1$ by which this quantity is determined uniquely. On the other hand, the value of $\operatorname{Im} d_1'$ does not affect the linear part of the right-hand side of (26).

The corresponding homeomorphism $h$ satisfies

$$h'(0) = 1 + O(\varepsilon^2),$$

and it remains to rescale $h$ by passing to $h(z)/h'(0)$. This completes the proof of Theorem 2. □

## 5. Extremal Holomorphic Covering Maps

**5.1.** We present an application of the above theorems to a well-known coefficient problem for holomorphic functions.

Let $F(z) = \sum_1^\infty C_k z^k$ be locally injective in the unit disk. Then its derivative $F'(z) = f(z)$ does not vanish in $\Delta$; thus $F$ is a nonbranched holomorphic covering map $\Delta \to F(\Delta)$ normalized by $F(0) = 0$. Assume that $F \in \tilde{B}^{2m}$ and consider the extremal problem to determine $\max |C_n|$ on the closed unit ball

$$\mathcal{B}_0(\tilde{B}^{2m}) = \{F(z) = \sum_1^\infty C_k z^k \in B^{2m} : \ \|F\| \le 1\}.$$

It is equivalent to the Hummel–Scheinberg–Zalcman problem for nonvanishing Bergman holomorphic functions $f$ in the disk (see [3]).

The first coefficient can be estimated in a standard way using Schwarz's lemma for holomorphic maps $\Delta \to \mathcal{B}_0(\tilde{B}^{2m})$. It gives, together with Parseval's equality applied to $f(z)^m$, that $|C_1| \le 1/\sqrt{\pi}$. The equality takes place only for the function

$$F(z) = \frac{\epsilon z}{\sqrt{\pi}}, \quad |\epsilon| = 1.$$

However there is no such connection for higher coefficients.

Boundedness in the $\tilde{B}^p$-norm yields, by the mean value inequality, the compactness in the topology of locally uniform convergence on $\Delta$, which, in turn, implies the existence of extremal covering maps $F_0$ maximizing $|C_n|$ on $\mathcal{B}_0(\tilde{B}^{2m})$. Note that $\|F_0\|_{2m} = 1$ since otherwise $\max |C_n|$ would increase by passing from $F_0$ to $(1 + r)F_0$ with appropriate $r > 0$.

**Theorem 4.** *Any extremal map $F_0$ maximizing $|C_n|$ is of the form*

$$F_0(z) = C_1^0 z + \sum_{k=n}^\infty C_k^0 z^k,$$

*i.e., it satisfies $C_2^0 = \cdots = C_{n-1}^0 = 0$ unless $F_0$ is a polynomial of degree $n$ (with nonvanishing derivative in $\Delta$).*

*Proof.* Let $F_0$ be different from a polynomial of degree at most $n$. Then its Taylor series in $\Delta$ contains nonzero powers $C_s^0$ with $s > n$. We assume the contrary that $C_j^0 \ne 0$ for some $1 < j < n$ and reach, by applying Theorem 1, a contradiction.

Observe that for any holomorphic function $h$ in a domain $D$ containing all the values $F(z)$ for $z \in \Delta$, we have $h(w) = \sum\limits_1^\infty d_k w^k$ in a neighborhood of the origin $w = 0$, and

$$\begin{aligned} h \circ F_0(z) &= d_1 \sum_1^\infty C_k^0 z^k + d_2 \Big( \sum_1^\infty C_k^0 z^k \Big)^2 + \cdots + d_n \Big( \sum_0^\infty C_k^0 z^k \Big)^n + \cdots \\ &=: \sum_1^\infty C_k^* z^k. \end{aligned}$$

Denoting $F_0'(z) = f_0(z) = \sum\limits_0^\infty c_k^0 z^k$ with $c_k^0 = (k+1) C_{k+1}^0$, we get

$$\begin{aligned} (h \circ F_0)'(z) &= d_1 \sum_1^\infty k C_k^0 z^{k-1} + 2d_2 \sum_1^\infty C_k^0 z^k \sum_1^\infty k C_k^0 z^{k-1} + \cdots \\ &\quad + n d_n \Big( \sum_1^\infty C_k^0 z^k \Big)^{n-1} \sum_1^\infty k C_k^0 z^{k-1} + \cdots \\ &= d_1 \sum_1^\infty c_{k-1}^0 z^{k-1} + 2d_2 \sum_1^\infty \frac{c_{k-1}^0}{k} z^k \sum_1^\infty c_{k-1}^0 z^{k-1} + \cdots \\ &\quad + n d_n \Big( \sum_1^\infty \frac{c_{k-1}^0}{k} z^k \Big)^{n-1} \sum_1^\infty c_{k-1}^0 z^{k-1} + \cdots \\ &=: \sum_0^\infty c_k^* z^k, \end{aligned}$$

where

$$c_0^* = d_1 c_0^0, \ c_1^* = d_1 c_1^0 + 2d_2 (c_0^0)^2, \ c_2^* = d_1 c_2^0 + 3d_2 c_0^0 c_1^0 + 3d_3 (c_0^0)^3, \ \dots . \tag{28}$$

If $h$ is, in addition, homeomorphic (thus conformal) on $F_0(\Delta)$, then the map $h \circ F_0$ is a local conformal homeomorphism of the disk $\Delta$.

Now, let $k = j_0 > 1$ and $s \geq n$ be the least indices of nonzero coefficients $C_k^0$ for $1 < k < n$ and for $k > n$, respectively.

We apply Theorem 1 starting with $j = j_0$ and $a = 0$, which provides a quasiconformal homeomorphism $h$ of $\hat{\mathbb{C}}$ satisfying the previous conformality assumptions and obtain from (28) that the coefficient $C_n^*$ becomes a holomorphic function of the independent parameters

$$d_1, d_{j_0+1}, \dots, d_{n+1}$$

defining $h$, because $d_{j_0}$, in itself, is a holomorphic function of these parameters. The openness of holomorphic maps allows us to choose these parameters (at least $d_1$ and $d_n$) so that

$$|C_n^*| = |d_1 C_n^0 + \cdots + d_n (C_0^0)^n| > |C_n^0| + O(\varepsilon) > |C_n^0|. \tag{29}$$

On the other hand, we have

$$\|(h \circ F_0)'\|_{2m} = \|f_0\|_{2m} + O(\varepsilon^2) = 1 + O(\varepsilon^2),$$

hence the map $h \circ F_0 + O(\varepsilon^2)$ also belongs to $\mathcal{B}_0(\tilde{B}^{2m})$. But then (29) violates the extremality of $F_0$ for $C_n$ by a suitable choice of $\varepsilon > 0$. This proves $C^0_{j_0} = 0$.

Applying Theorem 1 successively for $j_0 = 2, 3, \ldots, n-1$, one obtains the assertion of Theorem 4. $\square$

It seems likely that similar to what was established in [7] for the Hardy functions $f \in H^p(\Delta)$, the extremal map $F_0$ in $\mathcal{B}_0(\tilde{B}^{2m})$ cannot reduce to a polynomial of degree $n_1 \leq n$ either.

**5.2.** The results of this paper extend to arbitrary Hilbert spaces of holomorphic functions in the disk whose norm is real analytic.

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