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ON A REPRESENTATION OF THE DERIVATIVE OF A CONFORMAL MAPPING

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Abstract. Let ω conformally map the unit circle on a plane singly-connected domain D bounded by a simple rectifiable curve. It is shown that for the function $\lg\omega'$ to be represented in the unit circle by a Cauchy type A-integral with density $\arg\omega'$, it is necessary and sufficient that D be a Smirnov domain. In particular, for this representation to be done by a Cauchy–Lebesgue type integral with the same density, it is necessary and sufficient that the function $\lg\omega'$ belong to the Hardy class H_1.

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Let D be a finite singly-connected domain bounded by a simple rectifiable curve Γ, $\omega=\omega(z)$ be a function mapping conformally the unit circle U on D, and γ be the boundary of U. Then the derivative ω' belongs to the Hardy class H_1, and almost for all $\vartheta\in[0,2\pi]$ there exists an angular value of the function $\omega'(z)$ and

$$\lim_{z\overset{\lambda}{\longrightarrow}e^{i\vartheta}}\omega'(z)=\omega'(e^{i\vartheta})=-ie^{-i\vartheta}\,\frac{d\omega(e^{i\vartheta})}{d\vartheta} \tag{1}$$

(see, e.g., [1], Ch. III, §1, 1.1,1.6).

Throughout the paper it is assumed that $\omega'(0)>0$ and $\arg\omega'(0)=0$.

In view of (1), for almost all ϑ we have

$$\lim_{z\overset{\lambda}{\longrightarrow}e^{i\vartheta}}\arg\omega'(z)=\arg\omega'(e^{i\vartheta})=\arg\,\frac{d\omega(e^{i\vartheta})}{d\vartheta}-\vartheta-\frac{\pi}{2}\,, \tag{2}$$

where $\arg\frac{d\omega(e^{i\vartheta})}{d\vartheta}$ is one of the angles between the tangent to Γ at the point $\omega(e^{i\vartheta})$ and the abcissa axis.

The present paper is a continuation of [4]; it contains some comments on the well-known formula

$$\lg\omega'(z)=\lg\omega'(0)+\frac{i}{2\pi}\int\limits_0^{2\pi}\arg\omega'(e^{i\vartheta})\,\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma$$

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$$= \lg \omega'(0) + \frac{1}{\pi} \int\limits_{|\tau|=1} \frac{\arg \omega'(\tau)}{\tau - z}\, d\tau, \tag{3}$$

which is valid provided that $\lg \omega' \in H_1$ (this is the Schwarz formula applied to the function $i \lg \omega'$). In particular, using a certain extension of the Lebesgue integral, formula (3) is generalized here to Smirnov domains for which

$$\lg \omega'(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} \lg |\omega'(e^{i\sigma})| \, \frac{e^{i\sigma} + z}{e^{i\sigma} - z}\, d\sigma = -\lg \omega'(0) + \frac{1}{\pi i} \int\limits_{|\tau|=1} \frac{\ln |\omega'(\tau)|}{\tau - z}\, d\tau \tag{4}$$

(see, e.g., [1], Ch. III, §12).

Isolating in (3) the imaginary part, we obtain

$$\arg \omega'(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} \arg \omega'(e^{i\sigma}) \, \frac{1 - |z|^2}{|e^{i\sigma} - z|^2}\, d\sigma, \quad |z| < 1. \tag{5}$$

Having certain information on the properties of the function $\arg \omega'(e^{i\vartheta}) = \frac{d\omega(e^{i\vartheta})}{d\vartheta} - \vartheta - \frac{\pi}{2}$, we can establish, by formulas (3) and (5), the respective properties of the function ω' in the circle U and, conversely, knowing the properties of the function ω' in the circle, it is possible to establish some properties of the function $\frac{d\omega(e^{i\vartheta})}{d\vartheta}$ and, hence, the geometrical properties of the boundary of D, as we do, for instance, in the case of the Lindelöf theorem which states that the smoothness of the boundary of D (the continuity of the inclination angle of the tangent to Γ) is equivalent to the continuity of the function $\arg \omega'$ on the closed circle $\overline{U}$[1] (see, e.g., [2], pp. 42, 44). However the function $\arg \omega'(z)$ has been obtained as a boundary function of the harmonic function in U, which does not always give direct information on the properties of $\arg \omega'(e^{i\vartheta})$. Formulas (3) and (5) are useful if the function $\arg \omega'(e^{i\vartheta})$ is constructed using some other arguments as was done, for instance, in [3], [2] (§§3.2, 3.5), [4], [5] (Ch. III).

Even if it is assumed that D is a Smirnov domain, the function $\lg \omega'$ may not always belong to the Hardy class H_1 (see [6]), and the function $\arg \omega'(e^{i\vartheta})$ is not always summable (see equality (13) below). Hence formula (3) cannot be written even for all Smirnov domains[2] (if the consideration is restricted to the Lebesgue integral). However, if one uses certain generalized Lebesgue integrals in whose sense the conjugate function of the summable function is integrable (for instance, the A-integral, see [7] and [8], Ch. VIII, §18, or the B-integral, see [9], Ch. VII, §4), then formula (3) can be extended to Smirnov domains as well.

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[1] [12], p. 94, gives a wrong statement that the function $\arg \omega'(z)$ is continuous on the closed circle $\overline{U}$ under an assumption that there only exists a tangent at every point Γ.

[2] [12], pp. 90–92, gives a wrong statement that formula (3) is valid for all domains bounded by arbitrary rectifiable curves.

1. By the $\widetilde{L}$-integral we will mean a minimal extension of the Lebesgue integral in whose sense the conjugate functions

$$\widetilde{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} f(t)\,\mathrm{ctg}\,\frac{t-x}{2}\,dt$$

of the summable functions f on $[0,2\pi]$ are integrable, and the integral of the conjugate functions is equal to zero (see, for instance, [10], pp. 38, 88, or [5], Ch. I, §6). This is a class of functions of the form $f_1+\widetilde{f}_2$, where $f_1,\ f_2 \in L(0,2\pi)$.

Definition. A measurable function on $[a,b]$ is called A-integrable if

$$\Big|\{x\in[a,b];\ |f(x)|>\lambda\}\Big| = o(\lambda^{-1}) \tag{6}$$

and there exists a limit

$$\lim_{\lambda\to\infty}\int_{|f|\le\lambda} f(x)\,dx$$

which is called an A-integral of f with respect to $[a,b]$. We denote it by $(A)\int_a^b f(x)\,dx$.

$\widetilde{L}$-integrable functions are A-integrable (and B-integrable) and the integrales coincide (see the above-cited references).

We have equality

$$(A)\int_{|t|=1}\varphi(t)S(f)(t)\,dt = -\int_{|t|=1} S(\varphi)(t)f(t)\,dt, \tag{7}$$

where f is summable, φ satisfies the Lipshitz condition on γ,

$$S(f)(t) = \frac{1}{\pi i}\int_{|\tau|=1}\frac{f(\tau)}{\tau-t}\,d\tau$$

(see, for instance, [7], [8], Ch. VIII, §18). In equality (7), the A-integral can be replaced by any integral containing the $\widetilde{L}$-integral, say, by the B-integral (see, for instance, [10] or [5]).

Theorem. *In order that the formula*

$$\begin{aligned}\lg\omega'(z) &= \lg\omega'(0) + \frac{i}{2\pi}(A)\int_0^{2\pi}\arg\omega'(e^{i\sigma})\,\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma\\ &= \lg\omega'(0) + \frac{1}{\pi}(A)\int_{|\tau|=1}\frac{\arg\omega'(\tau)}{\tau-z}\,d\tau,\quad |z|<1,\end{aligned} \tag{8}$$

be valid, it is necessary and sufficient that D be a Smirnov domain.

Proof. Sufficiency. We will need the following two equalities which are easy to verify:

$$S(\lg|\omega'|)(t) = \frac{1}{\pi i}\int\limits_{|\tau|=1}\frac{\lg|\omega'(\tau)|}{\tau-t}\,d\tau = i\arg\omega'(t) + \frac{1}{2\pi}\int\limits_0^{2\pi}\lg|\omega'(e^{i\sigma})|\,d\sigma, \quad (9)$$

$$-(t-z)^{-1} = S\Big((\tau-z)^{-1}\Big)(t), \quad |z|<1. \quad (10)$$

Let D be a Smirnov domain. Then, taking into account (9), (10) and the equality

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\lg|\omega'(e^{i\sigma})|\,d\sigma = \lg\omega'(0)$$

(the latter equality is valid because D is a Smirnov domain; it follows from (4)), we obtain by virtue of (7)

$$\int\limits_{|\tau|=1}\frac{\lg|\omega'(\tau)|}{\tau-z}\,d\tau = -\int\limits_{|\tau|=1}\lg|\omega'(\tau)|S\Big((t-z)^{-1}\Big)(\tau)\,d\tau$$

$$= (A)\int\limits_{|\tau|=1}\frac{S(\lg|\omega'|)(\tau)}{\tau-z}\,d\tau = (A)i\int\limits_{|\tau|=1}\frac{\arg\omega'(\tau)}{\tau-z}\,d\tau + 2\pi i\lg\omega'(0). \quad (11)$$

Equalities (4) and (11) imply (8).

Necessity. Using the canonical expansion of a function from the class H_1, we can write

$$\lg\omega'(z) = \frac{1}{2\pi}\int\limits_0^{2\pi}\lg|\omega'(e^{i\sigma})|\,\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma + \frac{1}{2\pi}\int\limits_0^{2\pi}\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\psi(\sigma) \quad (12)$$

(again keeping in mind the branch for which $\arg\omega'(0)=0$), where ψ is a nondecreasing singular function (see, for instance, [1], p. 220). Isolating, in (12), the imaginary part and passing to the limit, we obtain

$$\lim_{r\to1}\arg\omega'(re^{i\vartheta}) = \arg\omega'(\vartheta) = \widetilde{\lg|\omega'|}(\vartheta) + \widetilde{d\psi}(\vartheta), \quad (13)$$

where

$$\widetilde{\lg|\omega'|}(\vartheta) = -\frac{1}{2\pi i}\int\limits_0^{2\pi}\ln|\omega'(e^{i\sigma})|\,\mathrm{ctg}\,\frac{\sigma-\vartheta}{2}\,d\sigma,$$

$$\widetilde{d\psi}(\vartheta) = -\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{ctg}\,\frac{\sigma-\vartheta}{2}\,d\psi(\sigma)$$

(the conjugate functions of $\lg|\omega'(e^{i\vartheta})|$ and $d\psi$, respectively).

Let (8) be valid, in particular, the function $\arg\omega'(e^{i\vartheta})$ be A-integrable. Then in view of (13) we can write

$$\Big|\{\vartheta\in[0,2\pi];\ |\widetilde{d\psi}(\vartheta)|>\lambda\}\Big|=o(\lambda^{-1}) \tag{14}$$

and therefore $\psi\equiv\text{const}$ (see, for instance, [10], p. 26), which means that D is a Smirnov domain. □

Corollary. *For* (8) *to hold with the summable function* $\arg\omega'(e^{i\vartheta})$*, it is necessary and sufficient that* $\lg\omega'$ *belong to the Hardy class* H_1*.*

The sufficiency is obvious. The necessity follows from equality (13), since, when $\arg\omega'(e^{i\vartheta})$ is summable, condition (14) is fulfilled and $\psi\equiv\text{const}$. Then $\widetilde{\lg|\omega'|}(\vartheta)=\arg\omega'(e^{i\vartheta})\in L(0,2\pi)$ and therefore $\lg\omega'\in H_1$.

Remark 1. By equality (3), from condition (14) it follows that the condition $|\{\vartheta\in[0,2\pi];|\arg\omega'(e^{i\vartheta})|>\lambda\}|=o(\lambda^{-1})$ is necessary and sufficient for D to belong to the Smirnov class.

Remark 2. One can easily verify that formula (8) remains in force if ω' is assumed to be an arbitrary function of the class H_1 which is different from zero in U. Hence we have the following assertion:

If $f\in H_1$ *and* $f(z)\neq 0$ *in* U*, then the parametric representation of* f (*see* [1]*, pp.* 110–111) *can be written in the form*

$$f(z)=f(0)\exp\left\{\frac{1}{2\pi}\,(A)\int_0^{2\pi}\arg f'(e^{i\sigma})\,\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma\right\}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\mu\right\}.$$

2. As follows from the arguments used in proving the theorem, the A-integral can be replaced by $\widetilde{L}$-integral. A further extension of the notion of the integral with an aim to extend formula (8) to non-Smirnov domains leads to a contradiction between the considered formula and the Cauchy and Schwarz integral formulas.

Indeed, let D be a non-Smirnov domain, i.e., the Schwarz formula (4) be invalid. Then we have

$$\lg\omega'(z)=\lg\omega'(0)+\frac{1}{2\pi}(X)\int_0^{2\pi}\arg\omega'(e^{i\sigma})\,\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\,d\sigma, \tag{15}$$

where $(X)\int\cdots$ is some extension of the $\widetilde{L}$-integral. From (15) in particular it follows that $(X)\int_0^{2\pi}\arg\omega'(e^{i\sigma})\,d\sigma=0$. By virtue of the latter equality, the formula of the mean

$$\lg\omega'(0)=\frac{1}{2\pi}\int_0^{2\pi}\lg\omega'(e^{i\sigma})\,d\sigma$$

and therefore the Cauchy formula

$$\lg \omega'(z) = (2\pi i)^{-1}(X) \int\limits_{|\tau|=1} (\tau - z)^{-1} \lg \omega'(\tau)\, d\tau \tag{16}$$

give

$$\lg \omega'(0) = \frac{1}{2\pi} \int\limits_0^{2\pi} \lg |\omega'(e^{i\sigma})|\, d\sigma$$

which, on account of equality (12), is valid if and only if D is a Smirnov domain.

3. In this subsection we give the proof of one well-known statement on $\omega'(z)$ (see [11]), based on representation (3).

Statement ([11]). *Let Γ be a closed smooth rectifiable curve with the equation $\zeta = \zeta(s)$, $0 \le s \le \ell$, (s is an arc abscissa), and $\delta = \delta(s)$ be a slope angle of the tangent to the point $\zeta(s)$ with the abscissa axis, which changes continuously on $[0,\ell]$. If the continuity modulus $\rho(\delta, t)$, $t \in (0,\ell)$ of the function δ satisfies the Dini condition*

$$\int\limits_0^{\ell} \frac{\rho(\delta,t)}{t}\, dt < \infty, \tag{17}$$

then the derivative of the conformal mapping of the unit circle on the finite domain bounded by Γ is continuous in the closed circle.

Proof. Let $\delta(s) = \delta(s(\zeta))$ and $\zeta = \omega(e^{i\vartheta})$. Then the function $\zeta = \zeta(\vartheta)$ is uniquely defined on $[0, 2\pi]$ and therefore, by the Lindelöf theorem, we have $\arg \omega'(e^{i\vartheta}) = \delta(\zeta(\vartheta)) - \vartheta - \frac{\pi}{2}$. Since Γ is a smooth curve, we can rewrite (3) as

$$\omega'(z) = \omega'(0) \exp\left\{ \frac{1}{\pi} \int\limits_{|\tau|=1} \frac{\delta(\zeta(\vartheta)) - \vartheta - \frac{\pi}{2}}{\tau - z}\, d\tau \right\}, \quad \tau = e^{i\vartheta}. \tag{18}$$

Let us set $\nu(\vartheta) = \delta(\zeta(\vartheta)) - \vartheta - \frac{\pi}{2}$ and show that the continuity modulus $\rho(\nu, t)$, $t \in (0, 2\pi)$, satisfies the Dini condition. We obtain

$$\Big|\nu(\vartheta + h) - \nu(\vartheta)\Big| = \Big|\delta(\zeta(\vartheta + h)) - \delta(\zeta(\vartheta)) - h\Big|$$

$$\le \rho\Big(\delta, \sup_{0<\sigma\le h} |\zeta(\vartheta+\sigma) - \zeta(\vartheta)| + h\Big) \le \rho\Bigg(\delta, \sup_{0<\sigma\le h} \int\limits_{\vartheta}^{\vartheta+h} |\omega'(e^{iu})|\, du + h\Bigg). \tag{19}$$

Since Γ is a smooth curve, we have $\omega' \in \bigcap\limits_{p>1} H_p$ (see, for instance, [2]). Therefore $\omega'(e^{iu}) \in \bigcap\limits_{p>1} L_p(0, 2\pi)$ and hence (19) implies that for any $\alpha \in (0,1)$ there exists M_α such that

$$\sup_{\rho\le h} \Big|\nu(\vartheta + \sigma) - \nu(\vartheta)\Big| \le \rho(\delta, M_\alpha h^\alpha) + h.$$

But then

$$\int_0^{2\pi} \frac{\rho(\nu,t)}{t}\,dt \le \int_0^{\ell} \frac{\rho(\nu,Mt^{\alpha})}{t}\,dt + 2\pi$$

$$\le k\int_0^{\ell} \frac{\rho(\delta,u)}{u^{1/\alpha}}\,u^{1/\alpha-1}\,du + 2\pi,\quad k=\frac{1}{\alpha M_{\alpha}^{1/\alpha}}\,. \tag{20}$$

Hence, by virtue of (20), we conclude that $\int\limits_0^{2\pi} \frac{\rho(\nu,t)}{t}\,dt < \infty$ and thus

$$\omega'(z) = \omega'(0)\exp\left\{\frac{1}{\pi}\int\limits_{|\tau|=1} \frac{\nu(\vartheta)}{\tau - z}\,d\tau\right\},\quad \tau = e^{i\vartheta}, \tag{21}$$

where condition (17), i.e., the Dini condition is fulfilled for the continuity modulus of the function ν. As is known, in that case a Cauchy type integral with density ν is a continuous function in the closed circle (see, for instance, [2]). Hence, by virtue of (21), it follows that the function $\omega'(z)$ is continuous too. □

Other applications of representation (3) can be found in [2]–[5], where the function $\arg\omega'(e^{i\vartheta})$ is assumed to be given.

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