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HYPER-HOLOMORPHIC CELLS AND FREDHOLM THEORY

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Abstract. We deal with differentiable cells defined by solutions to certain linear elliptic systems of first order. It turns out that in some cases families of such cells attached to a given submanifold may be described by Fredholm operators in appropriate function spaces. Using the previous results of the author on the existence of elliptic Riemann–Hilbert problems for generalized Cauchy–Riemann systems, we indicate some classes of systems which give rise to non-linear Fredholm operators of such type.

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1. Introduction

The classical boundary value problems (BVPs) for holomorphic functions, in particular the linear conjugation problem and the Riemann–Hilbert problem whose comprehensive theory owes much to the works of N. Muskhelishvili and his school (see, e.g., [24], [33], [34]), can be described by linear operators in appropriate function spaces (cf. the abstract operatorial setting developed in the book of S. Prössdorf [26]). Nowadays there also exist several interesting non-linear versions of these classical problems (see, e.g., [29], [35]).

One of the most spectacular generalizations of this kind was developed in the seminal paper of M. Gromov concerned with the pseudo-holomorphic curves [19]. Gromov's approach involves, in particular, two important new aspects: generalizing the equation satisfied by functions (which is in some sense equivalent to working with solutions of the Bers–Vekua equation [5], [33]) and consideration of non-linear boundary conditions in the spirit of "holomorphic discs attached to a totally real submanifold" [7].

All these results are concerned with functions which locally depend on two real variables and one may wonder if similar results can be obtained for functions of several real variables satisfying some elliptic system of equations. Such generalizations do not seem to have attracted much attention up to now, but the existing results about linear BVPs for elliptic systems (see [9], [31], [22]) suggested that some results of this type should be available for systems of Dirac type [17].

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The present paper may be considered as one of the first steps in this direction. To be more precise, we discuss some geometric properties of families of solutions to certain elliptic first order systems of linear partial differential equations with constant coefficients [36] (such systems are sometimes called "canonical first order systems" (CFOS) [3]). These systems were studied in many papers and they remain an object of a permanent interest (see [31] for a recent review of the topic).

An especially important class of such systems is provided by the so-called "generalized Cauchy–Riemann systems" (GCRS) [30]. Solutions to generalized Cauchy–Riemann systems are often called *hyper-holomorphic mappings* [28] and in many problems it is necessary to consider images of some standard domains (e.g., balls) under such mappings. Standard examples of such systems in low dimensions are the classical Cauchy–Riemann system in the plane, the Moisil–Theodoresco system in $\mathbf{R}^3$ [8], and Fueter system for functions of one quaternionic variable [10]. There exists a vast literature devoted to such equations (see references in [8] and [10]). In particular, some important results about the so-called *generalized analytic vectors* were obtained by georgian mathematicians [8], [25]. Similar systems emerged in the theory of para-analytic functions developed by M. Frechet [14].

The main paradigm we follow in this paper, has its origin in the theory of analytic (holomorphic) discs attached to a totally real submanifold [7]. One takes a smooth bounded domain homeomorphic to a ball of appropriate dimension and considers its images under solutions to a given CFOS. Such images (we call them *elliptic cells*) are our main concern in this paper.

More concretely, inspired by the theory of attached analytic discs [7], [13] we consider elliptic cells with boundaries in a fixed submanifold M of the target space of the elliptic system in question. They are called *elliptic cells attached to* M. For our purposes it appears useful to regard them as solutions of non-linear boundary value problems of Riemann–Hilbert type [35]. Accepting terminology from [35], M will be called a *target manifold* (for attached elliptic cells).

Actually, we only consider hyper-holomorphic cells, i.e., those which are defined by solutions to a given GCRS. Notice that except the aforementioned theory of attached analytic discs [7], there also exist important generalizations of this classical example in the framework of symplectic geometry [19].

Recently, similar objects appeared in mathematical physics under the name of *D-branes*. D-branes have already found interesting applications in topological field theory and string theory [6], [15]. It is worthy of noting that in those physical contexts there also appear D-branes attached to certain submanifolds. This confirms our trust that such objects deserve some attention by their own.

With this in mind, we investigate some situations where families of attached hyper-holomorphic cells can be locally described as kernels of some (non-linear) Fredholm operators. Such a phenomenon is well known in the case of analytic discs [2] and it plays an important role in M. Gromov's studies on pseudo-holomorphic curves [19]. In particular, one becomes able to use the well known concepts and techniques of Fredholm theory, which reveals some important topological aspects of the situation. We mimic Gromov's approach and establish some properties of emerging non-linear operators using the Fredholm theory of elliptic Riemann–Hilbert problems (RHPs) for GCRS developed in [31], [21].

In particular, we show that, for certain values of dimensions n, k, and m, there exist non-compact k-submanifolds in affine m-space such that families of hyper-holomorphic n-cells attached to such submanifolds are locally described by Fredholm operators. Borrowing again terminology from [35], such submanifolds are called *admissible targets* (for a given GCRS). Existence of admissible targets and Fredholmness of arising non-linear operators are derived from the recent results on the existence of elliptic boundary value problems for GCRS [31], [22] (cf. also [28]).

Such aspects of generalized Cauchy–Riemann systems seem to have never been discussed in the literature, so in this paper we pursue but a modest goal of describing and illustrating the framework which naturally stems from our previous results on generalized Cauchy–Riemann systems. Since the subject has many natural connections with other topics we give a sufficiently long list of references in order to indicate some of the related concepts and techniques.

2. Generalized Cauchy–Riemann Systems

We present here some relevant notions from [30] and [36].

Definition 1 ([30]**, cf.** [17]**).** An elliptic system of first order with constant coefficients is called a generalized Cauchy–Riemann system (GCRS) if it is invariant under the natural action of the orthogonal group on the source space and all components of its differentiable solutions are harmonic functions. Solutions of such systems are called hyper-holomorphic (hh) mappings. For a given such system S, its solutions will be also called S-mappings.

It is well known (see, e.g., [31]) that, without loss of generality, one may always assume that such a system in $\mathbf{R}^{n+1}$ may be written in a canonical form:

$$E\frac{\partial w}{\partial x_0} + A_1\frac{\partial w}{\partial x_1} + \ldots + A_n\frac{\partial w}{\partial x_n} + Dw = f, \tag{1}$$

where A_j, D are constant complex $(m \times m)$ matrices, $E = A_0$ is the identity matrix, and for all $i, j = 1, \ldots, n$, one has:

$$A_iA_j + A_jA_i = -2\delta_{ij}E. \tag{2}$$

We will consider such system in a smooth domain $U \Subset \mathbf{R}^{n+1}$ and assume that the unknown vector-function w belongs to class $C^1(U, \mathbf{C}^m)$.

It is also well known that system (1) is elliptic, in the usual sense [36], i.e.,

$$\det(t_0E + t_1A_1 + \ldots + t_nA_n) \neq 0,$$

for all $(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} - \{0\}$.

From (2) it is clear that such a system defines a representation of Clifford algebra Cl_n on $\mathbf{C}^m$ [17]. So the (complex) target dimension m, being the sum of dimensions of irreducible representations of Cl_n, is an integer multiple of $2^{[n/2]}$ [9]. If for a given system S this dimension is the minimal possible, $m(n) = 2^{[n/2]}$, we will say that system S is irreducible. In many situations it is sufficient to consider only irreducible GCRSs.

For the sake of visuality, we explicitly write down some examples of such systems in low dimensions. For $n = 1$, one has $m(1) = 1$ and the corresponding irreducible system is just the classical Cauchy–Riemann system for two real functions $u(x, y), v(x, y)$ of two real variables:

$$\begin{aligned} u_x - v_y &= 0, \\ u_y + v_x &= 0. \end{aligned}$$

For $n = 2$, one has $m(2) = 2$, and the corresponding irreducible systems for four real functions s, u, v, w of three real variables is the well-known Moisil-Theodoresco system which may be written using standard operators on vector-functions in $\mathbf{R}^3$ [17]:

$$\begin{aligned} \operatorname{div}(u, v, w) &= 0, \\ \operatorname{grad} s + \operatorname{rot}(u, v, w) &= 0. \end{aligned}$$

For $n = 3$, one has $m(3) = 2$, and the corresponding irreducible system is the so-called Fueter system for four real functions f_i of four real variables x_j [17]:

$$\begin{aligned}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} &= 0, \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} &= 0, \\
\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} &= 0.
\end{aligned}$$

As is well known, this system is a natural counterpart of Cauchy–Riemann system for a function of one quaternionic variable. Its solutions, called quaternionic-regular functions, have many interesting properties similar to those of usual holomorphic functions of one complex variable [17], [28].

The general theory of PDE yields that one can formulate various reasonable boundary value problems (BVPs) for such systems in bounded domains [36], [32]. For our purposes the most relevant are the classical local boundary value problems of Riemann–Hilbert type defined as follows. One searches for solutions of (1) satisfying a boundary condition of the form:

$$(B_1 B_2) \cdot w = g, \tag{3}$$

where B_1, B_2 are continuous complex $(\frac{m}{2} \times \frac{m}{2})$-matrix-functions on bU such that the rows of $(\frac{m}{2} \times m)$-matrix-function (B_1, B_2) are linearly independent at every boundary point, and g is a continuous vector-function with values in $\mathbf{C}^m$.

For our purposes especially appropriate are those RHPs which are elliptic in the usual sense (i.e., satisfy Shapiro-Lopatinski condition [36]) because then the problem (1), (3) is described by a Fredholm operator in appropriate function spaces [36]. It is well-known that not all systems of the type (1) admit elliptic boundary conditions (3) [8], [36], so the first natural problem is to investigate which GCRSs possess elliptic RHPs. We give an answer to this question in Section 4, which enables us to indicate cases in which hyper-holomorphic cells are described by Fredholm operators.

Remark 1. An important class of GCRS is associated with Euclidean Dirac operators [9]. The corresponding systems of the first order are called *systems of Dirac type* and their solutions are called *monogenic mappings* [17].

For notational convenience, in the sequel we denote by V the target space $\mathbf{C}^m$ of system (1).

3. Hyper-Holomorphic Cells

We fix a GCRS of the form (1) and denote by B a $(n+1)$-ball in its source space. We also take some submanifold M in V and refer to it as a target.

Definition 2. A hyper-holomorphic (hh) cell attached to M is defined as (the image of) a hyper-holomorphic mapping $u : \overline{B} \to V$ such that $u(bB) \in M$. For a fixed GCRS S, we will speak of S-cells attached to M.

The usual way of dealing with hh cells attached to a given submanifold is to consider families of cells attached at a given point. Such families may be described by certain non-linear operators in appropriate function spaces and if these operators appear to be Fredholm, then one may obtain a reasonable structural theory of such cells, as it happens, for example, for pseudo-holomorphic discs and curves [19], [2]. So it is natural to begin with looking for such situations where hh cells may be related to Fredholm operators. In order to make this idea precise, we need some constructions and definitions.

To this end, consider an irreducible GCRS S in $\mathbf{R}^{n+1}$ with values in V. Consider also some smooth (C^∞) submanifold M of V of the real dimension equal to the complex dimension $m(n)$ of V (in such case we speak of a *submanifold of middle dimension*). Let B denote an $(n+1)$-ball centered at the origin of the source space of S and let q be some fixed point on its boundary n-sphere bB. Furthermore, we fix a point $p \in M$ and a non-integer number $r > 1$, and let H^r denote the usual Hölder class.

Let F be the space of H^{r+1} maps $f : (B, bB, q) \to (V, M, p)$ which are homotopic to the constant map $f_p = p$ in $\pi_{n+1}(V, M, p)$ (such maps will be called *homotopically trivial*). In a standard way one checks that F is a smooth complex-Banach manifold (cf. [2]). Let G be the complex Banach space of all H^r maps $g : B \to V$. Define also a submanifold in $F \times G$ by putting

$$H = \{(f, g) \in F \times G : D(f) = g\}, \tag{4}$$

where by D is denoted the matricial partial differential operator defined by the left-hand-side of (1).

Then it is easy to see that H is a connected submanifold of $F \times G$ and one may define the projection map $L_p : H \to G$ given by $L_p(f, g) = g$. It is also easy to check that L_p is a smooth map of H into Banach space G.

Definition 3. Mapping L_p is called Gromov's operator of the pair (S, M) at point p. Manifold M is called an S-admissible target if Gromov's operator $L_p(S, M)$ is a Fredholm operator (mapping) for every $p \in M$.

Similar operators were introduced by M. Gromov for analytic discs [19] (cf. also [2]). General techniques of functional analysis (Fredholm theory, Sard-Smale theorem, theory of Fredholm structures) suggest that if this operator appears to be Fredholm, one may count for a reasonable structural theory for attached elliptic cells. In some sense this is the most natural way of formulating a version of Fredholm theory for elliptic cells.

We now present a typical result of this type available in our context. For us, of a special importance are those targets M for which L_p is Fredholm at any point $p \in M$, so we introduce a short-hand *admissible targets* for the targets possessing this property.

Recall that we are given an irreducible GCRS S in $\mathbf{R}^{n+1}$ with values in V. Construct another GCRS $D(S)$ with values in $W = V \times V$ which is a sort of "double" of S. If n is even, than $D(S)$ simply consists of two identical copies of S. If n is odd, then one adds to S the canonical GCRS corresponding to the second irreducible representation of Cl_n. Thus the complex dimension of the target space of $D(S)$ is $2m(n)$.

Remark 2. Consideration of such "doubles" is suggested by the results of Section 4. From the viewpoint of K-theory this may be considered as a sort of "stabilization" and it is quite natural that this operation improves certain properties of the system (see [9]).

We construct now non-compact admissible targets M in W as images of appropriate embeddings of $\mathbf{R}^{2m(n)}$. We assume that all spaces of smooth mappings are endowed with Whitney topology [20].

Theorem 1. *There exists an open set of embeddings $f : \mathbf{R}^{2m(n)} \to W$ such that, for every point p of $M = f(\mathbf{R}^{2m(n)})$, Gromov's operator of the pair $(D(S), M)$ at point p is a (non-linear) Fredholm operator of index zero.*

In other words, non-compact admissible targets exist for systems of the form $D(S)$. At the moment we do not have any general existence results for compact admissible targets. Notice that for every (compact or non-compact) admissible target, Fredholmness of Gromov's operators combined with a standard application of implicit function theorem for Banach spaces in the spirit of [20] implies that the homotopically trivial families of attached elliptic cells are locally finite-dimensional. Notice that here one need not restrict himself to systems of the form $D(S)$.

Corollary 1. *If M is a S-admissible target, then the family of homotopically trivial S-hyper-holomorphic cells attached to M at p is finite-dimensional.*

This result can be considered as a description of the subset of all hh cells attached to M which are close to a "degenerate" cell $f_p = p$. One obtains its natural generalization by considering the subset of all hh cells atached to M which are sufficiently close to an arbitrary given cell g attached to M. One need not even assume vanishing of the class of g in $\pi_{n+1}(V, M)$.

Corollary 2. *For a given S-cell g attached to an S-admissible target M, the set of all S-cells attached to M near g is finite-dimensional.*

In both these cases one encounters a natural problem of computing the "virtual dimension of nearby attached hh cells" (see [20]) in terms of S, M, and of the given cell g. Such formulae are available for (pseudo-)analytic discs (or Cauchy–Riemann cells, in our terminology) and they involve the notion of Maslov index of a curve along a totally real submanifold M [13], [19].

Up to the author's knowledge, in the general case this is an unsolved problem. As one can see from the discussion presented in the next section, progress in this problem depends on the availability of explicit index formulae for elliptic linear Riemann–Hilbert problems for GCRSs. Apparently in concrete cases one can successfully apply the analytic formulae for indices of elliptic boundary value problems in Euclidean space obtained by A.Dynin [11] and B.Fedosov [12].

The proof of Theorem 1 is presented in the next section. Using a natural linearization procedure it can be derived from general results on existence of elliptic boundary value problems for GCRSs which are also presented in the next section. An examination of the proof shows that for GCRSs in spaces of odd dimension (in our notation this means that n should be even) the same result can be obtained without passing to doubles, which yields the second main result of this paper.

Theorem 2. *For every irreducible GCRS on a space of odd dimension different from 5 and 7, there exists an open set of embeddings of $\mathbf{R}^{m(n)}$ into $\mathbf{C}^{m(n)}$ such that their images are admissible targets for attached S-cells.*

4. Elliptic Riemann–Hilbert Problems

We give now a comprehensive description of those GCRS which possess elliptic local boundary value problems. The results of this section were obtained in [31], [22] so in order to distinguish them from the results of the present paper, we present them as "statements".

By definition with every GCRS S there are associated integers m and n. Since the target $\mathbf{C}^m$ is a representation space for Clifford algebra Cl_n, its dimension m is an integer multiple l of the dimension of irreducible representations $m(n) = 2^{[\frac{n}{2}]}$ [31]. Thus there appears the third integer l (of course it is completely determined by m and n). For odd n, there also appear integers l_1, l_2 ($l_1 + l_2 = l$) showing how many irreducible representations of every of the two possible non-isomorphic types do participate in the direct sum decomposition of the representation defined by system S [31]. It turns out that these parameters completely determine existence of elliptic BVPs.

Statement 1 ([22]). *Suppose that n is odd and $n \geq 3$. If l is also odd, then there exist no elliptic RHPs for the given GCRS. If l is even, then elliptic RHPs exist if and only if $l_1 = l_2$. In the latter case, the set of elliptic boundary conditons is open and dense in the space of all local boundary conditions of the form* (3)*.*

Statement 2 ([31], [22]). *If $n \geq 2$ is even, $n \neq 4, 6$, then there always exist elliptic RHPs for GCRS as above and the set of elliptic boundary conditions is open and dense in the space of all local boundary conditions of the form* (3)*.*

We do not make here any comments on the proofs of these two results because they involve several non-trivial techniques from K-theory and differential equations (see [22]). We only mention that the basic technical tool is provided by K-homology theory developed in [4]. Both these theorems follow from the homological criterion for the existence of local boundary value problems for first order systems (which was first formulated P.Baum and R.Douglas as a conjecture and then proved by G.Gong in [18]). We had also to use a description of K-homological classes of Euclidean Dirac operators obtained in [4].

Remark 3. The restriction that $n \neq 4, 6$ results from the method of proof used in [22]. It comes from the paper [18] and it is related to some delicate questions of K-theory. At the moment it still remains unclear for us if this restriction is essential indeed.

Taking into account these results we can now prove Theorem 1 and in course of proving it we will also see the way of generalizing it to arbitrary GCRSs on odd-dimensional spaces, which is the second main result of this paper.

Proof of Theorem 1. Let us first determine the derivative (differential) of L_p at some point (f_0, g_0) and show that it may be interpreted as a boundary value problem (1),(3) for system $D(S)$, i.e., that it is an RHP for the GCRS $D(S)$.

Using the usual description of the tangent space to a manifold of mappings in terms of vector fields along a given mapping, it is easy to see that $T_{(f_0,g_0)}H$ may be identified with the space

$$Z = \{f : B \to W : f \in H^{r+1}(B, W), f(x) \in T_{f0(x)}M, \forall x \in bB, f(q) = p\}.$$

Granted that, it becomes clear that the derivative of L_p at (f_0, g_0) may be identified with the map $\delta : Z \to G$ given by $\delta(f) = Df$.

Let N_M denote the (geometric) normal bundle of M. This is a real vector bundle with fibre dimension $k = 2m(n)$. Consider its pull-back $E_0 = (f_0 \mid bB)^*(N_M)$. From the homotopy condition in the definition of F it follows that E_0 is a trivial bundle over bB, generated by k global sections, say, $p_1, \dots, p_k$. Using P_j as rows we may form the $(k \times 2k)$-matrix-function $p \in H^{r+1}(bB)$. By the very construction of P, $T_{f0(x)}M = \{w \in W : P(x)w = 0\}$ and one immediately observes that matrix P has exactly the same form as the matrix of boundary condition (3) for system $D(S)$.

Let us set $X = H^{r+1}(B, W), Y = H^r(B, W) \times H^{r+1}(bB, \mathbf{C}^{m(n)})$ and define a map $R : X \to Y$ by $R(f) = (D(f), (Pf)|bB)$. It is obvious that R is exactly the operator of a Riemann–Hilbert problem (1),(3) for system $D(S)$.

Our next goal is to understand under which conditions one may guarantee that R is a Fredholm operator. Notice that if the corresponding RHP is elliptic (i.e., satisfies the Shapiro-Lopatinski condition [36]), then R is a Fredholm operator in virtue of the general theory of elliptic linear boundary value problems [36]. So we should only arrange that matrix P defines an elliptic boundary condition for $D(S)$.

Statement 2 shows that in our situation (this is the crucial place where it is important that $D(S)$ is a "double") there is a plenty of elliptic boundary conditions. In particular, there exist constant matrices P_0 which define elliptic RHPs (1),(3). Let us embed $\mathbf{R}^{2m(n)}$ in W in such a way that the normal space of the image M is orthogonal to the subspace spanned by rows of such a P_0. For such target M, R is obviously Fredholm, so L_p is also Fredholm at any $p \in M$. Taking into account the stability of Fredholm property under small perturbations, we see that all sufficiently small perturbations of M will be admissible targets. The homotopy invariance ofthe Fredholm index implies that the index vanishes, which completes the proof. □

Remark 4. We used the fact that systems of the form $D(S)$ possess elliptic boundary conditions (3) defined by constant matrix-functions B_1, B_2. This fact is not self-evident but it follows from the results of [21]. The "raison d'être" of this result is the fact (see [21]) that RHPs for systems of the form $D(S)$ are equivalent to so-called transmission problems (also called linear conjugation problems) for system S [21]. Existence of constant elliptic coefficients for transmission problems was established in [21]. For $n = 1$ this is just a trivial consequence of the classical theory of linear conjugation problems for holomorphic functions [24] (actually every non-degenerate constant matrix generates an elliptic conjugation problem because its partial indices obviously vanish). For irreducible systems with $n = 2$ (Moisil-Theodoresco system) and $n = 3$ (Fueter system), existence of constant elliptic transmission conditions follows from the criteria of fredholmness for such problems obtained in [28] (cf. [32]).

Remark 5. The situation with compact admissible targets remains unclear. It is well known that there might be topological obstructions to existence of such targets, which happens already for the classical Cauchy–Riemann system [13]. In order to clarify this issue it is necessary to achieve better understanding of geometric conditions on admissible targets, which can be hopefully done in terms of transversality to certain subspace of the Grassmanian $G(2m(n), 4m(n))$. This may be done in some simple cases, for example, the necessary "algebraic analysis" of the Moisil-Theodoresco system is presented in [27]. In the general case this seems to be quite difficult and it is even unclear what is the dimension of the subset of "prohibited" $2m(n)$-subspaces. This point of view is related to some other approachs to the construction of admissible targets which will be mentioned in the last section.

Analyzing the proof of Theorem 1 and taking into account the previous remark, one sees that it may be extended to certain irreducible systems without taking their doubles, which leads to Theorem 2.

For "exceptional" values $n = 4$ and $n = 6$ the situation remains unclear, but we feel the statement of Theorem 2 should remain valid. This suggests that one should try to invent an explicit construction of constant elliptic boundary conditions (3) for irreducible systems in $\mathbf{R}^5$ and $\mathbf{R}^7$.

We would like to point our that despite certain analogies with analytic discs, the situation with hh cells is much more subtle. In particular, the restriction to systems of the $D(S)$ type cannot be just ommited.

For example, the most straightforward generalization of analytic discs attached to totally real surfaces [13] would be to consider the Fueter system in $\mathbf{R}^4 = \mathbf{H}$ (quaternionic regular functions [9]) and try to construct Fueter cells attached to 4-dimensional submanifolds in $\mathbf{R}^8$. However, it turns out that in this way one cannot obtain a reasonable Fredholm theory for such cells, since in this situation Gromov's operator is never Fredholm. The latter fact follows directly from Statement 1 because the resulting system has $2 = l_1 \neq l_2 = 0$.

We conclude the paper by discussing some geometric problems suggested by our approach.

5. Special Grassmanians

A natural problem raised by our results is to understand how can one characterize admissible targets geometrically. Gromov's general approach to solving of under-determinate systems [20], suggests that this issue should be related to certain special subsets of appropriate Grassmanians. Indeed, some first steps in this direction can be done in a quite natural way and we proceed by a brief discussion of these matters.

Actually, a more comprehensive investigation of these connections shows that they may be conveniently described in terms of so-called Grassmanian geometries and calibrations, in the sense of [16]. We do not describe relations to calibrated geometries in detail, but some of those ideas are implicitly present in the discussion below.

For a given GCRS, it is also interesting to investigate what can be the minimal possible dimensions k of target manifolds for which one can derive Fredholmness of Gromov's operators. Gromov's h-principle suggests that admissible targets should satisfy some transversality condition with respect to certain special subset of Grassmanian $Gr(k, 2m)$ defined by the characteristic matrix of the system in question.

In order to make this idea more precise let us first re-examine the classical case of analytic discs. Results of Gromov [20] and Alexander [2] translated to our language mean that admissible targets for analytic discs are precisely totally real submanifolds of $\mathbf{C}^k$. For $k = 2$, the condition of total reality means that the image of Gauss mapping Γ_M of a submanifold M does not intersect the subset of complex lines in $Gr_{\mathbf{R}}(2,4)$. Since target M is in this case two-dimensional, this suggests to consider generic targets M, such that Γ_M is transversal to the two-dimensional subset of complex lines $Gr_{\mathbf{C}}(1,2)$ in four-dimensional real grassmanian $Gr_{\mathbf{R}}(2,4)$.

For such generic targets, their tangent planes can coincide with complex lines only at isolated points and one may wish to eliminate these points by deforming M. For compact M, it is well known [7] that the only obstruction for elimination of points with complex tangencies is given by the Euler characteristic $\chi(M)$. It may be also shown that, for non-compact contractible M homeomorphic to $\mathbf{R}^2$, the set of embeddings into $\mathbf{R}^4$ without complex tangencies is open and dense in the set of all such embeddings. The latter statement is exactly the special case of Theorem 1 for the classical Cauchy–Riemann system in $\mathbf{R}^2$.

Thus it becomes clear that admissible targets may admit characterization by some genericity conditions like transversality and in order to find such conditions one should try to describe the subset of n-planes in $\mathbf{C}^k$ which can be represented as images of differentials of solutions to system S. Notice first that this is exactly what happens in the classical case, since for the usual Cauchy–Riemann systems these images are the complex lines.

Indeed, it is immediate to see that the most general form of a Jacobian matrix of a CR-solution (analytic disc) with values in $\mathbf{C}^2$ is:

$$\begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix}$$

where a, b, c, d are arbitrary real numbers. It is also clear that a vector expressed by the second column of this matrix is equal to $\mathbf{i}$ times vector expressed by the first column. So the image of the corresponding operator is a complex line and it is clear that every compex line may be obtained in this way. Of course the same holds for arbitrary value of the complex dimension k: the set of tangent planes to analytic discs coincides with the subset of complex lines in $Gr_{\mathbf{R}}(2,2k)$ and has codimension $2k-2$.

Admissible targets in this case coincide with totally real ($2k$-dimensional) submanifolds. Notice that they are not generic $2k$-dimensional manifolds because those may have complex tangencies and actually homological properties of the set of complex tangencies are closely related to the topology of the submanifold considered [13].

Similar considerations for the Fueter system show that tangent planes of its solutions are exactly invariant modules of the left action of $\mathbf{H}$ on $\mathbf{H}^k$. It is also instructive to have a look at the first irreducible system with non-equal (real) dimensions of the source and the target (i.e., $n+1 \neq 2m$ in our notation). This is of course the Moisil-Theodoresco system ($n=2$, $m=2$). A simple calculation shows that tangents to its solutions are precisely the 3-dimensional subspaces in $\mathbf{R}^4$ generated by columns of matrices of the form

$$\begin{pmatrix} a & b & e-k \\ -l-r & e & f \\ k & l & a+q \\ b+f & q & r \end{pmatrix}$$

Now it is quite simple to verify that every 3-dimensional subspace in $\mathbf{R}^4$ may be obtained as the tangent space of a MTS-solution. Obtaining a precise description of MTS-tangents in $\mathbf{R}^{4k}$ for $k \geq 2$ is a more delicate task. A further analysis of this problem shows that explicit description of this subset is closely related to certain homological problems. Similar problems are considered in see [27] where Gröbner bases and first syzygies for the Moisil-Theodoresco system are computed using computer programme CoCoA (cf. [1] where the same problems are studied in the case of Fueter system).

As was already noticed, for $n = 2, 3$ one can indicate explicit geometric conditions on T_pM for a target manifold M to be admissible. This follows from explicit criteria of Fredholmness for RHPs for Moisil-Theodoresco and Fueter systems obtained in [28]. It would be interesting to obtain similar results for general GCRSs.

We would like to mention also the general problem of computing the index of an elliptic RHP for GCRS. In principle this is possible using general results of Atiyah and Bott, which should lead to explicit formulae of Dynin-Fedosov type [11], [12], but it does not seem that somebody have ever written down those explicit formulae in terms of the characteristic matrix and boundary condition. Thus it would be illuminating to obtain an exact recipe, or even an algorithm applicable in concrete situations. In low dimensions, some useful preparatory work for developing such an algorithm was done in [28].

We conclude by mentioning another promising perspective which emerges from the aforementioned connection between special grassmanians and calibrated geometries. It is concerned with finding a proper calibration for a given GCRS. For the classical Cauchy–Riemann system this may be worked out in full detail and it turns out that the desired calibration is provided by the properly normalized Kähler form on the target space [16]. In fact, many properties of families of analytic discs may be derived directly from this interpretation, so one may hope that finding a proper calibration will be also useful in other cases.

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