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ON VECTOR SUMS OF MEASURE ZERO SETS

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Abstract. We consider the behaviour of measure zero subsets of a vector space under the operation of vector sum. The question whether the vector sum of such sets can be nonmeasurable is discussed in connection with the measure extension problem, and a certain generalization of the classical Sierpiński result [3] is presented.

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It is a well-known fact that some nice descriptive properties of subsets of a topological vector space are not preserved under the operation of vector sum. The following two examples are typical in this respect.

Example 1. There exist two Borel subsets X and Y of the real line R, such that $X+Y$ is not Borel (obviously, $X+Y$ is an analytic subset of R).

Example 2. There exist two sets $X \subset R$ and $Y \subset R$, both of Lebesgue measure zero, such that $X+Y$ is not Lebesgue measurable.

In connection with Example 1, see, e.g., [1] or [2], Example 2 is due to Sierpiński (see his early paper [3]). He gave this example starting with a simple observation that there are two Lebesgue measure zero sets $A \subset R$ and $B \subset R$ for which $A+B=R$, and utilizing some properties of Hamel bases in R.

This paper is devoted primarily to some generalizations of Example 2 for nonzero σ-finite quasi-invariant measures in vector spaces. In particular, we shall demonstrate that, for certain extensions of quasi-invariant measures, the phenomenon described in Example 2 can always be realized.

As a rule, the measures considered below are assumed to be defined on some σ-algebras of subsets of a given uncountable vector space E (over the field Q of all rationals) and are supposed to be quasi-invariant under the group of all nondegenerate rational homotheties of this space. More precisely, we shall say that a mapping $h: E \rightarrow E$ is a nondegenerate rational homothety of E if h can be represented as

$$h(x)=qx+x_0 \ (x \in E),$$

where q is a fixed nonzero rational number and x_0 is a fixed element of E. The family of all above-mentioned homotheties forms a group with respect to the

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usual composition operation. We denote this group by the symbol H_E (notice that H_E is not commutative).

If μ is a measure on E, then we put:

$\mathrm{dom}(\mu)$ = the domain of μ;

$I(\mu)$ = the σ-ideal of all μ-measure zero subsets of E.

We shall say that μ is H_E-quasi-invariant if both classes of sets $\mathrm{dom}(\mu)$ and $I(\mu)$ are invariant under all transformations from the group H_E.

Let μ be a nonzero σ-finite H_E-quasi-invariant measure on E. Motivated by Example 2, we may pose the following question: do there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ for which $X + Y \notin \mathrm{dom}(\mu)$? It can easily be shown that, in general, the answer to this question is negative. Moreover, various examples of a situation where

$$(\forall X \in I(\mu))(\forall Y \in I(\mu))(X + Y \in I(\mu))$$

can be constructed without any difficulties (see, e.g., [4]). Also, as demonstrated in the same monograph [4], the question posed above can be reduced to another much easier problem. More precisely, we have the following statement.

Theorem 1. *Let E be an uncountable vector space (over Q). Then, for any nonzero σ-finite H_E-quasi-invariant measure μ on E, the following eight assertions are equivalent:*

1) *there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ such that $X + Y \notin I(\mu)$;*

2) *there exists a set $X \in I(\mu)$ such that $X + X \notin I(\mu)$;*

3) *there exists a set $X \in I(\mu)$ such that $\mathrm{lin}_Q(X) \notin I(\mu)$ where $\mathrm{lin}_Q(X)$ stands for the linear hull (over Q) of X;*

4) *there exists a linearly independent (over Q) set $X \in I(\mu)$ such that $\mathrm{lin}_Q(X) \notin I(\mu)$;*

5) *there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ such that $X + Y \notin \mathrm{dom}(\mu)$;*

6) *there exists a set $X \in I(\mu)$ such that $X + X \notin \mathrm{dom}(\mu)$;*

7) *there exists a set $X \in I(\mu)$ such that $\mathrm{lin}_Q(X) \notin \mathrm{dom}(\mu)$;*

8) *there exists a linearly independent (over Q) set $X \in I(\mu)$ such that $\mathrm{lin}_Q(X) \notin \mathrm{dom}(\mu)$.*

The proof is presented in [4]. Notice that the argument is essentially based on some properties of the so-called Ulam transfinite matrix (see, e.g., [6]).

The equivalence of assertions 1)–8) shows us that, in order to obtain a positive answer to the question formulated above, we need only the existence of two sets $X \in I(\mu)$ and $Y \in I(\mu)$ for which $X + Y \notin I(\mu)$. Clearly, the question will be solved positively if the existence of two sets $X \in I(\mu)$ and $Y \in I(\mu)$ is established, for which $X + Y = E$.

Our goal is to demonstrate that there always exists an H_E-quasi-invariant extension μ' of μ such that the vector sum of some two μ'-measure zero subsets of E is identical with the whole space E. For this purpose, several auxiliary notions and propositions are necessary.

Let E be an arbitrary set, G be a group of transformations of E and let Y be a subset of E. We say (cf. [5]) that Y is G-absolutely negligible (in E) if, for any σ-finite G-quasi-invariant measure μ on E, there exists a G-quasi-invariant measure μ' on E extending μ and such that $\mu'(Y) = 0$. The concept of an absolutely negligible set is thoroughly discussed in monograph [5] where a significant role of this concept is emphasized for various questions concerning extensions of quasi-invariant (invariant) measures.

Below, the symbol ω denotes the first infinite ordinal (cardinal) and ω_1 stands for the first uncountable ordinal (cardinal).

Let E be a set and let $\{X_i : i \in I\}$ be a partition of E.

A set $X \subset E$ is called a partial selector of $\{X_i : i \in I\}$ if $\mathrm{card}(X \cap X_i) \leq 1$ for all indices $i \in I$. Accordingly, a set $X \subset E$ is called a selector of the same partition if $\mathrm{card}(X \cap X_i) = 1$ for all $i \in I$.

Our starting point is the following lemma (cf. [5]).

Lemma 1. *Let E be a set of cardinality ω_1, let G be a group of transformations of E, such that $\mathrm{card}(G) = \omega_1$ and*

$$\mathrm{card}(\{x \in E : g(x) = h(x)\}) \leq \omega$$

for any two distinct transformations $g \in G$ and $h \in G$. Further, let $\{G_\xi : \xi < \omega_1\}$ be an increasing (with respect to inclusion) ω_1-sequence of subgroups of G and let $\{X_\xi : \xi < \omega_1\}$ be a partition of E, such that:

1) $\mathrm{card}(G_\xi) \leq \omega$ *for all ordinals $\xi < \omega_1$;*
2) $\mathrm{card}(X_\xi) \leq \omega$ *for all ordinals $\xi < \omega_1$;*
3) $\cup\{G_\xi : \xi < \omega_1\} = G$;
4) *for each $\xi < \omega_1$, the set X_ξ is G_ξ-invariant.*

Then every partial selector of $\{X_\xi : \xi < \omega_1\}$ is a G-absolutely negligible subset of E.

A detailed proof of this proposition can be found in [5].

Let $(G, +)$ be a commutative group. The group of all translations of G is obviously isomorphic to G, and we can identify these two groups in our further considerations.

Lemma 2. *Let $(G, +)$ be a commutative group of cardinality ω_1 and let X be an arbitrary uncountable subset of G. Then there exists a G-absolutely negligible set $Y \subset G$ such that $X + Y = G$.*

Proof. Let $\{x_\xi : \xi < \omega_1\}$ be an injective family of all elements of G. Put $E = G$ and equip E with the group of all translations of G. Let $\{G_\xi : \xi < \omega_1\}$ and $\{X_\xi : \xi < \omega_1\}$ be two families satisfying the conditions of Lemma 1. We now define an injective ω_1-sequence $\{y_\xi : \xi < \omega_1\}$ of elements of E. Suppose that, for an ordinal $\xi < \omega_1$, the partial ξ-sequence $\{y_\zeta : \zeta < \xi\}$ has already been defined. For each ordinal $\zeta < \xi$, let $X_{\eta(\zeta)}$ be such that $y_\zeta \in X_{\eta(\zeta)}$. We denote

$$Z_\xi = \cup\{X_{\eta(\zeta)} : \zeta < \xi\}$$

and observe that $\mathrm{card}(Z_\xi) \le \omega$. Since the given set X is uncountable, we must have

$$(E \setminus Z_\xi) \cap (x_\xi - X) \ne \varnothing.$$

Choose any element of $(E \setminus Z_\xi) \cap (x_\xi - X)$ and denote it by y_ξ. Continuing in this manner, we will be able to construct the desired ω_1-sequence $\{y_\xi : \xi < \omega_1\}$ of elements of E. Now, putting

$$Y = \{y_\xi : \xi < \omega_1\}$$

and taking into account the relation

$$(\forall \xi < \omega_1)(x_\xi \in y_\xi + X),$$

we see that

$$X + Y = E = G.$$

Also, in accordance with Lemma 1, the set Y is G-absolutely negligible in E. This completes the proof of Lemma 2. □

Remark 1. It is not hard to verify that Lemma 2 remains true for an arbitrary group G of cardinality ω_1. In addition to this, suppose that E is a vector space over Q with $\mathrm{card}(E) = \omega_1$ and let $G = H_E$. Then $\mathrm{card}(G) = \omega_1$, too, and

$$\mathrm{card}(\{x \in E : g(x) = h(x)\}) \le 1$$

for any two distinct transformations $g \in G$ and $h \in G$. The argument used in the proof of Lemma 2 shows us that, for every uncountable set $X \subset E$, there exists an H_E-absolutely negligible set $Y \subset E$ for which we have $X + Y = E$.

Theorem 2. *Let* $(G, \cdot)$ *be a group of cardinality* ω_1 (*identified with the group of all its left translations*). *Then there exist two* G*-absolutely negligible sets* $X \subset G$ *and* $Y \subset G$ *such that* $X \cdot Y = G$. *In particular, for any nonzero* σ*-finite left* G*-quasi-invariant* (*respectively, left* G*-invariant*) *measure* μ *on* G, *there exists a left* G*-quasi-invariant* (*respectively, left* G*-invariant*) *measure* μ' *on* G *extending* μ *and satisfying the relations*

$$X \in I(\mu'), \ Y \in I(\mu'), \ X \cdot Y = G \notin I(\mu').$$

Proof. Take any uncountable G-absolutely negligible set $X \subset G$ (the existence of such a set easily follows from Lemma 1). In virtue of Lemma 2 (cf. Remark 1 above), there exists a G-absolutely negligible set $Y \subset G$ satisfying the equality $X \cdot Y = G$. □

Remark 2. It would be interesting to generalize Theorem 2 to those groups whose cardinalities are greater than ω_1. In this connection, it can be shown that if G is an uncountable group and $\mathrm{card}(G)$ is a regular cardinal, then, for each set $X \subset G$ with $\mathrm{card}(X) = \mathrm{card}(G)$, there exists a G-absolutely negligible set $Y \subset G$ such that $X \cdot Y = G$ (the argument is very similar to the proof of Lemma 2).

Lemma 3. *Let E be a vector space (over Q) and let*

$$E = E_1 + E_2 \quad (E_1 \cap E_2 = \{0\})$$

be a representation of E in the form of the direct sum of two vector subspaces E_1 and E_2 (over Q again). Suppose also that a set $Y \subset E_1$ is H_{E_1}-absolutely negligible in E_1. Then the set $Y + E_2$ turns out to be H_E-absolutely negligible in E.

The proof of this statement is presented in monograph [5].

Lemma 4. *Let E be a vector space (over Q). Then, for each uncountable set $X \subset E$, there exists an H_E-absolutely negligible set $Y \subset E$ such that $X+Y = E$.*

Proof. We may assume, without loss of generality, that $\mathrm{card}(X) = \omega_1$. Denote by E_1 the vector subspace of E (over Q again) generated by X. Evidently, we have $\mathrm{card}(E_1) = \omega_1$. Let us represent our E in the form of the direct sum of two vector subspaces:

$$E = E_1 + E_2 \quad (E_1 \cap E_2 = \{0\}).$$

Applying Lemma 2 (see also Remark 1), we can find an H_{E_1}-absolutely negligible set $Y_1 \subset E_1$ such that $X + Y_1 = E_1$. Let us put

$$Y = Y_1 + E_2.$$

Then, in view of Lemma 3, the set Y is H_E-absolutely negligible in E. Furthermore, we may write

$$X + Y = X + Y_1 + E_2 = E_1 + E_2 = E,$$

and the lemma is proved. □

From Lemma 4 we easily obtain the following statement.

Theorem 3. *Let E be a vector space (over Q) and let μ be a nonzero σ-finite H_E-quasi-invariant measure on E. Then, for each uncountable set $X \in I(\mu)$, there exist an H_E-quasi-invariant measure μ' on E extending μ and a set $Y \in I(\mu')$, for which we have $X + Y = E \notin I(\mu')$.*

Proof. Let Y be an H_E-absolutely negligible set in E such that $X + Y = E$ (the existence of Y was established in Lemma 4). The absolute negligibility of Y implies that there exists an H_E-quasi-invariant extension μ' of μ for which $\mu'(Y) = 0$. Thus, we see that the measure μ' and the set Y are the required ones. □

Finally, taking into account Theorem 1, we conclude that the following result is valid.

Theorem 4. *Let E be an uncountable vector space (over Q) and let μ be a nonzero σ-finite H_E-quasi-invariant measure on E. Then there exists an H_E-quasi-invariant measure μ' on E extending μ such that, for some sets $X \in I(\mu')$ and $Y \in I(\mu')$, we have $X + Y \notin \mathrm{dom}(\mu')$.*

The latter theorem can be regarded as a generalized version of Example 2 for nonzero σ-finite quasi-invariant measures in vector spaces. It would be interesting to extend this theorem to a wider class of uncountable groups equipped with nonzero σ-finite left quasi-invariant (in particular, left invariant) measures.

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