Georgian Mathematical Journal
Volume 8 (2001), Number 3, 455–468

MD-NUMBERS AND ASYMPTOTIC MD-NUMBERS OF OPERATORS

A. CASTEJÓN, E. CORBACHO, AND V. TARIELADZE

Abstract. First, the basic properties of mean dilatation (MD-) numbers for linear operators acting from a finite-dimensional Hilbert space are investigated. Among other results, in terms of first and second order MD-numbers, a characterization of isometries is obtained and a dimension-free estimation of the p-th order MD-number by means of the first order MD-number is established. After that asymptotic MD-numbers for a continuous linear operator acting from an infinite-dimensional Hilbert space are introduced and it is shown that in the case of an infinite-dimensional domain the asymptotic p-th order MD-number, rather unexpectedly, is simply the p-th power of the asymptotic first order MD-number (Theorem 3.1).

2000 Mathematics Subject Classification: Primary: 46C05; Secondary: 47A30, 47B06.
Key words and phrases: Grammian, pMD-number, MD-number, asymptotic upper MD-number, asymptotic lower MD-number.

1. Introduction

For a linear operator T from a Hilbert space H with $1 \le n = \dim H < \infty$ to an arbitrary Hilbert space Y and a natural number $p \le n$, the p-th order mean dilatation number (briefly, the pMD-number) $\delta_p(T)$ is defined by the equality

$$\delta_p(T) = \left(\frac{1}{c_{n,p}} \int\limits_S \cdots \int\limits_S G(T_1, \dots, Tx_p) ds(x_1) \dots ds(x_p) \right)^{1/2}, \tag{1.1}$$

where S is the unit sphere in H, s is the normalized isometrically invariant measure on it, $G(x_1, \dots, x_p)$ is the Grammian of $(x_1, \dots, x_p)$, and

$$c_{n,p} = \int\limits_S \cdots \int\limits_S G(x_1, \dots, x_p) ds(x_1) \dots ds(x_p) > 0$$

is a (natural) normalizing constant.

The definition of pMD-numbers was suggested in [6]. Afterwards they were considered also in [5].

ISSN 1072-947X / \$8.00 / ©

For simplicity, the first order MD-number $\delta_1(T)$ will be called the MD-number of T and, for it, (1.1) gives an expression

$$\delta_1(T) = \left(\int\limits_S \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}. \tag{1.2}$$

Using the equality (1.2), the MD-number $\delta_1(T)$ can also be defined when Y is an arbitrary Banach space and in this case it has already been used as a tool of the local theory of Banach spaces (see, e.g., [7, p.81]).

In Section 2, after recalling some known facts concerning the Grammian (Remark 2.1) which motivate the definition of pMD-numbers, their properties in the finite-dimensional case are analyzed. In particular, a concrete expression for $\delta_p(T)$ in terms of the eigenvalues of the operator T^*T is found (Proposition 2.6) and the following characterization of isometric operators is obtained: T is an isometry if and only if $\delta_1(T) = \delta_2(T) = 1$ (Proposition 2.8). Although the equality $\delta_p(T) = \delta_1^p(T)$ does not hold in general, the validity of the following "dimension-free" estimate

$$\delta_p(T) \le p^{\frac{p-1}{2}} \delta_1^p(T)$$

is proved (Lemma 2.10).

Possible infinite-dimensional extensions of pMD-numbers are introduced in Section 3. According to [6], we use the net $\mathcal{M}$ of all finite-dimensional subspaces of H to associate two quantities

$$\overline{\overline{\delta}}_p(T) := \limsup_{M \in \mathcal{M}} \delta_p(T|_M), \quad \underline{\underline{\delta}}_p(T) := \liminf_{M \in \mathcal{M}} \delta_p(T|_M)$$

to a given operator T (acting from H) and a natural number p. We call $\overline{\overline{\delta}}_p(T)$ the asymptotic upper pMD-number of T and $\underline{\underline{\delta}}_p(T)$ the asymptotic lower pMD-number of T. When they are equal, the operator is called asymptotically pMD-regular, their common value is denoted by $\overline{\delta}_p(T)$ and called the asymptotic pMD-number. Therefore for an asymptotic pMD-regular operator T we have

$$\overline{\delta}_p(T) := \lim_{M \in \mathcal{M}} \delta_p(T|_M).$$

When $p = 1$, we shall simply use the term "asymptotically MD-regular".

The main result of the section is Theorem 3.1, which asserts that, in the case of an infinite-dimensional domain, we have the following simple relation between the higher order and the first order asymptotic MD-numbers of a given operator T:

$$\overline{\overline{\delta}}_p(T) = (\overline{\overline{\delta}}_1(T))^p, \quad \underline{\underline{\delta}}_p(T) = (\underline{\underline{\delta}}_1(T))^p.$$

This result shows that in the case of the infinite-dimensional domain it is sufficient to study only the numbers $\overline{\overline{\delta}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$. Section 3 is concluded by Proposition 3.2, which implies, in particular, that not any operator is asymptotically MD-regular.

Notice, finally, that the numbers $\overline{\overline{\overline{\delta}}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$ can also be defined when Y is a general Banach space. The properties of the corresponding quantities in this general setting are investigated in [2].

2. Finite Dimensional Case

Throughout this paper the considered Hilbert spaces is supposed to be defined over the same field $\mathbb{K}$, real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. To simplify the notation, for the scalar product and the norm of different Hilbert spaces we use the same symbols $(\cdot|\cdot)$ and $\|\cdot\|$.

For the Hilbert spaces H, Y we denote by $L(H,Y)$ the set of all continuous linear operators $T : H \to Y$.

For a given operator $T \in L(H,Y)$ we denote by

– $\|T\|$ the ordinary norm of T;
– T^* the (Hilbert) adjoint operator of T;
– $|T|$ the unique self-adjoint positive square root of $T^*T : H \to H$.

Moreover, for a Hilbert–Schmidt operator T we denote by $\|T\|_{\mathrm{HS}}$ the *Hilbert–Schmidt norm* of T.

Fix a (non-zero) Hilbert space H and a natural number p such that $p \leq \dim H$. The p-th order *Grammian* G_p is the scalar-valued function, defined on H^p, which assigns the determinant of the matrix $(x_i|x_j)_{i,j=1}^p$ to any $(x_1,\dots,x_p) \in H^p$.

Evidently, $G_1(x) = \|x\|^2$ for each $x \in H$. Some other known properties of the Grammian are given in the next remark.

Remark 2.1. Fix a natural number $p > 1$ with $p \leq \dim H$ and a finite sequence $(x_1,\dots,x_p) \in H^p$.

(a) We have $G_p(x_1,\dots,x_p) \geq 0$. Moreover, $G_p(x_1,\dots,x_p) > 0$ if and only if $(x_1,\dots,x_p)$ is linearly independent.

(b) $G_p(x_1,\dots,x_p) \leq \|x_1\|^2 \dots \|x_p\|^2$. Moreover, if $x_1,\dots,x_p$ are nonzero elements, then we have $G_p(x_1,\dots,x_p) = \|x_1\|^2 \dots \|x_p\|^2$ iff $x_1,\dots,x_p$ are pairwise orthogonal.

(c) Let $P_{x_1,\dots,x_p} := \left\{ \sum\limits_{i=1}^{p} \alpha_i x_i : 0 \leq \alpha_i \leq 1,\, i = 1,\dots,p \right\}$ be the parallelepiped generated by the sequence $(x_1,\dots,x_p)$. Then $G(x_1,\dots,x_p) = \mathrm{vol}^2(P_{x_1,\dots,x_p})$, where $\mathrm{vol}(P_{x_1,\dots,x_p})$ stands for the (p-dimensional) volume of $P_{x_1,\dots,x_p}$ (i.e., the Lebesgue measure of $P_{x_1,\dots,x_p}$ in a p-dimensional vector subspace containing $(x_1,\dots,x_p)$. For this reason the quantity $g_p(x_1,\dots,x_p) := G_p^{1/2}(x_1,\dots,x_p)$ is often simply called "the hypervolume determined by the vectors $(x_1,\dots,x_p)$."

(d) Let $[x_1,\dots,x_{p-1}]$ be the linear span of $\{x_1,\dots,x_{p-1}\}$. Then

$$g_p(x_1,\dots,x_p) = g_{p-1}(x_1,\dots,x_{p-1}) \cdot \mathrm{dist}(x_p, [x_1,\dots,x_{p-1}]).$$

(e) If Y is another Hilbert space and $T : H \to Y$ is a continuous linear operator, then $g_p(Tx_1,\dots,Tx_p) \leq \|T\|^p g_p(x_1,\dots,x_p)$ (this is evident when $p = 1$, the rest follows from this and from (d) by induction, as $\mathrm{dist}(Tx_p, [Tx_1,\dots,Tx_{p-1}]) \leq \|T\| \,\mathrm{dist}(x_p, [x_1,\dots,x_{p-1}])$.

(f) In the notation of the previous item suppose that $\dim H = p$. Then $g_p(Tx_1, \ldots, Tx_p) = \det(|T|)g_p(x_1, \ldots, x_p)$. It follows that if $\dim H = p$ and $(x'_1, \ldots, x'_p)$, $(x_1, \ldots, x_p)$ are algebraic bases of H, then

$$\frac{g_p(Tx'_1, \ldots, Tx'_p)}{g_p(x'_1, \ldots, x'_p)} = \frac{g_p(Tx_1, \ldots, Tx_p)}{g_p(x_1, \ldots, x_p)},$$

i.e., the "volume ratio" is independent of a particular choice of basis of H and equals to $\det(|T|)$.

For a Hilbert space with $\dim H \geq 1$, its unit sphere with center at the origin is denoted by S_H or, simply, by S. When $n = \dim H < \infty$, the *uniform distribution* on S, i.e. the unique isometrically invariant probability measure given on the Borel σ-algebra of S, is denoted by s. The following key equality follows directly from the isometric invariance of s:

$$\int_S \overline{(x|h_1)}(x|h_2)ds(x) = \frac{1}{n}(h_1|h_2), \quad \forall h_1, h_2 \in H, \tag{2.1}$$

where 'bar' stands for complex conjugation.

In the rest of this section H will be a finite-dimensional Hilbert space with $\dim H = n \geq 1$, I will stand for the identity operator acting in H and Y will be another Hilbert space (*not necessarily finite-dimensional*).

Fix a number $p \in \{1, \ldots, n\}$ and introduce the functional $D_p : L(H, Y) \to \mathbb{R}_+$ defined by the equality

$$D_p(T) = \int_S \cdots \int_S G_p(Tx_1, \ldots, Tx_p)ds(x_1) \ldots ds(x_p), \quad T \in L(H, Y). \tag{2.2}$$

Therefore, for given T the number $D_p(T)$ can be viewed as the average of the squares of the volumes of the family of parallelepipeds

$$\left\{T(P_{x_1,\ldots,x_p}) : (x_1, \ldots, x_p) \in S \times \cdots \times S\right\}$$

with respect to the product $s \times \cdots \times s$ of the uniform distributions.[1]

Evidently, for $p = 1$ we get

$$D_1(T) = \int_S \|Tx\|^2 ds(x), \quad \forall T \in L(H, Y). \tag{2.3}$$

To make easier further references, we formalize some other easy observations in the next statement.

[1] The choice of s seems to be natural, since it gives no preference to any of the directions $(x_1, \ldots, x_p) \in S \times \cdots \times S$. Formally, the same definition can be given, taking any probability measure μ on S instead of s, but then the properties of $D_p(T)$ will depend on (the correlation operator of) μ.

Proposition 2.2. *Let $T \in L(H, Y)$.*

$$D_p(T) = D_p(|T|); \tag{2.4}$$

$$D_p(T) = 0 \iff \dim T(H) < p; \tag{2.5}$$

$$D_p(T) \le D_1^p(T); \tag{2.6}$$

$$D_2(T) = D_1^2(T) - \frac{1}{n} D_1(T^*T), \quad \textit{in particular,} \quad D_2(I) = 1 - \frac{1}{n}; \tag{2.7}$$

$$D_1(T^*T - I) = D_1(T^*T) - 2D_1(T) + 1. \tag{2.8}$$

Proof. (2.4) is true since $G_p(|T|x_1, \dots, |T|x_p) = G_p(Tx_1, \dots, Tx_p)$ for any $(x_1, \dots, x_p) \in H^p$.

(2.5) follows from Remark 2.1(a).

(2.6) follows from Remark 2.1(b) and (2.3).

(2.7): since $\forall x_1, x_2 \in H$

$$G_p(Tx_1, Tx_2) = \|Tx_1\|^2 \|Tx_2\|^2 - |(Tx_1|Tx_2)|^2 = \|Tx_1\|^2 \|Tx_2\|^2 - |(T^*Tx_1|x_2)|^2,$$

and since by (2.1)

$$\int\limits_S |(T^*Tx_1|x_2)|^2 ds(x_2) = \frac{1}{n} \|T^*Tx_1\|^2, \quad \forall x_1 \in H,$$

from (2.3) (using, of course, Fubini's theorem) we get (2.7).

(2.8) follows from (2.3) and from the evident equality $\|(T^*T - I)x\|^2 = \|T^*Tx\|^2 - 2\|Tx\|^2 + \|x\|^2$, $x \in H$. □

Remark 2.3. Fix $T \in L(H, Y)$.

(1) By means of the direct integration of the Grammian it is possible to get a "coordinate free" expression for $D_p(T)$ in terms of D_1 similar to (2.7) also for $p \ge 3$, e.g. we have

$$D_3(T) = D_1^3(T) - \frac{3}{n} D_1(T) D_1(T^*T) + \frac{2}{n^2} D_1(TT^*T).$$

However, the corresponding higher order combinatorial formula looks rather complicated. For this reason formula (2.12) from Proposition 2.4 is preferable.

(2) If $n \ge 2$ and $p = 2$, then (2.7) shows that in (2.6) we have the equality if and only if $T = 0$ (compare with Remark 2.1(b)).

Proposition 2.4. *Let* $\dim H = n$, $p \le n$, $T \in L(H, Y)$ *and* $(e_1, \dots, e_n)$ *be any orthonormal basis of* H. *Then:*

$$D_1(T) = \frac{1}{n} \left(\sum_{k=1}^{n} \|Te_k\|^2 \right) = \frac{1}{n} \|T\|_{HS}^2, \tag{2.9}$$

$$D_p(T) = \frac{1}{n^p} \sum_{j_1, \dots, j_p = 1}^{n} G_p(Te_{j_1}, \dots Te_{j_p}), \tag{2.10}$$

$$D_p(I) = \frac{n!}{(n-p)! n^p} \tag{2.11}$$

and

$$D_p(T) = \frac{p!}{n^p} \sum_{1 \le j_1 < \cdots < j_p \le n} \lambda_{j_1}^2 \ldots \lambda_{j_p}^2, \tag{2.12}$$

where $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of the operator* $|T| := (T^*T)^{\frac{1}{2}}$ *listed according to their multiplicities.*

Proof. Fix $x \in H$. We have $x = \sum_{k=1}^{n} (x|e_k)e_k$. Then $Tx = \sum_{k=1}^{n} (x|e_k)Te_k$ and

$$\|Tx\|^2 = \sum_{j,k=1}^{n} (x|e_j)\overline{(x|e_k)}(Te_j|Te_k).$$

Integrating this equality over S with respect to s and using (2.1) we get (2.9).

In the proof of (2.10) we will use the standard notation, namely, $\mathfrak{S}_p$ will stand for the set of all permutations $\sigma : \{1, \ldots, p\} \to \{1, \ldots, p\}$, and we will denote by $\operatorname{sgn} \sigma$ the sign of a fixed permutation σ. Fix finite sequences $(a_1, \ldots, b_p)$, $(b_1, \ldots, b_p)$ of the elements of H. We have the next "Parseval equality" for determinants, which follows from [1, p. V. 34, formula (26) and Prop. 5]:

$$\det((a_i|b_j)_{i,j=1}^p) = \sum_{1 \le j_1 < \cdots < j_p \le n} \left(\det((a_i|e_{j_k})_{i,k=1}^p)\right) \overline{\left(\det((b_i|e_{j_k})_{i,k=1}^p)\right)}. \tag{2.13}$$

Fix now a finite sequence $(x_1, \ldots, x_p)$ of the elements of H. Applying equality (2.13) to the matrix $(T^*Tx_i|x_j)_{i,j=1}^p$ we get

$$G(Tx_1, \ldots, Tx_p)$$
$$= \sum_{1 \le j_1 < \cdots < j_p \le n} \sum_{\sigma, \pi \in \mathfrak{S}_p} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\pi \prod_{k=1}^{p} (x_k|T^*Te_{j_{\sigma(k)}})\overline{(x_k|e_{j_{\pi(k)}})}. \tag{2.14}$$

Now integrating both sides of equality (2.14) with respect to the variables $x_1, \ldots, x_p$ and measure $s \times \cdots \times s$ and using formula (2.1) we obtain

$$D_p(T) = \frac{1}{n^p} \sum_{1 \le j_1 < \cdots < j_p \le n} \sum_{\sigma, \pi \in \mathfrak{S}_p} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\pi \prod_{k=1}^{p} (Te_{j_{\pi(k)}}|Te_{j_{\sigma(k)}})$$
$$= \frac{p!}{n^p} \sum_{1 \le j_1 < \cdots < j_p \le n} G(Te_{j_1}, \ldots, Te_{j_p}) = \frac{1}{n^p} \sum_{j_1, \ldots, j_p = 1}^{n} G(Te_{j_1}, \ldots Te_{j_p}). \tag{2.15}$$

The second equality in (2.15) is true since it can be easily seen that the relation

$$\sum_{\sigma, \pi \in \mathfrak{S}_p} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\pi \prod_{k=1}^{p} (y_{\sigma(k)}|y_{\pi(k)}) = p! G(y_1, \ldots, y_p)$$

holds for all finite sequences $(y_1, \ldots, y_p)$ of elements of Y.

Equality (2.11) follows from (2.10).

Equality (2.12) also follows from (2.10) since in (2.10) the elements $e_k, k = 1, \ldots, n$ can be taken the eigenvectors of T^*T. $\square$

Remark 2.5. (1) The validity of the important equality (2.10) was pointed out in [6], while the proof is given in [5, p. 113]; the above-given proof is different and, in a sense, less "combinatorial".

(2) Let

$$|||T|||_{p,e.} := \left(\sum_{j_1,\dots,j_p=1}^{n} G_p(Te_{j_1},\dots,Te_{j_p}) \right)^{1/2}.$$

Then from equality (2.10), inequality (2.6) and equality (2.9) we get

$$|||T|||^2_{p,e.} = n^p D_p(T) \le \|T\|^{2p}_{HS}.$$

This relation implies, in particular, that the value of $|||T|||_{p,e.}$ does not depend on a particular choice of an orthonormal basis $e. := (e_1,\dots,e_n)$ of H (compare with the next item).

(3) Suppose for a moment that H is an infinite-dimensional separable Hilbert space, $T : H \to Y$ is a continuous linear operator, $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H and put again

$$\begin{aligned} |||T|||_{p,e.} :&= \left(\sum_{j_1,\dots,j_p=1}^{\infty} G_p(Te_{j_1},\dots,Te_{j_p}) \right)^{1/2} \\ &= \lim_{n\to\infty} \left(\sum_{j_1,\dots,j_p=1}^{n} G_p(Te_{j_1},\dots Te_{j_p}) \right)^{1/2}. \end{aligned}$$

(3a) [3, p. 44, Prop. III.2.2] If T is a Hilbert–Schmidt operator, then

$$|||T|||_{p,e.} \le \|T\|^p_{HS} \tag{2.16}$$

and the value of $|||T|||_{p,e.}$ does not depend on a particular choice of an orthonormal basis $e. := (e_n)_{n\in\mathbb{N}}$ of H (note that inequality (2.16) follows from Remark 2.1(b), however the second statement now needs a separate proof).

(3b) [3, p. 42, Prop. III.2.1] If for given T there are a natural number p and an orthonormal basis $e. := (e_n)_{n\in\mathbb{N}}$ of H such that $|||T|||_{p,e.} < \infty$, then T is a Hilbert–Schmidt operator.

Using the functional D_p, the definition of the *p-th order mean dilatation number $\delta_p(T)$ of an operator $T \in L(H,Y)$*, given in the introduction, can be rewritten as

$$\delta_p(T) = \left(\frac{D_p(T)}{D_p(I)} \right)^{\frac{1}{2}}. \tag{2.17}$$

Evidently, when $p = 1$, (2.17) gives

$$\delta_1(T) = D_1^{1/2}(T) = \left(\int_S \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}. \tag{2.18}$$

The "normalized" version of Proposition 2.4 looks as follows:

Proposition 2.6. *Let* $\dim H = n$, $p \le n$, $T \in L(H,Y)$ *and* $(e_1, \dots, e_n)$ *be any orthonormal basis of* H. *Then:*

$$\delta_1(T) = \frac{1}{\sqrt{n}}\|T\|_{HS} = \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n}\|Te_k\|^2\right)^{\frac{1}{2}}, \tag{2.19}$$

$$\delta_p(T) = \binom{n}{p}^{-\frac{1}{2}}\left(\sum_{1\le j_1<\dots<j_p\le n}\lambda_{j_1}^2\dots\lambda_{j_p}^2\right)^{\frac{1}{2}}, \tag{2.20}$$

where $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of the operator* $|T| := (T^*T)^{\frac{1}{2}}$ *listed according to their multiplicities;*

$$\delta_n(T) = \det(|T|). \tag{2.21}$$

Proof. (2.19) follows from (2.9).
(2.20) follows from (2.12).
(2.21) follows from (2.20) by putting $p = n$. □

Remark 2.7. (a) Equality (2.21) can be derived directly from Remark 2.1(f), its validity was already noted in [6]. It shows that the normalization through $D_p(I)$ in the definition of the pMD-number is natural.

(b) Equality (2.19) implies that the functional $T \to \delta_1(T)$ is a norm on $L(H,Y)$ with the following property:

$$\frac{1}{\sqrt{n}}\|T\| \le \delta_1(T) \le \|T\|, \quad \forall\, T \in L(H,Y). \tag{2.22}$$

(c) Suppose $p > 1$, then the functional $T \to \delta_p(T)$ is absolutely p-homogeneous on $L(H,Y)$ (this follows, e.g., from (2.20)). Also (2.5) implies that $\delta_p(\cdot)$ vanishes on the operators with a rank $< p$.

(d) If for a given operator $T \in L(H,Y)$ we have $\|T\| \le 1$ and $\delta_1(T) = 1$, then T is an isometry (this is easy to see).

The following assertion shows that isometric operators can be characterized only in terms of δ_1 and δ_2.

Proposition 2.8. *Let* H *be a finite-dimensional Hilbert space with* $\dim H = n \ge 2$, Y *be any Hilbert space and* $T : H \to Y$ *be a linear operator. The following statements are equivalent:*
(i) T *is an isometry.*
(ii) $\delta_p(T) = 1 \quad \forall p \le n$.
(iii) $\delta_1(T) = \delta_2(T) = 1$.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are evident.

First proof of (iii) ⇒ (i). It is sufficient to verify that $T^*T = I$. For this it is enough to show that $\delta_1(T^*T - I) = 0$ (since δ_1 is a norm). Equalities (2.7) and (2.8) imply

$$\delta_2^2(T) = \frac{n}{n-1}\delta_1^4(T) - \frac{1}{n-1}\delta_1^2(T^*T) \tag{2.23}$$

and

$$\delta_1^2(T^*T - I) = \delta_1^2(T^*T) - 2\delta_1^2(T) + 1. \tag{2.24}$$

Now it is clear that (iii) and (2.23) imply $\delta_1(T^*T) = 1$. Thus the equality $\delta_1(T) = 1$ and (2.24) imply $\delta_1(T^*T - I) = 0$. Consequently, $T^*T = I$.

Second proof of (iii) $\Rightarrow$ (i).[2] Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities. Using (iii) and (2.20) we get

$$\sum_{k=1}^{n} {\lambda_k}^2 = n{\delta_1}^2(T) = n, \qquad \sum_{1\le j_1<j_2\le n} \lambda_{j_1}^2\lambda_{j_2}^2 = \binom{n}{2}\delta_2^2(T) = \binom{n}{2}.$$

Then

$$\sum_{k=1}^{n} {\lambda_k}^4 = \left(\sum_{k=1}^{n} {\lambda_k}^2\right)^2 - 2\sum_{1\le j_1<j_2\le n} \lambda_{j_1}^2\lambda_{j_2}^2 = n^2 - 2\binom{n}{2} = n$$

and

$$\sum_{k=1}^{n} ({\lambda_k}^2 - 1)^2 = \sum_{k=1}^{n} {\lambda_k}^4 - 2\sum_{k=1}^{n} {\lambda_k}^2 + n = 0.$$

Consequently, $\lambda_k = 1,\ k = 1, \dots, n$ and $|T| = I$. $\square$

Remark 2.9. It is interesting to note that in [6] only the validity of the implication (ii) $\Rightarrow$ (i) was conjectured. We see that even (iii) implies (i).

The following assertion will be used in the next section.

Lemma 2.10. *Let* $\dim H = n$, $T : H \to Y$ *be a linear operator and* $p \in \{1, \dots, n\}$. *Put*

$$c_{n,p} := D_p(I) = \frac{n!}{(n-p)!n^p}$$

and

$$r_n(p, T) := D_1^p(T) - D_p(T).$$

Then:

$$\frac{1}{p^{p-1}} \le c_{n,p} \le 1, \tag{2.25}$$

$$\delta_p(T) \le p^{\frac{p-1}{2}}\delta_1^p(T), \tag{2.26}$$

$$0 \le r_n(p, T) \le \frac{p!}{n}\|T\|^{2p}, \tag{2.27}$$

$$\delta_p^2(T) = \frac{1}{c_{n,p}}(\delta_1^{2p}(T) - r_n(p, T)). \tag{2.28}$$

[2] Suggested by the Referee.

Proof. Relation (2.25) is easy to verify. Inequality (2.26) follows from (2.6) and (2.25).

(2.27): Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities. Using (2.9), Newton's (polynomial) formula and (2.10) we can write:

$$\begin{aligned}
(D_1(T))^p - D_p(T) &= \frac{1}{n^p}(\sum_{k=1}^{n} \lambda_k^2)^p - \frac{p!}{n^p} \sum_{1 \le j_1 < \dots < j_p \le n} \lambda_{j_1}^2 \dots \lambda_{j_p}^2 \\
&= \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n} \\
&- \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p, \max k_i = 1} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n} \\
&= \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p, \max k_i > 1} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n}.
\end{aligned}$$

Consequently,

$$(D_1(T))^p - D_p(T) = \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p, \max k_i > 1} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n}.$$

From this equality, as $\lambda_k \le \|T\|$, $k = 1, \dots, n$, and

$$\sum_{k_1 + \dots + k_n = p, \max k_i > 1} \frac{1}{k_1! \dots k_n!} \le n^{p-1},$$

we obtain (2.27).

(2.28) follows from (2.27). □

Remark 2.11. (1) By (2.20) it can be seen that for a given operator T the equality $\delta_p(T) = \delta_1^p(T)$ holds if and only if T is a scalar multiplier of an isometry. In this connection the asymptotic versions behave better (see the next section).

(2) A simple relation between $\delta_p(T)$ and $\delta_1^p(T)$, can be written using the notation from the theory of outer products of Hilbert spaces. Namely, let H, Y be finite-dimensional Hilbert spaces, $T \in L(H, Y)$ and $\bigwedge^p T$ be the antisymmetric outer p-th power of the operator T. Then

$$\delta_p(T) = \binom{n}{p}^{-\frac{1}{2}} \| \textstyle\bigwedge^p T\|_{\mathrm{HS}} \text{ and } \delta_p(T) = \delta_1(\bigwedge^p T). \tag{2.29}$$

These formulas follow easily from the following relation, established in [5, p. 112–113, the proof of Prop. VI.2.2]: $\| \bigwedge^p T\|_{\mathrm{HS}}^2 = \frac{n^p}{p!} D_p(T)$. In [5] it is also shown that the antisymmetric outer p-th power of a Hilbert–Schmidt (or a nuclear) operator T acting between infinite-dimensional Hilbert spaces is again a Hilbert–Schmidt (a nuclear) operator (cf. also [4, Th. 2]).

3. Infinite-Dimensional Case

In this section H will be an infinite-dimensional Hilbert space and Y will be another Hilbert space. As before, $L(H,Y)$ will denote the set of all continuous linear operators $T : H \to Y$.

Fix a natural number p and consider the collection $\mathcal{M}_p$ of all finite-dimensional vector subspaces $M \subset H$ with $\dim M \geq p$. This collection is a directed set by set-theoretic inclusion.

Fix an operator $T \in L(H,Y)$. Then for any $M \in \mathcal{M}_p$ and the restriction $T|_M$ of T to M, the pMD-number $\delta_p(T|_M)$ is defined. In this way T generates with the net $(\delta_p(T|_M))_{M\in\mathcal{M}_p}$. Let us denote the upper limit of this net by $\overline{\overline{\delta}}_p(T)$ and call it the *asymptotic upper pMD-number* of T. In a similar manner, let us denote the lower limit of this net by $\underline{\underline{\delta}}_p(T)$ and call it the *asymptotic lower pMD-number* of T. Therefore

$$\overline{\overline{\delta}}_p(T) := \limsup_{M\in\mathcal{M}_p} \delta_p(T|_M) \quad \text{and} \quad \underline{\underline{\delta}}_p(T) := \liminf_{M\in\mathcal{M}_p} \delta_p(T|_M).$$

By formula (2.26) of Lemma 2.10 we have

$$\delta_p(T|_M) \leq p^{\frac{p-1}{2}} \|T\|^p, \quad \forall\ M \in \mathcal{M}_p.$$

This inequality shows that the net $(\delta_p(T|_M))_{M\in\mathcal{M}_p}$ is bounded and, consequently, always

$$\underline{\underline{\delta}}_p(T) \leq \overline{\overline{\delta}}_p(T) \leq p^{\frac{p-1}{2}} \|T\|^p < \infty.$$

In general, the net $(\delta_p(T|_M))_{M\in\mathcal{M}_p}$ may not be convergent (as we will see below). In the case of convergence we shall call the operator T *asymptotically pMD-regular*. For an asymptotically pMD-regular T we put

$$\overline{\delta}_p(T) = \lim_{M\in\mathcal{M}_p} \delta_p(T|_M),$$

and call $\overline{\delta}_p(T)$ the *asymptotic pMD-number* of T.

An asymptotically 1MD-regular operator T will be called *asymptotically MD-regular* and its 1MD-number will be called *asymptotic MD-number*.

The next result is somewhat unexpected. Its validity was not predicted in [6].

Theorem 3.1. *Let H be an infinite-dimensional Hilbert space, Y be a Hilbert space, $T \in L(H,Y)$ and $p > 1$ be a natural number. Then*

$$\overline{\overline{\delta}}_p(T) = (\overline{\overline{\delta}}_1(T))^p \ \text{ and } \ \underline{\underline{\delta}}_p(T) = (\underline{\underline{\delta}}_1(T))^p; \tag{3.1}$$

moreover, the operator T is asymptotically pMD-regular if and only if it is asymptotically MD-regular and in the case of asymptotic MD-regularity the equality

$$\overline{\delta}_p(T) = (\overline{\delta}_1(T))^p \tag{3.2}$$

holds.

Proof. Let $M \in \mathcal{M}_p$. We can write using formula (2.28) of Lemma 2.10:

$$\delta_p^2(T|_M) = \frac{1}{p!}\binom{\dim M}{p}^{-1}(\dim M)^p\left(\delta_1^{2p}(T|_M) - r_{\dim M}(p, T|_M)\right). \quad (3.3)$$

Evidently,

$$\lim_{M\in\mathcal{M}_p} \frac{1}{p!}\binom{\dim M}{p}^{-1}(\dim M)^p = 1. \quad (3.4)$$

Since according to inequality (2.27) of Lemma 2.10

$$r_{\dim M}(p, T|_M)| \leq \frac{p!}{\sqrt{\dim M}}\|T|_M\|^{2p} \leq \frac{p!}{\sqrt{\dim M}}\|T\|^{2p},$$

we also have

$$\lim_{M\in\mathcal{M}_p} r_{\dim M}(p, T|_M) = 0. \quad (3.5)$$

From (3.3) via (3.4) and (3.5) the we get (3.1). The "moreover" part now is clear. □

This theorem shows that in the case of an infinite-dimensional domain it is sufficient to study only the numbers $\overline{\overline{\delta}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$.

To simplify the notation, for a given operator $T \in L(H, Y)$ let us denote its

- asymptotic upper 1MD-number $\overline{\overline{\delta}}_1(T)$ by $\overline{\overline{\delta}}(T)$,
- asymptotic lower 1MD-number $\underline{\underline{\delta}}_1(T)$ by $\underline{\underline{\delta}}(T)$,
- asymptotic MD-number $\overline{\delta}_1(T)$ by $\overline{\delta}(T)$.

Using formula (2.19) from Proposition 2.6 the definition of these numbers can be formulated directly in terms of the Hilbert–Schmidt norm as follows:

$$\overline{\overline{\delta}}(T) = \limsup_{M\in\mathcal{M}_1} \frac{1}{\sqrt{\dim M}}\|T|_M\|_{HS}, \quad \underline{\underline{\delta}}(T) = \liminf_{M\in\mathcal{M}_1} \frac{1}{\sqrt{\dim M}}\|T|_M\|_{HS}.$$

For an operator $T \in L(H, Y)$ let us put

$$m(T) := \inf\{\|Tx\| : x \in H,\ \|x\| = 1\}.$$

The number $m(T)$ is sometimes called the lower bound of T.

The next statement implies, in particular, that a given operator T may not be asymptotically MD-regular.

Proposition 3.2. *Let H, Y be infinite-dimensional Hilbert spaces and $T : H \to Y$ be a continuous linear operator. Then:*

(a) *For any infinite-dimensional closed vector subspace $X \subset H$ the inequality*

$$\overline{\overline{\delta}}(T) \geq m(T|_X)$$

holds.

(b) *If* $\ker(T)$ *is infinite-dimensional, then* $\underline{\underline{\delta}}(T) = 0$.

(c) *If T is a partial isometry such that* $\ker(T)$ *and* $T(H)$ *are both infinite-dimensional, then* $\overline{\overline{\delta}}(T) = 1$, *while* $\underline{\underline{\delta}}(T) = 0$ *and so T is not asymptotically MD-regular.*

Proof. (a) Fix an infinite-dimensional $X \subset H$, a finite-dimensional vector subspace $M \subset H$ and put

$$\beta_M := \sup\{\delta_1(T|_N) : N \in \mathcal{M}_1, N \supset M\}.$$

Let us show that

$$\beta_M \geq m(T|_X). \tag{3.6}$$

To prove (3.6), fix a natural number n and an n-dimensional vector subspace X_n of X such that $X_n \cap M = \{0\}$ (such a choice is possible because M is finite-dimensional and X is infinite-dimensional). Let also $M_n := M + X_n$ and let M' be the vector subspace of M_n orthogonal to X_n. Using formula (2.19) from Proposition 2.6 we can write:

$$\delta_1^2(T|_{M_n}) = \frac{\|T|_{M'}\|_{HS}^2 + \|T|_{X_n}\|_{HS}^2}{\dim(M') + n} \geq \frac{\|T|_{X_n}\|_{HS}^2}{\dim(M') + n} \geq \frac{n}{\dim(M') + n}\, m(T|_X)^2.$$

As $\beta_M \geq \delta_1(T|_{M_n})$ and $\dim(M') \leq \dim(M)$, we get

$$\beta_M^2 \geq \frac{n}{\dim(M) + n}\, m^2(T|_X). \tag{3.7}$$

Since n is arbitrary, from (3.7) we have

$$\beta_M^2 \geq \sup_n \frac{nm^2(T|_X)}{\dim(M) + n} \geq \lim_n \frac{nm^2(T|_X)}{\dim(M) + n} = m^2(T|_X).$$

This relation, together with (3.6) and the definition of $\overline{\overline{\delta}}(T)$, implies (a).

(b) Let $X = \ker(T)$, fix a finite-dimensional vector subspace $M \subset H$ and put

$$\alpha_M := \inf\{\delta_1(T|_N) : N \in \mathcal{M}_1, N \supset M\}.$$

Let us show that

$$\alpha_M = 0. \tag{3.8}$$

To prove (3.8), fix a natural number n and an n-dimensional vector subspace X_n of X such that $X_n \cap M = \{0\}$ (such a choice is possible because M is finite-dimensional and X is infinite-dimensional). Let also $M_n := M + X_n$ and M' be the subspace of M_n orthogonal to X_n. Using formula (2.19) from Proposition 2.6 and taking into account that $T|_{X_n} = 0$ we can write:

$$\delta_1^2(T_{M_n}) = \frac{\|T|_{M'}\|_{HS}^2 + \|T|_{X_n}\|_{HS}^2}{\dim(M') + n} = \frac{\|T|_{M'}\|_{HS}^2}{\dim(M') + n}\,.$$

From this we get

$$\alpha_M^2 \leq \frac{\|T|_{M'}\|_{HS}^2}{\dim(M') + n}\,.$$

Observe now that since $\dim(M') \le \dim(M)$ and $\|T|_{M'}\|_{HS} \le \|T\|\sqrt{\dim(M')}$, we have

$$\alpha_M^2 \le \lim_n \frac{\|T|_{M'}\|^2}{\dim(M') + n} = 0.$$

This relation together with the definition of $\underline{\underline{\delta}}(T)$ implies (b).

(c) follows from (a) and (b). $\square$

The last proposition motivates the following

Problem. Give a characterization of asymptotically MD-regular operators in terms of some other known parameters (in terms of the spectrum, in terms of the diagonal (for the diagonal operators), etc.).

Acknowledgement

We are grateful to the referee for helpful comments and suggestions.

The third author acknowledges support from Xunta de Galicia (D.O.G. do 25-02-00) during his visit to Vigo.

References

1. N. Bourbaki, Espaces vectoriels topologiques. *Masson, Paris,* 1981.
2. A. Castejón, E. Corbacho, and V. Tarieladze, AMD numbers, compactness, strict singularity and essential spectrum of operators. Submitted to *Georgian Math. J.* in 2001.
3. E. Corbacho, Hipervolúmenes, su convergencia y sumabilidad en los operadores lineales acotados del espacio de Hilbert. *Ph.D. thesis, Zaragoza,* 1978.
4. S. Geiss, Antisymmetric tensor products of absolutely p-summing operators. *J. Approx. Theory* **68**(1992), No. 3, 223–246.
5. A. Martínez, Normas tensoriales e ideales de operadores. *Ph.D. thesis, Madrid,* 1986.
6. J. L. Rubio de Francia, Unpublished notes.
7. N. Tomczak-Jaegermann, Banach–Mazur distance and finite dimensional operator ideals. *Longman Scientific& Technical, Harlow,* 1989.

(Received 15.01.2001; revised 15.06.2001)

Authors' address:

Universidad de Vigo
Departamento de Matemática Aplicada
36280 Vigo, Galicia, España
E-mail: acaste@dma.uvigo.es
corbacho@uvigo.es
visit01@dma.uvigo.es

The permanent address of V. Tarieladze:
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093, Georgia
E-mail: tar@gw.acnet.ge