

MD-NUMBERS AND ASYMPTOTIC MD-NUMBERS OF OPERATORS

A. CASTEJÓN, E. CORBACHO, AND V. TARIELADZE

Abstract. First, the basic properties of mean dilatation (MD-) numbers for linear operators acting from a finite-dimensional Hilbert space are investigated. Among other results, in terms of first and second order MD-numbers, a characterization of isometries is obtained and a dimension-free estimation of the p -th order MD-number by means of the first order MD-number is established. After that asymptotic MD-numbers for a continuous linear operator acting from an infinite-dimensional Hilbert space are introduced and it is shown that in the case of an infinite-dimensional domain the asymptotic p -th order MD-number, rather unexpectedly, is simply the p -th power of the asymptotic first order MD-number (Theorem 3.1).

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1. INTRODUCTION

For a linear operator T from a Hilbert space H with $1 \leq n = \dim H < \infty$ to an arbitrary Hilbert space Y and a natural number $p \leq n$, the p -th order mean dilatation number (briefly, the pMD-number) $\delta_p(T)$ is defined by the equality

$$\delta_p(T) = \left(\frac{1}{c_{n,p}} \int_S \cdots \int_S G(T_1, \dots, Tx_p) ds(x_1) \dots ds(x_p) \right)^{1/2}, \quad (1.1)$$

where S is the unit sphere in H , s is the normalized isometrically invariant measure on it, $G(x_1, \dots, x_p)$ is the Grammian of (x_1, \dots, x_p) , and

$$c_{n,p} = \int_S \cdots \int_S G(x_1, \dots, x_p) ds(x_1) \dots ds(x_p) > 0$$

is a (natural) normalizing constant.

The definition of pMD-numbers was suggested in [6]. Afterwards they were considered also in [5].

For simplicity, the first order MD-number $\delta_1(T)$ will be called the MD-number of T and, for it, (1.1) gives an expression

$$\delta_1(T) = \left(\int_S \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}. \tag{1.2}$$

Using the equality (1.2), the MD-number $\delta_1(T)$ can also be defined when Y is an arbitrary Banach space and in this case it has already been used as a tool of the local theory of Banach spaces (see, e.g., [7, p.81]).

In Section 2, after recalling some known facts concerning the Grammian (Remark 2.1) which motivate the definition of pMD-numbers, their properties in the finite-dimensional case are analyzed. In particular, a concrete expression for $\delta_p(T)$ in terms of the eigenvalues of the operator T^*T is found (Proposition 2.6) and the following characterization of isometric operators is obtained: T is an isometry if and only if $\delta_1(T) = \delta_2(T) = 1$ (Proposition 2.8). Although the equality $\delta_p(T) = \delta_1^p(T)$ does not hold in general, the validity of the following “dimension-free” estimate

$$\delta_p(T) \leq p^{\frac{p-1}{2}} \delta_1^p(T)$$

is proved (Lemma 2.10).

Possible infinite-dimensional extensions of pMD-numbers are introduced in Section 3. According to [6], we use the net \mathcal{M} of all finite-dimensional subspaces of H to associate two quantities

$$\overline{\overline{\delta}}_p(T) := \limsup_{M \in \mathcal{M}} \delta_p(T|_M), \quad \underline{\underline{\delta}}_p(T) := \liminf_{M \in \mathcal{M}} \delta_p(T|_M)$$

to a given operator T (acting from H) and a natural number p . We call $\overline{\overline{\delta}}_p(T)$ the asymptotic upper pMD-number of T and $\underline{\underline{\delta}}_p(T)$ the asymptotic lower pMD-number of T . When they are equal, the operator is called asymptotically pMD-regular, their common value is denoted by $\overline{\delta}_p(T)$ and called the asymptotic pMD-number. Therefore for an asymptotic pMD-regular operator T we have

$$\overline{\delta}_p(T) := \lim_{M \in \mathcal{M}} \delta_p(T|_M).$$

When $p = 1$, we shall simply use the term “asymptotically MD-regular”.

The main result of the section is Theorem 3.1, which asserts that, in the case of an infinite-dimensional domain, we have the following simple relation between the higher order and the first order asymptotic MD-numbers of a given operator T :

$$\overline{\overline{\delta}}_p(T) = (\overline{\overline{\delta}}_1(T))^p, \quad \underline{\underline{\delta}}_p(T) = (\underline{\underline{\delta}}_1(T))^p.$$

This result shows that in the case of the infinite-dimensional domain it is sufficient to study only the numbers $\overline{\overline{\delta}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$. Section 3 is concluded by Proposition 3.2, which implies, in particular, that not any operator is asymptotically MD-regular.

Notice, finally, that the numbers $\overline{\delta}_1(T)$, $\underline{\delta}_1(T)$ and $\overline{\delta}_1(T)$ can also be defined when Y is a general Banach space. The properties of the corresponding quantities in this general setting are investigated in [2].

2. FINITE DIMENSIONAL CASE

Throughout this paper the considered Hilbert spaces is supposed to be defined over the same field \mathbb{K} , real \mathbb{R} or complex \mathbb{C} numbers. To simplify the notation, for the scalar product and the norm of different Hilbert spaces we use the same symbols $(\cdot|\cdot)$ and $\|\cdot\|$.

For the Hilbert spaces H, Y we denote by $L(H, Y)$ the set of all continuous linear operators $T : H \rightarrow Y$.

For a given operator $T \in L(H, Y)$ we denote by

- $\|T\|$ the ordinary norm of T ;
- T^* the (Hilbert) adjoint operator of T ;
- $|T|$ the unique self-adjoint positive square root of $T^*T : H \rightarrow H$.

Moreover, for a Hilbert–Schmidt operator T we denote by $\|T\|_{\text{HS}}$ the *Hilbert–Schmidt norm* of T .

Fix a (non-zero) Hilbert space H and a natural number p such that $p \leq \dim H$. The p -th order *Grammian* G_p is the scalar-valued function, defined on H^p , which assigns the determinant of the matrix $(x_i|x_j)_{i,j=1}^p$ to any $(x_1, \dots, x_p) \in H^p$.

Evidently, $G_1(x) = \|x\|^2$ for each $x \in H$. Some other known properties of the Grammian are given in the next remark.

Remark 2.1. Fix a natural number $p > 1$ with $p \leq \dim H$ and a finite sequence $(x_1, \dots, x_p) \in H^p$.

(a) We have $G_p(x_1, \dots, x_p) \geq 0$. Moreover, $G_p(x_1, \dots, x_p) > 0$ if and only if (x_1, \dots, x_p) is linearly independent.

(b) $G_p(x_1, \dots, x_p) \leq \|x_1\|^2 \dots \|x_p\|^2$. Moreover, if x_1, \dots, x_p are nonzero elements, then we have $G_p(x_1, \dots, x_p) = \|x_1\|^2 \dots \|x_p\|^2$ iff x_1, \dots, x_p are pairwise orthogonal.

(c) Let $P_{x_1, \dots, x_p} := \left\{ \sum_{i=1}^p \alpha_i x_i : 0 \leq \alpha_i \leq 1, i = 1, \dots, p \right\}$ be the parallelepiped generated by the sequence (x_1, \dots, x_p) . Then $G(x_1, \dots, x_p) = \text{vol}^2(P_{x_1, \dots, x_p})$, where $\text{vol}(P_{x_1, \dots, x_p})$ stands for the (p -dimensional) volume of P_{x_1, \dots, x_p} (i.e., the Lebesgue measure of P_{x_1, \dots, x_p} in a p -dimensional vector subspace containing (x_1, \dots, x_p)). For this reason the quantity $g_p(x_1, \dots, x_p) := G_p^{1/2}(x_1, \dots, x_p)$ is often simply called “the hypervolume determined by the vectors (x_1, \dots, x_p) .”

(d) Let $[x_1, \dots, x_{p-1}]$ be the linear span of $\{x_1, \dots, x_{p-1}\}$. Then

$$g_p(x_1, \dots, x_p) = g_{p-1}(x_1, \dots, x_{p-1}) \cdot \text{dist}(x_p, [x_1, \dots, x_{p-1}]).$$

(e) If Y is another Hilbert space and $T : H \rightarrow Y$ is a continuous linear operator, then $g_p(Tx_1, \dots, Tx_p) \leq \|T\|^p g_p(x_1, \dots, x_p)$ (this is evident when $p = 1$, the rest follows from this and from (d) by induction, as $\text{dist}(Tx_p, [Tx_1, \dots, Tx_{p-1}]) \leq \|T\| \text{dist}(x_p, [x_1, \dots, x_{p-1}])$).

(f) In the notation of the previous item suppose that $\dim H = p$. Then $g_p(Tx_1, \dots, Tx_p) = \det(|T|)g_p(x_1, \dots, x_p)$. It follows that if $\dim H = p$ and $(x'_1, \dots, x'_p), (x_1, \dots, x_p)$ are algebraic bases of H , then

$$\frac{g_p(Tx'_1, \dots, Tx'_p)}{g_p(x'_1, \dots, x'_p)} = \frac{g_p(Tx_1, \dots, Tx_p)}{g_p(x_1, \dots, x_p)},$$

i.e., the “volume ratio” is independent of a particular choice of basis of H and equals to $\det(|T|)$.

For a Hilbert space with $\dim H \geq 1$, its unit sphere with center at the origin is denoted by S_H or, simply, by S . When $n = \dim H < \infty$, the *uniform distribution* on S , i.e. the unique isometrically invariant probability measure given on the Borel σ -algebra of S , is denoted by s . The following key equality follows directly from the isometric invariance of s :

$$\int_S \overline{(x|h_1)}(x|h_2) ds(x) = \frac{1}{n} (h_1|h_2), \quad \forall h_1, h_2 \in H, \quad (2.1)$$

where ‘bar’ stands for complex conjugation.

In the rest of this section H will be a finite-dimensional Hilbert space with $\dim H = n \geq 1$, I will stand for the identity operator acting in H and Y will be another Hilbert space (not necessarily finite-dimensional).

Fix a number $p \in \{1, \dots, n\}$ and introduce the functional $D_p : L(H, Y) \rightarrow \mathbb{R}_+$ defined by the equality

$$D_p(T) = \int_S \cdots \int_S G_p(Tx_1, \dots, Tx_p) ds(x_1) \dots ds(x_p), \quad T \in L(H, Y). \quad (2.2)$$

Therefore, for given T the number $D_p(T)$ can be viewed as the average of the squares of the volumes of the family of parallelepipeds

$$\{T(P_{x_1, \dots, x_p}) : (x_1, \dots, x_p) \in S \times \cdots \times S\}$$

with respect to the product $s \times \cdots \times s$ of the uniform distributions.¹

Evidently, for $p = 1$ we get

$$D_1(T) = \int_S \|Tx\|^2 ds(x), \quad \forall T \in L(H, Y). \quad (2.3)$$

To make easier further references, we formalize some other easy observations in the next statement.

¹The choice of s seems to be natural, since it gives no preference to any of the directions $(x_1, \dots, x_p) \in S \times \cdots \times S$. Formally, the same definition can be given, taking any probability measure μ on S instead of s , but then the properties of $D_p(T)$ will depend on (the correlation operator of) μ .

Proposition 2.2. *Let $T \in L(H, Y)$.*

$$D_p(T) = D_p(|T|); \tag{2.4}$$

$$D_p(T) = 0 \iff \dim T(H) < p; \tag{2.5}$$

$$D_p(T) \leq D_1^p(T); \tag{2.6}$$

$$D_2(T) = D_1^2(T) - \frac{1}{n}D_1(T^*T), \text{ in particular, } D_2(I) = 1 - \frac{1}{n}; \tag{2.7}$$

$$D_1(T^*T - I) = D_1(T^*T) - 2D_1(T) + 1. \tag{2.8}$$

Proof. (2.4) is true since $G_p(|T|x_1, \dots, |T|x_p) = G_p(Tx_1, \dots, Tx_p)$ for any $(x_1, \dots, x_p) \in H^p$.

(2.5) follows from Remark 2.1(a).

(2.6) follows from Remark 2.1(b) and (2.3).

(2.7): since $\forall x_1, x_2 \in H$

$$G_p(Tx_1, Tx_2) = \|Tx_1\|^2\|Tx_2\|^2 - |(Tx_1|Tx_2)|^2 = \|Tx_1\|^2\|Tx_2\|^2 - |(T^*Tx_1|x_2)|^2,$$

and since by (2.1)

$$\int_S |(T^*Tx_1|x_2)|^2 ds(x_2) = \frac{1}{n}\|T^*Tx_1\|^2, \quad \forall x_1 \in H,$$

from (2.3) (using, of course, Fubini's theorem) we get (2.7).

(2.8) follows from (2.3) and from the evident equality $\|(T^*T - I)x\|^2 = \|T^*Tx\|^2 - 2\|Tx\|^2 + \|x\|^2, x \in H$. \square

Remark 2.3. Fix $T \in L(H, Y)$.

(1) By means of the direct integration of the Grammian it is possible to get a "coordinate free" expression for $D_p(T)$ in terms of D_1 similar to (2.7) also for $p \geq 3$, e.g. we have

$$D_3(T) = D_1^3(T) - \frac{3}{n}D_1(T)D_1(T^*T) + \frac{2}{n^2}D_1(TT^*T).$$

However, the corresponding higher order combinatorial formula looks rather complicated. For this reason formula (2.12) from Proposition 2.4 is preferable.

(2) If $n \geq 2$ and $p = 2$, then (2.7) shows that in (2.6) we have the equality if and only if $T = 0$ (compare with Remark 2.1(b)).

Proposition 2.4. *Let $\dim H = n, p \leq n, T \in L(H, Y)$ and (e_1, \dots, e_n) be any orthonormal basis of H . Then:*

$$D_1(T) = \frac{1}{n} \left(\sum_{k=1}^n \|Te_k\|^2 \right) = \frac{1}{n} \|T\|_{HS}^2, \tag{2.9}$$

$$D_p(T) = \frac{1}{n^p} \sum_{j_1, \dots, j_p=1}^n G_p(Te_{j_1}, \dots, Te_{j_p}), \tag{2.10}$$

$$D_p(I) = \frac{n!}{(n-p)!n^p} \tag{2.11}$$

and

$$D_p(T) = \frac{p!}{n^p} \sum_{1 \leq j_1 < \dots < j_p \leq n} \lambda_{j_1}^2 \dots \lambda_{j_p}^2, \quad (2.12)$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities.

Proof. Fix $x \in H$. We have $x = \sum_{k=1}^n (x|e_k)e_k$. Then $Tx = \sum_{k=1}^n (x|e_k)Te_k$ and

$$\|Tx\|^2 = \sum_{j,k=1}^n (x|e_j)\overline{(x|e_k)}(Te_j|Te_k).$$

Integrating this equality over S with respect to s and using (2.1) we get (2.9).

In the proof of (2.10) we will use the standard notation, namely, \mathfrak{S}_p will stand for the set of all permutations $\sigma : \{1, \dots, p\} \rightarrow \{1, \dots, p\}$, and we will denote by $\text{sgn } \sigma$ the sign of a fixed permutation σ . Fix finite sequences (a_1, \dots, a_p) , (b_1, \dots, b_p) of the elements of H . We have the next ‘‘Parseval equality’’ for determinants, which follows from [1, p. V. 34, formula (26) and Prop. 5]:

$$\det((a_i|b_j)_{i,j=1}^p) = \sum_{1 \leq j_1 < \dots < j_p \leq n} \left(\det((a_i|e_{j_k})_{i,k=1}^p) \right) \overline{\left(\det((b_i|e_{j_k})_{i,k=1}^p) \right)}. \quad (2.13)$$

Fix now a finite sequence (x_1, \dots, x_p) of the elements of H . Applying equality (2.13) to the matrix $(T^*Tx_i|x_j)_{i,j=1}^p$ we get

$$\begin{aligned} & G(Tx_1, \dots, Tx_p) \\ &= \sum_{1 \leq j_1 < \dots < j_p \leq n} \sum_{\sigma, \pi \in \mathfrak{S}_p} \text{sgn } \sigma \cdot \text{sgn } \pi \prod_{k=1}^p (x_k|T^*Te_{j_{\sigma(k)}})\overline{(x_k|e_{j_{\pi(k)}})}. \end{aligned} \quad (2.14)$$

Now integrating both sides of equality (2.14) with respect to the variables x_1, \dots, x_p and measure $s \times \dots \times s$ and using formula (2.1) we obtain

$$\begin{aligned} D_p(T) &= \frac{1}{n^p} \sum_{1 \leq j_1 < \dots < j_p \leq n} \sum_{\sigma, \pi \in \mathfrak{S}_p} \text{sgn } \sigma \cdot \text{sgn } \pi \prod_{k=1}^p (Te_{j_{\pi(k)}}|Te_{j_{\sigma(k)}}) \\ &= \frac{p!}{n^p} \sum_{1 \leq j_1 < \dots < j_p \leq n} G(Te_{j_1}, \dots, Te_{j_p}) = \frac{1}{n^p} \sum_{j_1, \dots, j_p=1}^n G(Te_{j_1}, \dots, Te_{j_p}). \end{aligned} \quad (2.15)$$

The second equality in (2.15) is true since it can be easily seen that the relation

$$\sum_{\sigma, \pi \in \mathfrak{S}_p} \text{sgn } \sigma \cdot \text{sgn } \pi \prod_{k=1}^p (y_{\sigma(k)}|y_{\pi(k)}) = p!G(y_1, \dots, y_p)$$

holds for all finite sequences (y_1, \dots, y_p) of elements of Y .

Equality (2.11) follows from (2.10).

Equality (2.12) also follows from (2.10) since in (2.10) the elements $e_k, k = 1, \dots, n$ can be taken the eigenvectors of T^*T . \square

Remark 2.5. (1) The validity of the important equality (2.10) was pointed out in [6], while the proof is given in [5, p. 113]; the above-given proof is different and, in a sense, less “combinatorial”.

(2) Let

$$|||T|||_{p,e} := \left(\sum_{j_1, \dots, j_p=1}^n G_p(Te_{j_1}, \dots, Te_{j_p}) \right)^{1/2}.$$

Then from equality (2.10), inequality (2.6) and equality (2.9) we get

$$|||T|||_{p,e}^2 = n^p D_p(T) \leq \|T\|_{HS}^{2p}.$$

This relation implies, in particular, that the value of $|||T|||_{p,e}$ does not depend on a particular choice of an orthonormal basis $e := (e_1, \dots, e_n)$ of H (compare with the next item).

(3) Suppose for a moment that H is an infinite-dimensional separable Hilbert space, $T : H \rightarrow Y$ is a continuous linear operator, $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of H and put again

$$\begin{aligned} |||T|||_{p,e} &:= \left(\sum_{j_1, \dots, j_p=1}^{\infty} G_p(Te_{j_1}, \dots, Te_{j_p}) \right)^{1/2} \\ &= \lim_{n \rightarrow \infty} \left(\sum_{j_1, \dots, j_p=1}^n G_p(Te_{j_1}, \dots, Te_{j_p}) \right)^{1/2}. \end{aligned}$$

(3a) [3, p. 44, Prop. III.2.2] If T is a Hilbert–Schmidt operator, then

$$|||T|||_{p,e} \leq \|T\|_{HS}^p \tag{2.16}$$

and the value of $|||T|||_{p,e}$ does not depend on a particular choice of an orthonormal basis $e := (e_n)_{n \in \mathbb{N}}$ of H (note that inequality (2.16) follows from Remark 2.1(b), however the second statement now needs a separate proof).

(3b) [3, p. 42, Prop. III.2.1] If for given T there are a natural number p and an orthonormal basis $e := (e_n)_{n \in \mathbb{N}}$ of H such that $|||T|||_{p,e} < \infty$, then T is a Hilbert–Schmidt operator.

Using the functional D_p , the definition of the *p-th order mean dilatation number* $\delta_p(T)$ of an operator $T \in L(H, Y)$, given in the introduction, can be rewritten as

$$\delta_p(T) = \left(\frac{D_p(T)}{D_p(I)} \right)^{\frac{1}{2}}. \tag{2.17}$$

Evidently, when $p = 1$, (2.17) gives

$$\delta_1(T) = D_1^{1/2}(T) = \left(\int_S \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}. \tag{2.18}$$

The “normalized” version of Proposition 2.4 looks as follows:

Proposition 2.6. *Let $\dim H = n$, $p \leq n$, $T \in L(H, Y)$ and (e_1, \dots, e_n) be any orthonormal basis of H . Then:*

$$\delta_1(T) = \frac{1}{\sqrt{n}} \|T\|_{HS} = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n \|Te_k\|^2 \right)^{\frac{1}{2}}, \quad (2.19)$$

$$\delta_p(T) = \binom{n}{p}^{-\frac{1}{2}} \left(\sum_{1 \leq j_1 < \dots < j_p \leq n} \lambda_{j_1}^2 \dots \lambda_{j_p}^2 \right)^{\frac{1}{2}}, \quad (2.20)$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities;

$$\delta_n(T) = \det(|T|). \quad (2.21)$$

Proof. (2.19) follows from (2.9).

(2.20) follows from (2.12).

(2.21) follows from (2.20) by putting $p = n$. \square

Remark 2.7. (a) Equality (2.21) can be derived directly from Remark 2.1(f), its validity was already noted in [6]. It shows that the normalization through $D_p(I)$ in the definition of the pMD-number is natural.

(b) Equality (2.19) implies that the functional $T \rightarrow \delta_1(T)$ is a norm on $L(H, Y)$ with the following property:

$$\frac{1}{\sqrt{n}} \|T\| \leq \delta_1(T) \leq \|T\|, \quad \forall T \in L(H, Y). \quad (2.22)$$

(c) Suppose $p > 1$, then the functional $T \rightarrow \delta_p(T)$ is absolutely p -homogeneous on $L(H, Y)$ (this follows, e.g., from (2.20)). Also (2.5) implies that $\delta_p(\cdot)$ vanishes on the operators with a rank $< p$.

(d) If for a given operator $T \in L(H, Y)$ we have $\|T\| \leq 1$ and $\delta_1(T) = 1$, then T is an isometry (this is easy to see).

The following assertion shows that isometric operators can be characterized only in terms of δ_1 and δ_2 .

Proposition 2.8. *Let H be a finite-dimensional Hilbert space with $\dim H = n \geq 2$, Y be any Hilbert space and $T : H \rightarrow Y$ be a linear operator. The following statements are equivalent:*

- (i) T is an isometry.
- (ii) $\delta_p(T) = 1 \quad \forall p \leq n$.
- (iii) $\delta_1(T) = \delta_2(T) = 1$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are evident.

First proof of (iii) \Rightarrow (i). It is sufficient to verify that $T^*T = I$. For this it is enough to show that $\delta_1(T^*T - I) = 0$ (since δ_1 is a norm). Equalities (2.7) and (2.8) imply

$$\delta_2^2(T) = \frac{n}{n-1} \delta_1^4(T) - \frac{1}{n-1} \delta_1^2(T^*T) \quad (2.23)$$

and

$$\delta_1^2(T^*T - I) = \delta_1^2(T^*T) - 2\delta_1^2(T) + 1. \tag{2.24}$$

Now it is clear that (iii) and (2.23) imply $\delta_1(T^*T) = 1$. Thus the equality $\delta_1(T) = 1$ and (2.24) imply $\delta_1(T^*T - I) = 0$. Consequently, $T^*T = I$.

*Second proof of (iii) \Rightarrow (i).*² Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities. Using (iii) and (2.20) we get

$$\sum_{k=1}^n \lambda_k^2 = n\delta_1^2(T) = n, \quad \sum_{1 \leq j_1 < j_2 \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 = \binom{n}{2} \delta_2^2(T) = \binom{n}{2}.$$

Then

$$\sum_{k=1}^n \lambda_k^4 = \left(\sum_{k=1}^n \lambda_k^2 \right)^2 - 2 \sum_{1 \leq j_1 < j_2 \leq n} \lambda_{j_1}^2 \lambda_{j_2}^2 = n^2 - 2 \binom{n}{2} = n$$

and

$$\sum_{k=1}^n (\lambda_k^2 - 1)^2 = \sum_{k=1}^n \lambda_k^4 - 2 \sum_{k=1}^n \lambda_k^2 + n = 0.$$

Consequently, $\lambda_k = 1, k = 1, \dots, n$ and $|T| = I$. \square

Remark 2.9. It is interesting to note that in [6] only the validity of the implication (ii) \Rightarrow (i) was conjectured. We see that even (iii) implies (i).

The following assertion will be used in the next section.

Lemma 2.10. *Let $\dim H = n, T : H \rightarrow Y$ be a linear operator and $p \in \{1, \dots, n\}$. Put*

$$c_{n,p} := D_p(I) = \frac{n!}{(n-p)!n^p}$$

and

$$r_n(p, T) := D_1^p(T) - D_p(T).$$

Then:

$$\frac{1}{p^{p-1}} \leq c_{n,p} \leq 1, \tag{2.25}$$

$$\delta_p(T) \leq p^{\frac{p-1}{2}} \delta_1^p(T), \tag{2.26}$$

$$0 \leq r_n(p, T) \leq \frac{p!}{n} \|T\|^{2p}, \tag{2.27}$$

$$\delta_p^2(T) = \frac{1}{c_{n,p}} (\delta_1^{2p}(T) - r_n(p, T)). \tag{2.28}$$

²Suggested by the Referee.

Proof. Relation (2.25) is easy to verify. Inequality (2.26) follows from (2.6) and (2.25).

(2.27): Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities. Using (2.9), Newton’s (polynomial) formula and (2.10) we can write:

$$\begin{aligned} (D_1(T))^p - D_p(T) &= \frac{1}{n^p} \left(\sum_{k=1}^n \lambda_k^2\right)^p - \frac{p!}{n^p} \sum_{1 \leq j_1 < \dots < j_p \leq n} \lambda_{j_1}^2 \dots \lambda_{j_p}^2 \\ &= \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n} \\ &\quad - \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p, \max k_i = 1} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n} \\ &= \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p, \max k_i > 1} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n}. \end{aligned}$$

Consequently,

$$(D_1(T))^p - D_p(T) = \frac{p!}{n^p} \sum_{k_1 + \dots + k_n = p, \max k_i > 1} \frac{1}{k_1! \dots k_n!} \lambda_1^{2k_1} \dots \lambda_n^{2k_n}.$$

From this equality, as $\lambda_k \leq \|T\|$, $k = 1, \dots, n$, and

$$\sum_{k_1 + \dots + k_n = p, \max k_i > 1} \frac{1}{k_1! \dots k_n!} \leq n^{p-1},$$

we obtain (2.27).

(2.28) follows from (2.27). \square

Remark 2.11. (1) By (2.20) it can be seen that for a given operator T the equality $\delta_p(T) = \delta_1^p(T)$ holds if and only if T is a scalar multiplier of an isometry. In this connection the asymptotic versions behave better (see the next section).

(2) A simple relation between $\delta_p(T)$ and $\delta_1^p(T)$, can be written using the notation from the theory of outer products of Hilbert spaces. Namely, let H, Y be finite-dimensional Hilbert spaces, $T \in L(H, Y)$ and $\wedge^p T$ be the antisymmetric outer p -th power of the operator T . Then

$$\delta_p(T) = \binom{n}{p}^{-\frac{1}{2}} \|\wedge^p T\|_{\text{HS}} \text{ and } \delta_p(T) = \delta_1(\wedge^p T). \tag{2.29}$$

These formulas follow easily from the following relation, established in [5, p. 112–113, the proof of Prop. VI.2.2]: $\|\wedge^p T\|_{\text{HS}}^2 = \frac{n^p}{p!} D_p(T)$. In [5] it is also shown that the antisymmetric outer p -th power of a Hilbert–Schmidt (or a nuclear) operator T acting between infinite-dimensional Hilbert spaces is again a Hilbert–Schmidt (a nuclear) operator (cf. also [4, Th. 2]).

3. INFINITE-DIMENSIONAL CASE

In this section H will be an infinite-dimensional Hilbert space and Y will be another Hilbert space. As before, $L(H, Y)$ will denote the set of all continuous linear operators $T : H \rightarrow Y$.

Fix a natural number p and consider the collection \mathcal{M}_p of all finite-dimensional vector subspaces $M \subset H$ with $\dim M \geq p$. This collection is a directed set by set-theoretic inclusion.

Fix an operator $T \in L(H, Y)$. Then for any $M \in \mathcal{M}_p$ and the restriction $T|_M$ of T to M , the p MD-number $\delta_p(T|_M)$ is defined. In this way T generates with the net $(\delta_p(T|_M))_{M \in \mathcal{M}_p}$. Let us denote the upper limit of this net by $\overline{\delta}_p(T)$ and call it the *asymptotic upper p MD-number* of T . In a similar manner, let us denote the lower limit of this net by $\underline{\delta}_p(T)$ and call it the *asymptotic lower p MD-number* of T . Therefore

$$\overline{\delta}_p(T) := \limsup_{M \in \mathcal{M}_p} \delta_p(T|_M) \quad \text{and} \quad \underline{\delta}_p(T) := \liminf_{M \in \mathcal{M}_p} \delta_p(T|_M).$$

By formula (2.26) of Lemma 2.10 we have

$$\delta_p(T|_M) \leq p^{\frac{p-1}{2}} \|T\|^p, \quad \forall M \in \mathcal{M}_p.$$

This inequality shows that the net $(\delta_p(T|_M))_{M \in \mathcal{M}_p}$ is bounded and, consequently, always

$$\underline{\delta}_p(T) \leq \overline{\delta}_p(T) \leq p^{\frac{p-1}{2}} \|T\|^p < \infty.$$

In general, the net $(\delta_p(T|_M))_{M \in \mathcal{M}_p}$ may not be convergent (as we will see below). In the case of convergence we shall call the operator T *asymptotically p MD-regular*. For an asymptotically p MD-regular T we put

$$\overline{\delta}_p(T) = \lim_{M \in \mathcal{M}_p} \delta_p(T|_M),$$

and call $\overline{\delta}_p(T)$ the *asymptotic p MD-number* of T .

An asymptotically 1MD-regular operator T will be called *asymptotically MD-regular* and its 1MD-number will be called *asymptotic MD-number*.

The next result is somewhat unexpected. Its validity was not predicted in [6].

Theorem 3.1. *Let H be an infinite-dimensional Hilbert space, Y be a Hilbert space, $T \in L(H, Y)$ and $p > 1$ be a natural number. Then*

$$\overline{\delta}_p(T) = (\overline{\delta}_1(T))^p \quad \text{and} \quad \underline{\delta}_p(T) = (\underline{\delta}_1(T))^p; \tag{3.1}$$

moreover, the operator T is asymptotically p MD-regular if and only if it is asymptotically MD-regular and in the case of asymptotic MD-regularity the equality

$$\overline{\delta}_p(T) = (\overline{\delta}_1(T))^p \tag{3.2}$$

holds.

Proof. Let $M \in \mathcal{M}_p$. We can write using formula (2.28) of Lemma 2.10:

$$\delta_p^2(T|_M) = \frac{1}{p!} \binom{\dim M}{p}^{-1} (\dim M)^p \left(\delta_1^{2p}(T|_M) - r_{\dim M}(p, T|_M) \right). \quad (3.3)$$

Evidently,

$$\lim_{M \in \mathcal{M}_p} \frac{1}{p!} \binom{\dim M}{p}^{-1} (\dim M)^p = 1. \quad (3.4)$$

Since according to inequality (2.27) of Lemma 2.10

$$r_{\dim M}(p, T|_M) \leq \frac{p!}{\sqrt{\dim M}} \|T|_M\|^{2p} \leq \frac{p!}{\sqrt{\dim M}} \|T\|^{2p},$$

we also have

$$\lim_{M \in \mathcal{M}_p} r_{\dim M}(p, T|_M) = 0. \quad (3.5)$$

From (3.3) via (3.4) and (3.5) we get (3.1). The “moreover” part now is clear. \square

This theorem shows that in the case of an infinite-dimensional domain it is sufficient to study only the numbers $\overline{\delta}_1(T)$, $\underline{\delta}_1(T)$ and $\overline{\delta}_1(T)$.

To simplify the notation, for a given operator $T \in L(H, Y)$ let us denote its

- asymptotic upper 1MD-number $\overline{\delta}_1(T)$ by $\overline{\delta}(T)$,
- asymptotic lower 1MD-number $\underline{\delta}_1(T)$ by $\underline{\delta}(T)$,
- asymptotic MD-number $\overline{\delta}_1(T)$ by $\overline{\delta}(T)$.

Using formula (2.19) from Proposition 2.6 the definition of these numbers can be formulated directly in terms of the Hilbert–Schmidt norm as follows:

$$\overline{\delta}(T) = \limsup_{M \in \mathcal{M}_1} \frac{1}{\sqrt{\dim M}} \|T|_M\|_{HS}, \quad \underline{\delta}(T) = \liminf_{M \in \mathcal{M}_1} \frac{1}{\sqrt{\dim M}} \|T|_M\|_{HS}.$$

For an operator $T \in L(H, Y)$ let us put

$$m(T) := \inf \{ \|Tx\| : x \in H, \|x\| = 1 \}.$$

The number $m(T)$ is sometimes called the lower bound of T .

The next statement implies, in particular, that a given operator T may not be asymptotically MD-regular.

Proposition 3.2. *Let H, Y be infinite-dimensional Hilbert spaces and $T : H \rightarrow Y$ be a continuous linear operator. Then:*

- (a) *For any infinite-dimensional closed vector subspace $X \subset H$ the inequality*

$$\overline{\delta}(T) \geq m(T|_X)$$

holds.

- (b) *If $\ker(T)$ is infinite-dimensional, then $\underline{\delta}(T) = 0$.*

(c) If T is a partial isometry such that $\ker(T)$ and $T(H)$ are both infinite-dimensional, then $\overline{\delta}(T) = 1$, while $\underline{\delta}(T) = 0$ and so T is not asymptotically MD-regular.

Proof. (a) Fix an infinite-dimensional $X \subset H$, a finite-dimensional vector subspace $M \subset H$ and put

$$\beta_M := \sup\{\delta_1(T|_N) : N \in \mathcal{M}_1, N \supset M\}.$$

Let us show that

$$\beta_M \geq m(T|_X). \tag{3.6}$$

To prove (3.6), fix a natural number n and an n -dimensional vector subspace X_n of X such that $X_n \cap M = \{0\}$ (such a choice is possible because M is finite-dimensional and X is infinite-dimensional). Let also $M_n := M + X_n$ and let M' be the vector subspace of M_n orthogonal to X_n . Using formula (2.19) from Proposition 2.6 we can write:

$$\delta_1^2(T|_{M_n}) = \frac{\|T|_{M'}\|_{HS}^2 + \|T|_{X_n}\|_{HS}^2}{\dim(M') + n} \geq \frac{\|T|_{X_n}\|_{HS}^2}{\dim(M') + n} \geq \frac{n}{\dim(M') + n} m(T|_X)^2.$$

As $\beta_M \geq \delta_1(T|_{M_n})$ and $\dim(M') \leq \dim(M)$, we get

$$\beta_M^2 \geq \frac{n}{\dim(M) + n} m^2(T|_X). \tag{3.7}$$

Since n is arbitrary, from (3.7) we have

$$\beta_M^2 \geq \sup_n \frac{nm^2(T|_X)}{\dim(M) + n} \geq \lim_n \frac{nm^2(T|_X)}{\dim(M) + n} = m^2(T|_X).$$

This relation, together with (3.6) and the definition of $\overline{\delta}(T)$, implies (a).

(b) Let $X = \ker(T)$, fix a finite-dimensional vector subspace $M \subset H$ and put

$$\alpha_M := \inf\{\delta_1(T|_N) : N \in \mathcal{M}_1, N \supset M\}.$$

Let us show that

$$\alpha_M = 0. \tag{3.8}$$

To prove (3.8), fix a natural number n and an n -dimensional vector subspace X_n of X such that $X_n \cap M = \{0\}$ (such a choice is possible because M is finite-dimensional and X is infinite-dimensional). Let also $M_n := M + X_n$ and M' be the subspace of M_n orthogonal to X_n . Using formula (2.19) from Proposition 2.6 and taking into account that $T|_{X_n} = 0$ we can write:

$$\delta_1^2(T|_{M_n}) = \frac{\|T|_{M'}\|_{HS}^2 + \|T|_{X_n}\|_{HS}^2}{\dim(M') + n} = \frac{\|T|_{M'}\|_{HS}^2}{\dim(M') + n}.$$

From this we get

$$\alpha_M^2 \leq \frac{\|T|_{M'}\|_{HS}^2}{\dim(M') + n}.$$

Observe now that since $\dim(M') \leq \dim(M)$ and $\|T|_{M'}\|_{HS} \leq \|T\|\sqrt{\dim(M')}$, we have

$$\alpha_M^2 \leq \lim_n \frac{\|T|_{M'}\|^2}{\dim(M') + n} = 0.$$

This relation together with the definition of $\underline{\underline{\delta}}(T)$ implies (b).

(c) follows from (a) and (b). \square

The last proposition motivates the following

Problem. Give a characterization of asymptotically MD-regular operators in terms of some other known parameters (in terms of the spectrum, in terms of the diagonal (for the diagonal operators), etc.).

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REFERENCES

1. N. BOURBAKI, Espaces vectoriels topologiques. *Masson, Paris*, 1981.
2. A. CASTEJÓN, E. CORBACHO, and V. TARIELADZE, AMD numbers, compactness, strict singularity and essential spectrum of operators. Submitted to *Georgian Math. J.* in 2001.
3. E. CORBACHO, Hipervolumenes, su convergencia y sumabilidad en los operadores lineales acotados del espacio de Hilbert. *Ph.D. thesis, Zaragoza*, 1978.
4. S. GEISS, Antisymmetric tensor products of absolutely p -summing operators. *J. Approx. Theory* **68**(1992), No. 3, 223–246.
5. A. MARTÍNEZ, Normas tensoriales e ideales de operadores. *Ph.D. thesis, Madrid*, 1986.
6. J. L. RUBIO DE FRANCIA, Unpublished notes.
7. N. TOMCZAK-JAEGERMANN, Banach–Mazur distance and finite dimensional operator ideals. *Longman Scientific & Technical, Harlow*, 1989.

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Authors' address:

Universidad de Vigo
 Departamento de Matemática Aplicada
 36280 Vigo, Galicia, España
 E-mail: acaste@dma.uvigo.es
 corbacho@uvigo.es
 visit01@dma.uvigo.es

The permanent address of V. Tarieladze:
 N. Muskhelishvili Institute of Computational Mathematics
 Georgian Academy of Sciences
 8, Akuri St., Tbilisi 380093, Georgia
 E-mail: tar@gw.acnet.ge