Georgian Mathematical Journal
Volume 8 (2001), Number 3, 447–453

EULER POLYNOMIALS AND THE RELATED QUADRATURE RULE

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Dedicated to the Memory of Prof. Niko Muskhelishvili

Abstract. The use of Euler polynomials and Euler numbers allows us to construct a quadrature rule similar to the well-known Euler–MacLaurin quadrature formula, using Euler (instead of Bernoulli) numbers, and even (instead of odd) order derivatives of a given function evaluated at the extrema of the considered interval. An expression of the remainder term and numerical examples are also given.

2000 Mathematics Subject Classification. 33C45, 33F05, 65D32.
Key words and phrases. Euler polynomials, Euler numbers, Euler–MacLaurin quadrature formula, quadrature rules.

1. Introduction

Bernoulli polynomials $B_n(x)$ are usually defined (see, e.g., [1], p. xxix) starting from the generating function

$$G(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{1.1}$$

and, consequently, Bernoulli numbers $\beta_n := B_n(0)$ can be obtained by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \beta_n\,\frac{t^n}{n!}. \tag{1.2}$$

Bernoulli numbers (see [2]–[3]) are contained in many mathematical formulas such as the Taylor expansion in a neighborhood of the origin of trigonometric and hyperbolic tangent and cotangent functions, the sums of powers of natural numbers, the residual term of the Euler–MacLaurin quadrature formula.

Bernoulli polynomials, first studied by Euler (see [4]–[5]–[6]), are employed in the integral representation of differentiable periodic functions, and play an important role in the approximation of such functions by means of polynomials. They are also used in the remainder term of the composite Euler–MacLaurin quadrature formula (see [7]).

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\,\frac{t^n}{n!}, \qquad |t| < \pi. \tag{1.3}$$

Euler numbers $\mathcal{E}_n$ can be obtained by the generating function

$$\frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n}{n!} t^n, \tag{1.4}$$

and the connection between Euler numbers and Euler polynomials is given by

$$E_n\left(\frac{1}{2}\right) = 2^{-n}\mathcal{E}_n, \qquad n = 0, 1, 2, \dots .$$

The first Euler numbers are given by

$$\mathcal{E}_0 = 1,\ \mathcal{E}_1 = 0,\ \mathcal{E}_2 = -1,\ \mathcal{E}_3 = 0,\ \mathcal{E}_4 = 5,\ \mathcal{E}_5 = 0,\ \mathcal{E}_6 = -61,\ \mathcal{E}_7 = 0,\ \dots .$$

For further values see [8], p. 810.

Euler polynomials are strictly connected with Bernoulli ones, and are used in the Taylor expansion in a neighborhood of the origin of trigonometric and hyperbolic secant functions.

Recursive computation of Bernoulli and Euler polynomials can be obtained by using the following formulas:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = nx^{n-1}, \qquad n = 2, 3, \dots ; \tag{1.5}$$

$$E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) = 2x^n, \qquad n = 1, 2, \dots . \tag{1.6}$$

Further results can be found in a recent article by M. X. He and P. E. Ricci [9].

In this note, starting from the Euler polynomials, we construct a quadrature rule similar to the well-known Euler–MacLaurin quadrature formula, using Euler (instead of Bernoulli) numbers in the remainder term, and even (instead of odd) order derivatives of a given function evaluated at the extrema of the considered interval.

In a forthcoming paper the same procedure will be applied in a more general framework of Appell polynomials, including both the Euler–MacLaurin quadrature formula and the results of this article.

2. Definition and Simple Properties of Euler Polynomials

Euler polynomials $E_n(t)$ can also be defined as polynomials of degree $n \geq 0$ satisfying the conditions

(i) $E_m'(t) = mE_{m-1}(t), \quad m \geq 1$;

(ii) $E_m(t+1) + E_m(t) = 2t^m, \quad m \geq 1$.

Note that a polynomial set $E_m(t)$ satisfying (i) and (ii) is such that

$$E_m(t+1) = \sum_{k=0}^{m} \frac{1}{k!} m(m-1)\cdots(m-k+1)E_{m-k}(t)$$
$$= \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(t) = \sum_{s=0}^{m} E_s(t) = \sum_{s=0}^{m} \binom{m}{s} E_s(t),$$

and, consequently, from (ii), assuming $m \geq 1$,

$$2t^m = E_m(t+1) + E_m(t) = \sum_{k=0}^{m} E_k(t) + E_m(t)$$
$$= \sum_{k=0}^{m-1} \binom{m}{k} E_k(t) + 2E_m(t)$$

so that

$$E_m(t) = t^m - \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} E_m(t), \qquad m \geq 1, \tag{2.1}$$

which is equivalent to (1.6).

Then $E_m(t)$ are determined if we know $E_0(t)$ and $E_1(t)$. The latter are obtained by using the following procedure. Put $E_0(t) = a$ and $E_1(t) = at + b$; writing condition (ii), for $m = 1$, yields

$$2t = E_1(t+1) + E_1(t) = a(t+1) + b + at + b = 2at + a + 2b.$$

Now, equating the first and the last term in t, we find

$$a = 1 \qquad \text{and} \qquad b = -\frac{1}{2}$$

and, consequently,

$$E_0(t) = 1 \qquad \text{and} \qquad E_1(t) = t - \frac{1}{2}. \tag{2.2}$$

The above considerations show that a polynomial set $\{E_m(t)\}$ satisfying (i) and (ii), where $E_m(t)$ is of degree $m \geq 0$, is uniquely determined by the recurrence relation (2.1) with the initial conditions $E_0(t)$ and $E_1(t)$ given by (2.2). The existence of such a set of polynomials is trivial since the set defined by (2.1) and (2.2) satisfies conditions (i) and (ii).

Condition (i), for $m = 2$, gives

$$E_2'(t) = 2E_1(t),$$

and, after integrating term by term,

$$E_2(t) = t^2 - t,$$

and, for $m = 3$,

$$E_3'(t) = 3E_2(t),$$

and, after integrating,

$$E_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{4},$$

and so on.

Euler polynomials satisfy the symmetry relation

$$E_m(1-t) = (-1)^m E_m(t).$$

Since $E_m(0) = E_m(1) = 0$, for even m, we put, if m is odd:

$$e_m = E_m(0) = -E_m(1).$$

The connection of e_m with Euler numbers is given by the formula

$$e_m = -\frac{1}{2^m}\sum_{h=0}^{m}\binom{m}{h}\mathcal{E}_{m-h}.$$

3. The Quadrature Formula

The main result of this paper is expressed by the following theorem.

Theorem 3.1. *Consider a function $f(x) \in C^{2m}[a,b]$ and the corresponding integral over $[a,b]$. Let $x_i = a + ih$, $i = 0,1,\dots,n$, where $h := \frac{b-a}{n}$, and $f_i = f(x_i)$, $f_i^{(p)} = f^{(p)}(x_i)$, $p = 1,2,\dots,2m$. Then the following composite trapezoidal rule holds true:*

$$\int_a^b f(x)dx = h\left(\frac{1}{2}f(a) + f_1 + \cdots + f_{n-1} + \frac{1}{2}f(b)\right)$$
$$-\sum_{k=1}^{m-1}\frac{e_{2k+1}}{(2k+1)!}h^{2k+1}\left[f^{(2k)}(b) + f^{(2k)}(a)\right] + R_E[f], \tag{3.1}$$

where the correction terms are expressed by means of the even derivatives of the given funcion $f(x)$ at the extrema, and the error term is given by

$$R_E[f] = \frac{h^{2m}}{(2m)!}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f^{(2m)}(x)E_{2m}\left(\frac{x-x_i}{h}\right)dx. \tag{3.2}$$

Proof. We consider first the trapezoidal rule extended to the whole interval.

Put $f(x) = f(a+ht) = g(t)$, where $h := b-a$ denotes the length of a given interval, and $t = \frac{x-a}{h}$. By using successive integration by parts we find

$$\int_a^b f(x)dx = h\int_0^1 g(t)dt = h\int_0^1 g(t)dE_1(t)$$
$$= h\left\{[g(t)E_1(t)]_0^1 - \int_0^1 g'(t)E_1(t)dt\right\}$$

$$= h\left\{\frac{1}{2}[g(1)+g(0)] - \frac{1}{2}\left[g'(t)E_2(t)\right]_0^1 + \frac{1}{2}\int_0^1 g''(t)E_2(t)dt\right\}$$

$$= h\left\{\frac{1}{2}[g(1)+g(0)] - \frac{1}{6}\left[g''(t)E_3(t)\right]_0^1 - \frac{1}{6}\int_0^1 g'''(t)E_3(t)dt\right\}$$

$$= h\left\{\frac{1}{2}[g(1)+g(0)] - \frac{e_3}{6}\left[g''(1)+g''(0)\right] - \frac{1}{24}\left[g'''(t)E_4(t)\right]_0^1 \right.$$
$$\left. + \frac{1}{24}\int_0^1 g^{(iv)}(t)E_4(t)dt\right\}$$

$$= h\left\{\frac{1}{2}[g(1)+g(0)] - \frac{e_3}{6}\left[g''(1)+g''(0)\right] + \frac{1}{120}\left[g^{(iv)}(t)E_5(t)\right]_0^1 \right.$$
$$\left. - \frac{1}{120}\int_0^1 g^{(v)}(t)E_5(t)dt\right\} = \dots$$

$$= h\left\{\frac{1}{2}[g(1)+g(0)] - \sum_{k=1}^{m-1}\frac{e_{2k+1}}{(2k+1)!}\left[g^{(2k)}(1)+g^{(2k)}(0)\right] \right.$$
$$\left. + \frac{1}{(2m)!}\int_0^1 g^{(2m)}(t)E_{2m}(t)dt\right\}.$$

Recalling that $f(x) = g(t)$, the preceding formula yields

$$\int_a^b f(x)dx = h\left\{\frac{1}{2}[f(b)+f(a)] - \sum_{k=1}^{m-1}\frac{e_{2k+1}}{(2k+1)!}h^{2k}\left[f^{(2k)}(b)+f^{(2k)}(a)\right] \right.$$
$$\left. + \frac{h^{2m-1}}{(2m)!}\int_a^b f^{(2m)}(x)E_{2m}\left(\frac{x-a}{h}\right)dx\right\}. \tag{3.3}$$

Consider now the partition of the interval $[a, b]$ into n partial intervals by means of equidistant knots $x_i = a + ih$, $i = 0, 1, \dots, n$, where $h := \frac{b-a}{n}$, $f_i = f(x_i)$, $f_i^{(p)} = f^{(p)}(x_i)$, $p = 1, 2, \dots, 2m$. By (3.3) we have

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx = h\left(\frac{1}{2}f(a) + f_1 + \dots + f_{n-1} + \frac{1}{2}f(b)\right)$$
$$- \sum_{k=1}^{m-1}\frac{e_{2k+1}}{(2k+1)!}h^{2k+1}\left[f^{(2k)}(b)+f^{(2k)}(a)\right]$$
$$+ \frac{h^{2m}}{(2m)!}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f^{(2m)}(x)E_{2m}\left(\frac{x-x_i}{h}\right)dx.$$

Note that the remainder term of the above procedure is expressed by

$$R_E[f] = \frac{h^{2m}}{(2m)!} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f^{(2m)}(x) E_{2m}\left(\frac{x - x_i}{h}\right) dx. \quad \square$$

Remark 3.1. The above quadrature rule is similar to the well known Euler–MacLaurin quadrature formula, but the correction terms of the trapezoidal rule are expressed by the *even* (instead of odd) derivatives of the function f computed at the extrema of the interval $[a, b]$, and it makes use of Euler (instead of Bernoulli) numbers.

By using equations (3.1)–(3.3), it is possible to write explicitly summation formulas for values of an analytic function (in particular of a polynomial) on equidistant knots similar to those which appear in [10], pp. 158–159, formulas (5.8.11)–(5.8.15).

4. Numerical Examples

We consider here two simple examples of the application of formula (3.1).

Consider the elementary integral

$$I := \int_0^1 \frac{1}{1+x} dx = \ln 2 \simeq 0.69314718 \ldots .$$

By using (3.1) with $n = 90$ and $m = 6$ we have found numerical approximation

$$I \simeq 0.69315476 \ldots$$

so that an absolute error is less than 10^{-5}, and a relative error is less than 2×10^{-5}.

Consider now the integral

$$J := \int_1^2 \frac{e^x}{x} dx \simeq 3.0591166 \ldots .$$

By using (3.1) with $n = 90$ and $m = 7$ we have found numerical approximation

$$J \simeq 3.0591352 \ldots$$

so that an absolute error is less than 2×10^{-5}, and a relative error is less than 10^{-5}.

Acknowledgement

This paper was prepared (concluded) in the framework of the Italian National Group for Scientific Computation (G.N.C.S.) whose partial support we wish to acknowledge.

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(Received 10.03.2001)

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