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ON VOLTERRA TYPE SINGULAR INTEGRAL EQUATIONS

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Abstract. Conditions for the boundedness are established, and the norms of Volterra type one-dimensional integral operators with fixed singularities of first order in the kernel are calculated in the space L_2 with weight. Integral equations of second order, containing the said operators, are investigated. Conditions for the solvability and solution formulas are given.

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1. Introduction

It is the well-known fact that classical Volterra equations of second order with continuous kernels or potential type kernels are uniquely solvable in Lebesgue spaces. However, when the kernel has nonintegrable singularities, the Fredholm properties of equations largely depend on the functional spaces in which they are considered. The equations dealt with in this paper contain Volterra type integral operators with fixed singularities of first order. It is proved that these equations considered in the space L_2 with power weight with exponent β are uniquely solvable only when $\beta > 1/2$; for $\beta < 1/2$ they have negative indices, while for $\beta = 1/2$ the normal solvability property does not hold.

Since the considered integral operators have homogeneous kernels of the order -1, we can apply the Wiener–Hopf method and reduce the equations to boundary value problems of analytic functions in the Hardy class H_2. A more extensive application of this method is described in the monograph [3]. Integral equations with fixed singularities are treated in the monograph [2].

Boundedness and compactness criteria for a wide class of integral operators (this class contains the operators considered in this paper) are given in [5].

Interest in the equations considered in this paper is due to their application in the theory of computer-aided design of closed-circuit crushing and grinding processes and processes of granulation (see [1], Ch. 3, §1).

2. Boundedness

Let V_α and W_α be the integral operators defined by the equalities

$$(V_\alpha\varphi)(x) = \int_x^a \frac{x^{\alpha-1}}{y^\alpha}\,\varphi(y)\,dy, \quad x \in (0,1), \quad \alpha \in \mathbb{R}, \tag{1}$$

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$$(W_\alpha\varphi)(x)=\int_x^a \frac{(y-x)^{\alpha-1}}{y^\alpha}\,\varphi(y)\,dy,\quad x\in(0,1),\quad \alpha>0, \tag{2}$$

where $0<a<\infty$.

Denote by $L_{2,\beta}$, $\beta\in\mathbb{R}$, the Banach space of measurable functions on the interval $(0,a)$ with the norm

$$\|\varphi\|_{2,\beta}:=\Big(\int_0^a |x^\beta\varphi(x)|^2\,dx\Big)^{1/2}.$$

Lemma 1. *For the operator V_α to be bounded in the space $L_{2,\beta}$, it is necessary and sufficient that the inequality $\alpha+\beta>1/2$ be fulfilled, and at that we have*

$$\|V_\alpha\|_{2,\beta}=\frac{1}{\alpha+\beta-1/2}\,. \tag{3}$$

Proof. Necessity. Let χ be the characteristic function of the interval $(a/2,a)$. It is obvious that $\chi\in L_{2,\beta}$ for all $\beta\in\mathbb{R}$, while for $\alpha+\beta\le 1/2$ we have $V_\alpha\chi\notin L_{2,\beta}$.

Sufficiency. Let us introduce the operators

$$(Z\varphi)(t)=(a\cdot e^{-t})^{1/2+\beta}\varphi(a\cdot e^{-t}),\quad t\in\mathbb{R}^+,$$
$$(Z^{-1}\psi)(x)=x^{-(1/2+\beta)}\psi\Big(-\ln\frac{x}{a}\Big),\quad x\in(0,a).$$

Here Z and Z^{-1} are the isometric inverse operators acting from the space $L_{2,\beta}$ into the space $L_2(\mathbb{R}^+)$, and from the space $L_2(\mathbb{R}^+)$ into the space $L_{2,\beta}$.

Let $\alpha+\beta>1/2$ and

$$\widetilde{V}_\alpha=ZV_\alpha Z^{-1}. \tag{4}$$

After some simple transformations we obtain

$$(\widetilde{V}_\alpha\psi)(t)=\int_0^t e^{(1/2-\alpha-\beta)(t-\tau)}\psi(\tau)\,d\tau=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} v(t-\tau)\psi_+(\tau)\,d\tau,$$

where

$$v(t)=\begin{cases}\sqrt{2\pi}e^{(1/2-\alpha-\beta)t} & \text{for } t>0\\ 0 & \text{for } t\le 0\end{cases},\quad \psi_+(t)=\begin{cases}\psi(t) & \text{for } t>0\\ 0 & \text{for } t<0\end{cases}. \tag{5}$$

The operator $\widetilde{V}_\alpha$ is a convolution operator with kernel $v\in L_1(\mathbb{R})$, therefore it is bounded in the space $L_2(\mathbb{R}^+)$ and

$$\|\widetilde{V}_\alpha\|_2=\|\mathcal{F}v\|_\infty, \tag{6}$$

where $\mathcal{F}$ is the Fourier transform

$$(\mathcal{F}v)(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{it\tau}v(\tau)\,d\tau=\frac{i}{t-i(1/2-\alpha-\beta)}\,, \tag{7}$$

From equalities (4), (6) and (7) we obtain (3). □

Lemma 2. *For the operator W_α to be bounded in the space $L_{2,\beta}$, it is necessary and sufficient that $\beta > -1/2$, and at that we have*

$$\|W_\alpha\|_{2,\beta} = B\Big(\frac{1}{2} + \beta, \alpha\Big), \tag{8}$$

where $B(\cdot,\cdot)$ is a beta-function.

Proof. The *necessity* is proved like in the case of the operator V_α.

Sufficiency. Let $\beta > -1/2$,

$$\widetilde{W}_\alpha = Z W_\alpha Z^{-1}. \tag{9}$$

Then

$$(\widetilde{W}_\alpha\psi)(t) = \int_0^t e^{(1/2-\alpha-\beta)(t-\tau)}(e^{t-\tau}-1)^{\alpha-1}\psi(\tau)\,d\tau$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} w(t-\tau)\psi_+(\tau)\,d\tau,$$

where

$$w(t) = \begin{cases} \sqrt{2\pi}e^{(1/2-\alpha-\beta)t}(e^t-1)^{\alpha-1} & \text{for } t>0 \\ 0 & \text{for } t \le 0 \end{cases},$$

and from the condition $\alpha > 0$, $\beta > -1/2$ we obtain $w \in L_1(\mathbb{R})$. Hence $\|\widetilde{W}_\alpha\|_2 = \|\mathcal{F}w\|_\infty \le \frac{1}{\sqrt{2\pi}}\|w\|_1$, but since $w(t) \ge 0$, we have $\frac{1}{\sqrt{2\pi}}\|w\|_1 = (\mathcal{F}w)(0) \le \|\mathcal{F}w\|_\infty$, Therefore

$$\|\widetilde{W}_\alpha\|_2 = (\mathcal{F}w)(0). \tag{10}$$

On the other hand (see [4], p. 309, 3.251.(3)),

$$(\mathcal{F}w)(t) = \int_0^\infty e^{(1/2-\alpha-\beta+it)\tau}(e^\tau-1)^{\alpha-1}\,d\tau = B\Big(\frac{1}{2}+\beta-it, \alpha\Big), \quad t \in \mathbb{R}. \tag{11}$$

Equality (8) is obtained from (9), (10), (11). □

3. Equations

Let us consider the equation

$$\varphi(x) - \alpha\int_x^a \frac{(y-x)^{\alpha-1}}{y^\alpha}\varphi(y)\,dy = f(x), \quad 0 < x < a. \tag{12}$$

Applying the operator Z to both sides of equation (12) we obtain an equivalent equation in the space $L_2(\mathbb{R}^+)$

$$\psi(x) - \alpha(\widetilde{W}_\alpha\psi)(t) = g(t), \quad t > 0, \tag{13}$$

where $\widetilde{W}_\alpha$ is defined by equality (9), $\psi = Z\varphi$, $g = Zf$.

Equation (13) is a Wiener–Hopf equation. By extending its definition to a convolution equation along the entire axis we obtain

$$\psi_+(t) - \alpha(\widetilde{W}_\alpha \psi_+)(t) = g_+(t), \quad t \in \mathbb{R}, \tag{14}$$

where ψ_+ and g_+ are respectively the sought and the known functions which are defined by rule (5) (here use has been made of the obvious equality $(\widetilde{W}_\alpha \psi_+)(t)=0$ for $t < 0$).

Applying the Fourier transform to equation (14), we obtain an equivalent equation in the Hardy class H_2: $s(x)\Psi^+(t) = G^+(t)$, $t \in \mathbb{R}$, where Ψ_+ and G^+ are the tranformed Fourier functions ψ_+ and g_+, respectively; $s(t) = 1 - \alpha(\mathcal{F}w)(t)$ is a function from the class H_∞. Thus, this equation has a unique solution $\Psi^+(t) = s^{-1}(t)G^+(t)$ and, after restoring the solution of equation (12), we obtain

$$\varphi = Z^{-1}\mathcal{F}^{-1}s^{-1}\mathcal{F}Zf. \tag{15}$$

But for $\varphi \in L_{2,\beta}$, it is necessary and sufficient that the function $s^{-1}\mathcal{F}Zf$ belong to the Hardy class H_2, which, in turn, depends on whether the function $s(z)$ has zeros at $\operatorname{Im} z \geq 0$.

Let $\beta > 1/2$. By (8)

$$\|W_\alpha\|_{2,\beta} = B\Big(\frac{1}{2} + \beta, \alpha\Big) = \int\limits_0^1 x^{\beta-1/2}(1-x)^{\alpha-1}\,dx < \int\limits_0^1 (1-x)^{\alpha-1}\,dx = \frac{1}{\alpha}\,,$$

i.e., $\|\alpha W_\alpha\|_{2,\beta} < 1$.

Therefore the operator $I - \alpha W_\alpha$ is invertible in the space $L_{2,\beta}$. By virtue of the continuity, the solution will have the form of (15).

Let $\beta < 1/2$. Then by (11)

$$s\Big(i\Big(\frac{1}{2} - \beta\Big)\Big) = 1 - \alpha \int\limits_0^1 (1-x)^{\alpha-1}\,dx = 0$$

and for $s^{-1}\mathcal{F}Zf \in H_2$ it is necessary that $(\mathcal{F}Zf)(i(1/2-\beta)) = 0$, which is equivalent to condition

$$\int\limits_0^a f(x)\,dx = 0, \tag{16}$$

i.e., in this case the latter condition is the necessary one.

For $\beta = 1/2$, $s(0) = 0$ and, as is well-known, equation (12) will not be normally solvable.

Thus we have proved

Theorem. *Let $\beta > -1/2$. Then the integral equation* (12):

(a) *when $\beta > 1/2$, for any function $f \in L_{2,\beta}$ has a unique solution $\varphi \in L_{2,\beta}$ calculated by formula* (15);

(b) *for* $\beta < 1/2$, *to have a solution it is necessary that condition* (16) *be fulfilled. The unique solution* (*if it exists*) *is calculated by formula* (15);

(c) *for* $\beta = 1/2$ *is not normally solvable.*

Remark. For equation (12) to be solvable in the space $L_{2,\beta}$ for $\beta < 1/2$, condition (16) is not sufficient for arbitrary α. One can prove that for sufficiently large natural α, the equation $s(z) = 0$ has roots in the upper half-plane which are different from $z = i(1/2-\beta)$ (their number has order $\ln\alpha$). Thus, depending on α, equation (12) may have an arbitrarily large negative index.

Example 1. Let us consider equation (12) in greater detail for the case $\alpha = 2$.

Then equation (12) takes the form

$$\varphi(x) - 2\int_x^a \frac{y-x}{y^2}\,\varphi(y)\,dy = f(x), \quad 0 < x < 1,$$

and we obtain

$$s(t) = \frac{(t - i(1/2-\beta))(t + i(\frac{5}{2}+\beta))}{(t + i(1/2+\beta))(t + i(\frac{3}{2}+\beta))}, \quad s^{-1}(t) = 1 + s_0(t).$$

For $\beta > 1/2$ we have

$$(\mathcal{F}^{-1}s_0)(t) = \begin{cases} \dfrac{2\sqrt{2\pi}}{3}\Big(e^{(1/2-\beta)t} - e^{-(\frac{5}{2}+\beta)t}\Big) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases},$$

and from (15) it follows that

$$\varphi(x) = f(x) + \frac{2}{3x}\int_x^a f(y)\,dy - \frac{2x^2}{3}\int_x^a \frac{f(y)}{y^3}\,dy.$$

If $\beta < 1/2$, then

$$(\mathcal{F}^{-1}s_0)(t) = \begin{cases} -\dfrac{2\sqrt{2\pi}}{3}\,e^{-(\frac{5}{2}+\beta)t} & \text{if } t > 0 \\ -\dfrac{2\sqrt{2\pi}}{3}\,e^{(1/2-\beta)t} & \text{if } t < 0 \end{cases}$$

and we obtain

$$\varphi(x) = f(x) - \frac{2}{3x}\int_0^x f(y)\,dy - \frac{2x^2}{3}\int_x^a \frac{f(y)}{y^3}\,dy.$$

Example 2. Let now $\alpha + \beta > 1/2$. In the same space $L_{2,\beta}$ we consider the equation

$$\varphi - \alpha V_\alpha \varphi = f, \tag{17}$$

where V_α is the operator defined by (1).

For the symbol of (17) (see (7)) we have

$$s(t) = 1 - \alpha(\mathcal{F}v)(t) = \frac{t - i(1/2 - \beta)}{t - i(1/2 - \alpha - \beta)}\,.$$

After easy calculation from (15) we obtain a unique solution

$$\varphi(x) = f(x) + \frac{\alpha}{x}\int\limits_{x}^{a} f(y)\,dy, \;\; 0 < x < a.$$

Note that, when $\beta < 1/2$ for (17) to be solvable it is necessary and sufficient that condition (16) be fulfilled.

Example 3. When designing granulation processes, one deals with the integral equations (see [1], p. 142)

$$\varphi(x) - \alpha\int\limits_{0}^{x} \frac{(a-x)^{\alpha-1}}{(a-y)^{\alpha}}\,\varphi(y)\,dy = f(x), \;\; 0 < x < a, \;\; \alpha \in \mathbb{R}, \tag{18}$$

$$\varphi(x) - \alpha\int\limits_{0}^{x} \frac{(x-y)^{\alpha-1}}{(a-y)^{\alpha}}\,\varphi(y)\,dy = f(x), \;\; 0 < x < a, \;\; \alpha > 0. \tag{19}$$

As different from the above kernels which had integrable singularities at zero, the considered kernels in have analogous singularities at a. Hence it is natural to consider equation (18), (19) in the space $L^1_{2,\beta} = \{\varphi|\ (a-x)^{\beta}\varphi(x) \in L_2(0,a)\}$.

Applying a simple transformation $(C\varphi)(x) = \varphi(a-x)$, equations (18), (19) are reduced to the above-considered equations and therefore all the statements proved in Sections 2 and 3 hold for them as well with the only difference that the solution formulas have to be appropriately modified.

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