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PLURIREGULAR, PLURIGENERALIZED REGULAR EQUATIONS IN CLIFFORD ANALYSIS

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Abstract. Problems for pluriregular equations in Clifford analysis are solved effectively. They are related to the equations called polyharmonic, polywave, polyheat, harmonic-heat, harmonic-wave, wave-heat and harmonic-wave-heat equations which are considered here for the first time.

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§ 1. Elliptic and Plurielliptic Equations in Clifford Analysis

Equations, for which the boundary value problems (b.v.p.) and initial value problems are considered in what follows, can be obtained by the Dirac operator in Clifford analysis. That is why first some basic notions and definitions of Clifford algebra and Clifford analysis will be given (see, e.g., [1, 4]).

Let $R_{(n)}$, $R_{(n,n-1)}$ and $R^0_{(n)}$ $(n \geq 1)$ be Clifford algebras. These are associative algebras with generators $e_0, e_1, \dots, e_n$ and relations

$$\begin{gathered} e_0^2 = e_0, \quad e_j^2 = -e_0 \ \text{ for } \ j = 1, \dots, n-1, \\ e_j e_k + e_k e_j = 0 \ \text{ for } \ j,k = 1, \dots, n \ \text{ and } \ j \neq k, \\ e_n^2 = \begin{cases} -e_0 & \text{in the case of } \ R_{(n)}, \\ e_0 & \text{in the case of } \ R_{(n,n-1)}, \\ 0 & \text{in the case of } \ R^0_{(n)}, \end{cases} \end{gathered}$$

where e_0 is the identity element. The relations imply the existence of a basis $\{e_A\}$, $A \subseteq \{1, \dots, n\}$, of the form $e_A = e_{\alpha_1} \cdots e_{\alpha_k}$ for $A = \{\alpha_1, \dots, \alpha_k\}$, $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$, $e_\varnothing = e_0$. These algebras are 2^n-dimensional and noncommutative (for $n \geq 2$), and any element can be represented as

$$u = \sum_{A \subseteq \{1,\dots,n\}} u_A e_A, \quad u_\varnothing = u_0.$$

An element u is called vectorial if

$$u = \sum_{k=0}^{n} u_k e_k.$$

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For every u two conjugates are defined:

$$\bar{u} = \sum_A u_A \bar{e}_A, \quad \widetilde{u} = \sum_A u_A \widetilde{e}_A,$$

where $\bar{e}_0 = e_0$, $\bar{e}_j = \widetilde{e}_j = -e_j$, $j = 1, \dots, n$, and

$$\bar{e}_A = \bar{e}_{\alpha_k} \cdots \bar{e}_{\alpha_1} = (-1)^{k(k+1)/2} e_A, \quad \widetilde{e}_A = \bar{e}_{\alpha_1} \cdots \bar{e}_{\alpha_k} = (-1)^k e_A.$$

$R_{(1)}$ is the space of complex numbers, $R_{(1,0)}$ is called the space of double numbers, $R^0_{(1)}$ is the space of dual numbers.

Consider a modification of the Dirac operator [1],

$$\overline{\partial} = \sum_{k=0}^{n} \frac{\partial}{\partial x_k} e_k = \frac{\partial}{\partial x_0} e_0 + D, \quad \partial = \frac{\partial}{\partial x_0} e_0 - D, \quad x = \sum_{k=0}^{n} x_k e_k,$$

where D is the Dirac operator. In $R_{(n)}$, $R_{(n,n-1)}$ and $R^0_{(n)}$, one has respectively

$$\partial\overline{\partial} = \overline{\partial}\partial = \Delta_{(n)}, \quad \partial\overline{\partial} = \Delta_{(n-1)} - \frac{\partial^2}{\partial x_n^2}, \quad \partial\overline{\partial} = \Delta_{(n-1)},$$

where $\Delta_{(k)}$ $(k = n, n-1)$ is the Laplace operator with respect to the variables $x_0, \dots, x_k$. From these equalities it follows that the equations

$$\overline{\partial} u = 0, \quad \overline{\partial} u + \widetilde{u} h = 0, \quad h = const, \tag{1.1}$$

are elliptic in $R_{(n)}$ and hyperbolic in $R_{(n,n-1)}$.

Pluriregular and plurigeneralized regular equations in $R_{(n)}$ and $R_{(n,n-1)}$ are written as

$$\overline{\partial}^m u = 0, \tag{1.2}$$

$$P^m u = 0, \quad Pu = \overline{\partial} u + \widetilde{u} h. \tag{1.3}$$

If $u(x)$ is a solution of (1.2) or (1.3), then it is a solution of the polyharmonic or the polyHelmholtz equation in $R_{(n)}$ and the polywave equation or the polyKlein–Gordon equation in $R_{(n,n-1)}$:

$$\Delta^m u = 0,$$
$$(\Delta - |h|^2)^m u = 0, \quad u = u(x), \quad x = (x_0, \dots, x_n),$$

and

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right)^m u(x) = 0,$$
$$\left(\Delta - h^2 - \frac{\partial^2}{\partial t^2}\right)^m u(x) = 0, \quad x_n \equiv t.$$

1.1. B.V.P. for pluriregular and plurigeneralized regular functions in elliptic case. Let $u(x)$ be a regular function or generalized regular function, i.e., a solution of the equation defined by (1.1) in the space $R_{(n)}$. Then $u(x)$, $x = (x_0, \dots, x_n)$ is also a solution of the equation

$$\Delta u = 0, \quad \text{or} \tag{1.4}$$

$$\Delta u - |h|^2 u = 0, \tag{1.5}$$

where h is assumed to be a vectorial constant $h = \sum_{k=0}^{n} h_k e_k$.

Solutions of Dirichlet and Neumann problems for equation (1.4) in the half-space $x_n > 0$ can accordingly be represented as

$$u(x) = \frac{2x_n}{\omega_n} \int_{R^n} \frac{u(\xi)\, d\xi}{(r^2 + x_n^2)^{\frac{n+1}{2}}}, \tag{1.6}$$

$$u(x) = \frac{1}{(n-1)\omega_n} \int_{R^n} \frac{\varphi(\xi)\, d\xi}{(r^2 + x_n^2)^{\frac{n-1}{2}}}, \tag{1.7}$$

where $r^2 = (x_0 - \xi_0)^2 + \dots + (x_{n-1} - \xi_{n-1})^2$,

$$\frac{\partial u}{\partial \xi_n} = \varphi(\xi) \quad \text{for} \quad \xi_n = 0.$$

Here and below the boundary data are supposed to be sufficiently smooth and vanishing at infinity for the case of a half-space as a boundary.

Solutions of Dirichlet and Neumann problems for equation (1.4) in the ball $|x| < 1$ can respectively be represented as

$$u(x) = \frac{1}{\omega_n} \int_{|\xi|=1} \frac{(1 - |x|^2) u(\xi)\, dS_\xi}{|\xi - x|^{(n+1)}}, \tag{1.8}$$

$$u(x) = -\frac{1}{4\pi} \int_{|\xi|=1} f(\xi) \Big\{ \ln[1 - r\cos(x\xi) + |x - \xi|] - \frac{2}{|x - \xi|} \Big\} dS_\xi. \tag{1.9}$$

Dirichlet and Neumann problems for the Helmholtz equation. Let R_+^{n+1} be the half-space $x_n > 0$, $x = (x_0, \dots, x_{n-1})$, and let $u(x, x_n)$ be a real function. In this space the solution of (1.5), vanishing at infinity and satisfying on the boundary $x_n = 0$ the conditions

$$u(x, 0) = f(x) \ \text{ for the Dirichlet problem,} \tag{1.10}$$

$$\frac{\partial u}{\partial x_n} = \varphi(x) \ \text{ for the Neumann problem,} \tag{1.11}$$

can respectively be represented as

$$u(x,x_n)=\frac{(-1)^{m+1}}{\pi^m h}\frac{\partial}{\partial x_n}\frac{\partial^m}{\partial(x_n^2)^m}\int\limits_{R^n} f(y)e^{-|h|r}dy \ \text{ for } \ n=2m,$$
$$u(x,x_n)=\frac{(-1)^{m+1}i}{2\pi^m}\frac{\partial}{\partial x_n}\frac{\partial^m}{\partial(x_n^2)^m}\int\limits_{R^n} f(y)H_0^{(1)}(i|h|r)dy \ \text{ for } \ n=2m+1, \tag{1.12}$$

and in the case of (1.11) as

$$u(x,x_n)=\frac{(-1)^{m+1}}{\pi^m h}\frac{\partial^m}{\partial(x_n^2)^m}\int\limits_{R^n} \varphi(y)e^{-|h|r}dy \ \text{ for } \ n=2m,$$
$$u(x,x_n)=\frac{(-1)^{m+1}i}{2\pi^m}\frac{\partial^m}{\partial(x_n^2)^m}\int\limits_{R^n} \varphi(y)H_0^{(1)}(i|h|r)dy \ \text{ for } \ n=2m+1,$$

where $r^2=|x-y|^2+x_n^2$. For formulas (1.6)–(1.9) and (1.12) see, e.g., [4], $H_0^{(1)}$ is the Hankel function of zero order.

From these representations remarkable properties follow: if the given functions f, φ are odd with respect to some fixed variable x_k $(0\le k\le n-1)$, then

$$u(x,x_n)=0 \quad \text{for} \quad x_k=0, \quad x_n>0,$$

and if they are even, then

$$\frac{\partial u}{\partial x_k}=0, \quad x_k=0, \quad x_n>0.$$

The problems solved above for harmonic functions can be used to solve the corresponding problems for the regular equation (1.1).

Let $u(x)$ be a pluriregular function in $R_{(n)}$, i.e., a solution of the pluriregular equation

$$\overline{\partial}^m u=0, \quad m\ge 2. \tag{1.13}$$

Then $u(x)$ is also a solution of the polyharmonic equation

$$\Delta^m u=0, \tag{1.14}$$

where Δ is the Laplace operator with respect to $x_0,x_1,\dots,x_n$.

To consider a b.v.p. for (1.13), first of all we will consider a b.v.p. for (1.14). A solution of (1.14) can be represented as

$$u(x)=\sum_0^{m-1} x_n^k u_k(x) \tag{1.15}$$

or

$$u(x)=\sum_0^{m-1}(r^2-1)^k u_k(x), \quad r^2=|x|^2, \tag{1.16}$$

where $u_k(x)$ are harmonic functions.

Problem. Define the solution of (1.14) in the half-space $x_n > 0$ which vanishes at infinity and satisfies the conditions

$$\frac{\partial^k u}{\partial x_n^k} = f_k(x_0, \dots, x_{n-1}), \quad k = 0, 1, \dots, m-1, \quad \text{for } x_n = 0. \tag{1.17}$$

Solution. By these conditions, using (1.15), we will obtain the boundary conditions for u_k:

$$\begin{aligned} &u_0(x_0, \dots, x_{n-1}, 0) = f_0, \\ &u_1(x_0, \dots, x_{n-1}, 0) + \frac{\partial u_0}{\partial x_n} = f_1, \\ &2u_2 + 2\frac{\partial u_1}{\partial x_n} + \frac{\partial^2 u_0}{\partial x_n^2} = f_2, \end{aligned}$$

and so on. Then as the left-hand sides in (1.17) are the boundary data for harmonic functions, by (1.6) one can determine all of them. For instance, if $m = 2$, we obtain

$$u(x) = \frac{2(n+1)x_n^3}{\omega_{n+1}} \int\limits_{R^n} \frac{f_0(\xi)\, d\xi}{r^{n+3}} + \frac{2x_n^2}{\omega_{n+1}} \int\limits_{R^n} \frac{f_1(\xi)\, d\xi}{r^{n+1}}.$$

For any $m \geq 2$ $u(x)$ can be represented as

$$u(x) = \sum_{k=0}^{m-1} \frac{2}{\omega_{n+1} k!} x_n^k \sum_{l=0}^{k} (-1)^l C_k^l \frac{d^l}{dx_n^l} x_n \int\limits_{R^n} f_{k-l}(\xi) \frac{d\xi}{r^{n+1}};$$

here $r^2 = (x_0 - \xi_0)^2 + \dots + (x_{n-1} - \xi_{n-1})^2 + x_n^2$. These representations are also obtained in [4] by using the Fourier integral transform.

For the ball $|x| \leq 1$, formula (1.16) will be used.

Problem. Define a polyharmonic function $u(x)$ in the ball $|x| < 1$ with the boundary conditions

$$\left.\frac{\partial^k u}{\partial r^k}\right|_{r=1} = f_k(x), \quad k = 0, 1, \dots, m-1.$$

Solution. By force of (1.16) for harmonic functions $u_k(x)$, $k = 0, \dots, m-1$, we have the conditions:

$$\begin{aligned} &u_0(x) = f_0(x), \quad |x| = 1, \\ &\frac{\partial u}{\partial r} = 2u_1 + \frac{\partial u_0}{\partial r} = f_1 \quad \text{for} \quad r = 1, \\ &\frac{\partial^2 u}{\partial r^2} = 8u_2 + 2u_1 + 4\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_0}{\partial r^2} = f_2 \quad \text{and so on.} \end{aligned} \tag{1.18}$$

First we prove that if φ is a harmonic function, then $r\frac{\partial \varphi}{\partial r}$ is harmonic too. Indeed,

$$r\frac{\partial \varphi}{\partial r} = \sum_0^n \frac{\partial \varphi}{\partial x_k} x_k$$

and

$$\Delta\Big(r\,\frac{\partial\varphi}{\partial r}\Big)=2\sum_0^n\frac{\partial^2\varphi}{\partial x_k^2}=0.$$

Now by the induction method one can prove that if $r^{k-1}\frac{\partial^{k-1}\varphi}{\partial r^{k-1}}$ is harmonic, then $r^k\frac{\partial^k\varphi}{\partial r^k}$ is harmonic too, as

$$r^k\,\frac{\partial^k\varphi}{\partial r^k}=r\frac{\partial}{\partial r}\Big(r^{k-1}\,\frac{\partial^{k-1}\varphi}{\partial r^k}\Big)-(k-1)r^{k-1}\frac{\partial^{k-1}\varphi}{\partial r^{k-1}}\,.$$

For this reason (1.18) are boundary conditions for the harmonic functions

$$u_0(x),\quad 2u_1+r\,\frac{\partial u_0}{\partial r},\quad 8u_2+2u_1+4r\,\frac{\partial u_1}{\partial r}+r^2\,\frac{\partial^2u_0}{\partial r^2},\quad\text{and so on.}$$

Hence these harmonic functions are defined by (1.8), and $u_0,u_1,u_2,\dots$ are defined gradually and correspondingly, by (1.16) u is represented in quadratures. For instance, for a biharmonic function we have

$$u(x)=\frac{1}{2}(r^2-1)Pf_1-\frac{1}{2}(r^2-1)r\,\frac{\partial}{\partial r}\,Pf_0+Pf_0,$$

where

$$Pf\equiv\frac{1-r^2}{\omega_n}\int\limits_{|\xi|=1}\frac{f(\xi)\,d\xi}{|\xi-x|^{n+1}}\,,\qquad x\in R^{n+1}.$$

Problem. Consider equation (1.13) for $m=2$ and $n=2$, $u=u_0e_0-u_1e_1-u_2e_2-u_{12}e_1e_2$. Find its solution in the half-space $x_2>0$ vanishing at infinity and satisfying the conditions

$$u_0(x_0,x_1,0)=f_0(x_0,x_1),\qquad \frac{\partial u_2}{\partial x_2}\Big|_{x_2=0}=f_1(x_0,x_1),$$
$$u_1(x_0,x_1,0)=\varphi_0(x_0,x_1),\qquad \frac{\partial u_{12}}{\partial x_2}\Big|_{x_2=0}=\varphi_1(x_0,x_1).$$

Solution. Equation (1.13) can be rewritten as

$$\overline{\partial}u=F,\quad \overline{\partial}F=0,\quad F=F_0e_0-F_1e_1-F_2e_2-F_{12}e_1e_2.$$

It is clear that

$$F_1=\frac{\partial u_0}{\partial x_0}+\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2},\quad F_2=\frac{\partial u_{12}}{\partial x_2}-\frac{\partial u_0}{\partial x_1}+\frac{\partial u_1}{\partial x_0}.$$

Thus by the given conditions, F_1, F_2 are given for $x_2=0$. Then F, as a solution of $\overline{\partial}F=0$, is defined in the half-space $x_2>0$ and u is defined from the nonhomogeneous equation $\overline{\partial}u=F$ with the conditions $u_0=f_0$, $u_1=f_1$ for $x_2=0$.

Thus for the equation $\overline{\partial}^2u=0$ it is sufficient to have four conditions, i.e., for this equation one can solve all the above problems solved for the ball.

Now consider the plurigeneralized regular equation of mth order

$$P^mu=0,\tag{1.19}$$

where

$$Pu = \overline{\partial} u + \tilde{u}h, \quad h = \sum_0^n h_k e_k.$$

For $m = 2$ we have the bigeneralized regular equation.

As the solution of the equation $\overline{\partial} u + \tilde{u}h = 0$ with constant h is also a solution of the Helmholtz equation, from (1.19) one can obtain that u is also a solution of the polyHelmholtz equation:

$$(\Delta - |h|^2)^m u = 0 \tag{1.20}$$

which, for $m = 2$, is called the biHelmholtz equation.

Note that if u_k $(k=0, 1, \dots, m-1)$ are solutions of the equation $\Delta u_k - |h|^2 u_k = 0$, then

$$u = \sum_{k=0}^{m-1} x_n^k u_k(x)$$

is a solution of equation (1.20).

Hence the b.v.p. with the conditions

$$\frac{\partial^k u}{\partial x_n^k} = f_k(x_0, \dots, x_{n-1}), \quad x_n = 0, \quad k = 0, 1, \dots, m-1,$$

in the half-space $x_n > 0$ can be reduced to the Dirichlet problem for u_k $(k = 0, \dots, m-1)$ whose solution is represented in quadratures by (1.12). If we have the boundary conditions (Riquie)

$$\Delta^k u = f_k(x_0, \dots, x_{n-1}), \quad k = 0, \dots, m-1, \quad x_n = 0, \tag{1.21}$$

then equation (1.20) is represented in the form

$$(\Delta - |h|^2)^{m-1} u = F, \tag{1.22}$$

$$\Delta F - |h|^2 F = 0. \tag{1.23}$$

By conditions (1.21) one can define F for $x_n = 0$. Thus F is defined by (1.12). Then u is defined from (1.22) gradually.

It is interesting to consider the equation

$$\Delta(\Delta - |h|^2)u = 0, \quad u(x), \quad x = (x_0, \dots, x_n), \tag{1.24}$$

which can be called harmonic-Helmholtz equation.

Dirichlet Problem. Define a regular solution of (1.24) for $x_n > 0$, vanishing at infinity, by the conditions

$$u(x_0, \dots, x_{n-1}, 0) = \varphi(x_0, \dots, x_{n-1}), \quad \frac{\partial^2 u}{\partial x_n^2} = \psi(x_0, \dots, x_{n-1}), \quad x_n = 0. \tag{1.25}$$

Solution. Let

$$\Delta u = F(x), \tag{1.26}$$

$$\Delta F - |h|^2 F = 0, \quad x_n > 0. \tag{1.27}$$

Then by force of (1.25), (1.26) we get

$$F(x_0,\dots,x_{n-1},0)=\Delta\varphi(x_0,\dots,x_{n-1})+\psi(x_0,\dots,x_{n-1})\equiv f(x_0,\dots,x_{n-1}).$$

Thus with this condition F is defined as the solution of (1.27) by (1.12). Now with the condition $u(x_0,\dots,x_{n-1},0)=\varphi(x_0,\dots,x_{n-1})$, u can be defined effectively by (1.26).

As we see, all problems which are solved for the Helmholtz equation can be solved for equations (1.20), (1.24).

§ 2. Hyperbolic and Plurihyperbolic Equations in Clifford Analysis

First, consider the wave equation

$$\Delta u=\frac{\partial^2 u}{\partial t^2},\quad x_n\equiv t,\quad u=u(x,t),\ \ t>0,\ \ x\in R^n,\ \ n\geq 1. \tag{2.1}$$

Cauchy Problem. Find a solution $u(x,t)$, a function of the class $C^2(t>0)\cap C^1(t\geq 0)$, by the initial conditions

$$u(x,0)=\varphi(x),\quad \frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x),\quad x=(x_0,x_1,\dots,x_{n-1}), \tag{2.2}$$

where $\varphi,\psi\in L(R^n)$, i.e., are absolutely integrable.

Solution. Without loss of generality one can consider $u(x,0)=0$. Indeed, let $u_1(x,t)$, $u_2(x,t)$ be a solution of (2.1) with the conditions

$$\begin{aligned} u_1(x,0)&=0,\quad \frac{\partial u_1}{\partial t}\Big|_{t=0}=\psi(x),\quad x\in R^n,\\ u_2(x,0)&=0,\quad \frac{\partial u_2}{\partial t}\Big|_{t=0}=\varphi(x). \end{aligned} \tag{2.3}$$

Then it is easy to see that the function

$$u(x,t)=u_1(x,t)+\frac{\partial u_2}{\partial t} \tag{2.4}$$

is a solution of the problem with conditions (2.2) because by the condition $u_2(x,0)=0$ and (2.1) one has $\dfrac{\partial^2 u_2}{\partial t^2}\Big|_{t=0}=0$. Thus we can consider equation (2.1) with conditions (2.3).

Using the Fourier integral transform with respect to the variables $x_0,x_1,\dots,x_{n-1}$ of equation (2.1) and conditions (2.3) one can obtain [2, 4]

$$u(x,t)=\frac{1}{2\pi^m}\,\frac{d^{m-1}}{d(t^2)^{m-1}}\left[t^{n-1}\int\limits_{|y|\leq 1}\frac{\psi(x-ty)}{\sqrt{1-|y|^2}}\,dy\right]\ \text{ for }\ n=2m, \tag{2.5}$$

$$u(x,t)=\frac{1}{4\pi^m}\,\frac{d^{m-1}}{d(t^2)^{m-1}}\left[t^{n-2}\int\limits_{|y|=1}\psi(x-ty)ds_y\right]\ \text{ for }\ n=2m+1. \tag{2.6}$$

For the above representations for $u(x,t)$ to be a regular solution one can suppose that $\psi(x)$ is a function with continuous partial derivatives of order up to $(n+1)/2$ when n is odd and of order up to $(n+2)/2$ when n is even.

Cauchy problem for the Klein–Gordon equation. Find a regular solution of the equation

$$\Delta u - h^2 u = \frac{\partial^2 u}{\partial t^2}, \quad x \in R^n, \quad t > 0, \tag{2.7}$$

by the conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t} = \psi(x) \quad \text{for} \quad t = 0. \tag{2.8}$$

For equation (2.7) the problem with conditions (2.2) can be solved as for (2.4) if we know its solution with conditions (2.8).

The solution can be represented as

$$u(x,t) = \frac{1}{2\pi^m} \frac{d^{m-1}}{d(t^2)^{m-1}} \int\limits_{|y|\le t} \psi(x-y) \frac{\cos h\sqrt{t^2-\rho^2}}{\sqrt{t^2-\rho^2}}\, dy, \quad n = 2m, \tag{2.9}$$

$$u(x,t) = \frac{1}{2\pi^m} \frac{d^{m}}{d(t^2)^{m}} \int\limits_{|y|\le t} \psi(x-y) J_0\Big(h(\sqrt{t^2-|y|^2}\,)\Big) dy, \quad n = 2m+1 \tag{2.10}$$

(see [3], [5]), where J_0 is the Bessel function of zero order.

Let $u(x,t) : R^{n+1} \to R_{(n,n-1)}$ be a solution of the equation

$$\overline{\partial} u = 0. \tag{2.11}$$

Cauchy Problem. Find a regular solution $u(x,t)$ of (2.11), with values in $R_{(n,n-1)}$ for $x = (x_0, \dots, x_{n-1}) \in R^n$ and $x_n \equiv t \ge 0$, subject to the condition

$$u(x,0) = \varphi(x), \tag{2.12}$$

where the given function $\varphi(x)$ with values in $R_{(n,n-1)}$ has continuous partial derivatives of required order.

Solution. Condition (2.12) gives that

$$\frac{\partial u}{\partial x_k} = \frac{\partial \varphi}{\partial x_k} \quad \text{for} \quad t = 0, \quad k = 0, 1, \dots, n-1.$$

By equation (2.11) we can derive

$$\frac{\partial u}{\partial t} = -e_n \sum_{k=0}^{n-1} e_k \frac{\partial u}{\partial x_k} \quad \text{for} \quad t = 0. \tag{2.13}$$

As u is also a solution of the wave equation, due the given Cauchy conditions (2.12), (2.13) it can be represented explicitly in quadratures by (2.5), (2.6).

Consider the hyperbolic h-regular equation

$$\overline{\partial} u + \tilde{u} h = 0, \tag{2.14}$$

where h is a vectorial constant. Then $u(x)$ is also a solution of the Klein–Gordon equation

$$\Delta u - |h|^2 u = \frac{\partial^2 u}{\partial x_n^2}, \quad n \geq 1, \quad x_n \equiv t \geq 0. \tag{2.15}$$

Cauchy initial value problem. Define a regular solution $u(x,t)$ of (2.14) with values in $R_{(n,n-1)}$ for $x \in R^n$ subject to the condition

$$u(x,0) = \varphi(x), \quad x = (x_0, \dots, x_{n-1}). \tag{2.16}$$

Solution. By (2.16) the following quantities are given:

$$\frac{\partial u}{\partial x_k} = \frac{\partial \varphi}{\partial x_k} \quad \text{for} \quad t = 0, \quad k = 0, 1, \dots, n-1.$$

Then from (2.14) it follows that

$$e_n \frac{\partial u}{\partial t} = -\sum_{k=0}^{n-1} e_k \frac{\partial \varphi}{\partial x_k} - \tilde{\varphi} h \quad \text{for} \quad t = 0. \tag{2.17}$$

As u is at the same time a solution of (2.15), by (2.16), (2.17) it is represented explicitly in form (2.9), (2.10).

If we consider the nonhomogeneous equation

$$\overline{\partial} u + \tilde{u} h = f(x,t),$$

with the condition

$$u(x,0) = 0,$$

then the solution can be defined by the representation

$$u(x,t) = \int_0^t v(x,t,\tau)\, d\tau,$$

where $v(x,t,\tau)$, $t > \tau$, is a solution of (2.14) with the condition

$$v(x,\tau,\tau) = e_n f(x,\tau).$$

For some hyperbolic systems the solvability of Cauchy problem was studied in [5].

2.1. Boundary-initial value problems for pluriregular, plurigeneralized regular hyperbolic systems, polywave and polyKlein–Gordon equations. Let $u(x,t) : R^{n+1} \to R_{(n,n-1)}$ and consider a higher order equation

$$\overline{\partial}^m u = 0, \quad m \geq 1. \tag{2.18}$$

It is clear that $u(x,t)$ is also a solution of the polywave equation

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right)^m u = 0 \tag{2.19}$$

called the plurihyperbolic equation.

(2.18) for $m = 2$ is called the biregular system and (2.19) the biwave equation. Consider the biwave equation of another form

$$\Big(\Delta - a^2 \frac{\partial^2}{\partial t^2}\Big)\Big(\Delta - b^2 \frac{\partial^2}{\partial t^2}\Big)u = 0. \tag{2.20}$$

This equation has application in the theory of elasticity. Indeed, consider the moving equations of an isotropic elastic body in displacement coordinates [3]:

$$\begin{gathered}(\lambda + \mu)\frac{\partial \theta}{\partial x_k} + \mu\Delta u_k = \rho \frac{\partial^2 u_k}{\partial t^2}, \quad k = 0, 1, 2,\\ \theta = \frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2};\end{gathered} \tag{2.21}$$

then from (2.21) follows

$$\begin{gathered}\Delta\theta = \frac{\rho}{\lambda + 2\mu}\frac{\partial^2 \theta}{\partial t^2},\\ (\lambda + \mu)\frac{\partial \Delta\theta}{\partial x_k} + \mu\Delta\Delta u_k - \rho\frac{\partial^2 \Delta u_k}{\partial t^2} = 0,\\ (\lambda + \mu)\frac{\partial^3 \theta}{\partial x_k \partial t^2} + \mu\frac{\partial^2 \Delta u_k}{\partial t^2} = \rho\frac{\partial^4 u_k}{\partial t^4},\end{gathered}$$

i.e., one has

$$\Delta\Delta u_k - \frac{\rho}{\mu}\frac{\lambda + 3\mu}{\lambda + 2\mu}\frac{\partial^2 \Delta u_k}{\partial t^2} + \frac{\rho^2}{\mu(\lambda + 2\mu)}\frac{\partial^4 u_k}{\partial t^4} = 0$$

which can be rewritten as

$$\Big(\Delta - a^2 \frac{\partial^2}{\partial t^2}\Big)\Big(\Delta - b^2 \frac{\partial^2}{\partial t^2}\Big)u = 0,$$

where $a^2 = \frac{\rho}{\mu}$, $b^2 = \frac{\rho}{\lambda+2\mu}$.

Cauchy problem for (2.20). Define a regular solution $u(x, t)$ of equation (2.20) for $x = (x_0, \dots, x_{n-1}) \in R^n$, $t \equiv x_n \geq 0$, with the conditions

$$u(x, 0) = f_0(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = f_1(x), \quad \frac{\partial^2 u}{\partial t^2}\Big|_{t=0} = f_2(x), \quad \frac{\partial^3 u}{\partial t^3}\Big|_{t=0} = f_3(x). \tag{2.22}$$

Solution. By these conditions, for $t = 0$ we can define

$$v(x, t) \equiv \Delta u - b^2 \frac{\partial^2 u}{\partial t^2} \quad \text{and} \quad \frac{\partial v}{\partial t} = \frac{\partial \Delta u}{\partial t} - b^2 \frac{\partial^3 u}{\partial t^3}.$$

Thus for the equation

$$\Delta v - a^2 \frac{\partial^2 v}{\partial t^2} = 0$$

we have the Cauchy problem whose solution is given as (2.5), (2.6). Then for the equation

$$\Delta u - b^2 \frac{\partial^2 u}{\partial t^2} = v(x, t)$$

we have the Cauchy problem and its solution is again represented in quadratures.

In such a way the Cauchy problem can be solved for the equation

$$\Big(\Delta - a_1^2\,\frac{\partial^2}{\partial t^2}\Big)\cdots\Big(\Delta - a_m^2\,\frac{\partial^2}{\partial t^2}\Big)u = 0$$

with the conditions

$$u(x,0) = f_0(x), \quad \frac{\partial u}{\partial t} = f_1(x), \dots, \frac{\partial^{2m-1}u}{\partial t^{2m-1}} = f_{2m-1} \quad \text{for } t = 0. \tag{2.23}$$

Now consider (2.18) in the case $m = 2$.

Cauchy problem for (2.18). Let $u(x,t) : R^{n+1} \to R_{(n,n-1)}$. Define regular solution of (2.18) with the conditions

$$u(x,0) = f_0(x), \quad \frac{\partial u}{\partial t} = f_1(x) \quad \text{for } t = 0. \tag{2.24}$$

Solution. By these conditions one can define $\overline{\partial}u$ for $t = 0$; thus we have the Cauchy problem (2.12) for equation (2.11) which is already solved. Then we obtain a nonhomogeneous equation with the condition $u(x,0) = f_0(x)$. Representing the solution in the form

$$u(x,t) = \overset{1}{u}(x,t) + \overset{2}{u}(x,t),$$

where $\overset{1}{u}(x,t)$ is a solution of the homogeneous equation (2.11) with the nonhomogeneous condition $\overset{1}{u}(x,0) = f_0(x)$, and $\overset{2}{u}(x,t)$ is a solution of the nonhomogeneous equation with the homogeneous condition $\overset{2}{u}(x,0) = 0$, hence both $\overset{1}{u}$, $\overset{2}{u}$ can be defined in quadratures.

Now consider the plurigeneralized regular equation of mth order:

$$P^m u = 0, \quad Pu = \overline{\partial}u + \tilde{u}h, \quad h = \sum_0^n h_k e_k.$$

In the particular case, for $m = 2$ we have a bigeneralized regular equation which can be written as

$$\overline{\partial}(\overline{\partial}u + \tilde{u}h) + (\partial\tilde{u} + u\bar{h})h = 0. \tag{2.25}$$

$u(x,t)$ will also be a solution of the biKlein–Gordon equation:

$$\Big(\Delta - |h|^2 - \frac{\partial^2}{\partial t^2}\Big)^2 u = 0.$$

If $m > 2$, one can obtain the polyKlein–Gordon equation

$$\Big(\Delta - |h|^2 - \frac{\partial^2}{\partial t^2}\Big)^m u = 0.$$

The Cauchy problem for these equations is posed as (2.22) and (2.23) and using (2.9), (2.10) the solution is represented in quadratures. The Cauchy problem for equation (2.25) with conditions (2.24) is also solved as (2.24) for equation (2.18). For any m one can solve the Cauchy problem in the same way.

2.2. Plurielliptic-hyperbolic equation. Consider the equation

$$\overline{\partial}^m \Big(\overline{\partial} + \frac{\partial}{\partial t}\, e_n\Big)^m u(x,t) = 0, \quad m \geq 1, \tag{2.26}$$

where

$$\overline{\partial} = \sum_0^{n-1} \frac{\partial}{\partial x_k}\, e_k, \quad e_k^2 = -e_0, \quad k = 1, \dots, n-1, \quad e_n^2 = e_0.$$

This equation is called the plurielliptic-hyperbolic equation. It is clear that $u(x,t)$ is at the same time a solution of the equation

$$\Delta^m \Big(\Delta - \frac{\partial^2}{\partial t^2}\Big)^m u(x,t) = 0,$$

which can be called the polyharmonic-wave equation.

Like of biharmonic, biwave equations it is interesting to consider $m = 1$, i.e.,

$$\Delta\Big(\Delta - \frac{\partial^2}{\partial t^2}\Big) u(x,t) = 0, \quad x = (x_0, \dots, x_{n-1}). \tag{2.27}$$

called harmonic-wave equation. The following problems are correctly posed and are solved in quadratures.

Dirichlet–Cauchy problem. Define a regular solution of (2.27) for $t > 0$, $x_{n-1} > 0$, vanishing at infinity, by the conditions

$$u(x,0) = \varphi_1(x), \quad \frac{\partial u}{\partial t} = \varphi_2(x), \quad t = 0, \tag{2.28}$$

$$u(x,t) = \varphi(x_0, \dots, x_{n-2}, t), \quad x_{n-1} = 0, t > 0. \tag{2.29}$$

Solution. Let

$$\Delta u(x,t) = F(x,t), \tag{2.30}$$

$$\Delta F - \frac{\partial^2 F}{\partial t^2} = 0, \tag{2.31}$$

then by force of (2.28), (2.30) the unknown function F satisfies

$$F(x,0) = \Delta\varphi_1(x) \equiv f_1(x), \quad \frac{\partial F}{\partial t} = \Delta\varphi_2(x) \equiv f_2(x), \quad t = 0,$$

i.e., to determine F we have the Cauchy initial value problem for the wave equation and it is represented in quadratures as above.

To define $u(x,t)$ we have the Dirichlet problem for the nonhomogeneous equation (2.30) with condition (2.29). The solution is given above too.

Neumann–Cauchy problem. Find a solution of (2.27) for $t > 0$, $x_{n-1} > 0$, vanishing at infinity and satisfying conditions (2.28) and

$$\frac{\partial u}{\partial x_{n-1}} = \varphi(x_0, \dots, x_{n-2}, t), \quad x_{n-1} = 0, \quad t > 0. \tag{2.32}$$

The solution can be reduced to the Neumann problem for equation (2.30) and is represented in quadratures.

It is obvious that all problems which are considered for harmonic functions can be considered here in a similar manner.

In the same way, the harmonic-Klein–Gordon equation

$$\Delta\Big(\Delta - k^2 - \frac{\partial^2}{\partial t^2}\Big)u(x,t) = 0$$

can be considered, for which problems (2.28), (2.29) or (2.30) can be solved.

It is clear that this equation is connected with the elliptic regular and hyperbolic generalized regular equation

$$\overline{\partial}\Big(\overline{\partial}u + e_n\frac{\partial u}{\partial t} + \tilde{u}h\Big) = 0.$$

Moreover, we can consider the Helmholtz-wave equation or the Helmholtz–Klein–Gordon equation

$$(\Delta - k_1^2)\Big(\Delta - \frac{\partial^2}{\partial t^2}\Big)u(x,t) = 0,$$
$$(\Delta - k_1^2)\Big(\Delta - k_2^2 - \frac{\partial^2}{\partial t^2}\Big)u(x,t) = 0.$$

For such equations problems (2.28), (2.29) or (2.32) are correctly posed and can be solved in quadratures too.

For equation (2.26), boundary-initial value problems can be considered similarly, for instance, in the case $m = 1$

$$\overline{\partial}\Big(\overline{\partial} + \frac{\partial}{\partial t}e_n\Big)u(x,t) = 0,$$

which can be rewritten as

$$\overline{\partial}u = F(x,t), \tag{2.33}$$

$$\Big(\overline{\partial} + \frac{\partial}{\partial t}e_n\Big)F = 0. \tag{2.34}$$

It is clear that the corresponding problems for (2.33) and (2.34) will be correctly posed and solved in quadratures.

The boundary-initial value problems for nonhomogeneous equations corresponding to the above homogeneous equations will be solved too. In this case the boundary-initial conditions can be supposed homogeneous. We will consider only one of them since others can be solved in the same way.

Problem. Define a solution of the equation

$$\Delta\Big(\Delta - \frac{\partial^2}{\partial t^2}\Big)u(x,t) = F(x,t),$$
$$x = (x_0,\dots,x_{n-1}) \in R^n, \quad t > 0, \quad x_{n-1} > 0,$$

which vanishes at infinity and satisfies the conditions:

$$u(x,0)=0, \quad \frac{\partial u}{\partial t}=0, \quad t=0, \tag{2.35}$$

$$u(x,t)=0, \quad x_{n-1}=0, \quad t>0. \tag{2.36}$$

Solution. Let

$$\Delta u = F_1(x,t), \tag{2.37}$$

$$\Delta F_1 - \frac{\partial^2 F_1}{\partial t^2} = F(x,t), \tag{2.38}$$

then by force of (2.35) for $F_1(x,t)$ one has Cauchy homogeneous conditions for the nonhomogeneous wave equation (2.38), thus the solution will be given in quadratures. Hence $u(x,t)$ is a solution of (2.37) with condition (2.36).

§ 3. Pluriparabolic Equations in Clifford Analysis

Systems of parabolic equations are related to heat and polyheat equations. First consider the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \quad u=u(x,t), \quad x_n \equiv t, \quad t>0, \quad x \in R^n, \quad n \geq 1, \tag{3.1}$$

where Δ is the Laplace operator with respect to $x_0, \dots, x_{n-1}$, $x=(x_0,\dots,x_{n-1})$.

Cauchy problem. The solution of (3.1) with the condition

$$u(x,0)=\varphi(x)$$

can be represented in the form

$$u(x,t)=\frac{1}{(2\sqrt{\pi t})^n}\int\limits_{R^n}\varphi(y)\exp\Big[-\frac{|x-y|^2}{4t}\Big]dy. \tag{3.2}$$

It is known as Poisson's formula.

Now the Cauchy problem for the nonhomogeneous equation

$$\frac{\partial u}{\partial t} = \Delta u + f(x,t) \tag{3.3}$$

can be solved easily. Indeed, if $v(x,t,\tau)$ is a solution of the homogeneous equation (3.1) for $t>\tau$, $x \in R^n$, with the condition

$$v(x,\tau,\tau)=f(x,\tau)$$

then the solution of (3.3) with the condition

$$u(x,0)=0$$

can be defined as

$$u(x,t)=\int\limits_0^t v(x,t,\tau)d\tau$$

i.e., by force of (3.2) it is represented as

$$u(x,t)=\frac{1}{(2\sqrt{\pi})^n}\int_0^t\frac{d\tau}{(t-\tau)^{n/2}}\int_{R^n}f(y,\tau)\exp\Big[-\frac{|x-y|^2}{4(t-\tau)}\Big]dy. \tag{3.4}$$

Consider Clifford algebras $R^0_{(n)}$ $(n\geq 1)$ and the equation [4]

$$\overline{\partial}u-P_nu=0,\quad u(x)=u(x_0,x_1,\dots,x_n),\quad x_n\equiv t, \tag{3.5}$$

where the linear operator P_n is defined by the relation

$$\partial P_nu=\frac{\partial u}{\partial t}, \tag{3.6}$$

i.e., the solution of (3.5) is a solution of the heat equation (3.1) too. $u(x)$ can be represented in the form

$$u(x)=\sum_{A\subseteq\{1,\dots,n-1\}}u_Ae_A+\sum_{A\subseteq\{1,\dots,n-1\}}u_{A\cup\{n\}}e_Ae_n. \tag{3.7}$$

Let $u(x):R^{n+1}\to R^0_{(n)}$ be a solution of equation (3.5), where

$$P_nu=-\sum_{A\subseteq\{1,\dots,n-1\}}(-1)^{|A|}u_{A\cup\{n\}}e_A \tag{3.8}$$

and $|A|$ stands for the cardinality of a set A. Then (3.6) takes place. Moreover, by (3.6) P_nu is defined uniquely.

Equation (3.5) with (3.7), (3.8) can be rewritten as

$$\sum_{\substack{j=0\\A\subseteq\{1,\dots,n-1\}}}^{n-1}\frac{\partial u_A}{\partial x_j}e_je_A+\sum_{A\subseteq\{1,\dots,n-1\}}(-1)^{|A|}u_{A\cup\{n\}}e_A=0, \tag{3.9}$$

$$\begin{aligned}&\sum_{A\subseteq\{1,\dots,n-1\}}\frac{\partial u_A}{\partial x_n}e_A+\sum_{A\subseteq\{1,\dots,n-1\}}(-1)^{|A|}\frac{\partial u_{A\cup\{n\}}}{\partial x_0}e_A\\&\quad-\sum_{\substack{j=1\\A\subseteq\{1,\dots,n-1\}}}^{n-1}(-1)^{|A|}\frac{\partial u_{A\cup\{n\}}}{\partial x_j}e_je_A=0.\end{aligned} \tag{3.10}$$

Let u satisfy (3.9) and each of u_A, $A\subseteq\{1,\dots,n-1\}$, be a solution of the heat equation (3.1), then u is a solution of (3.10) too (cf. [4]).

Cauchy problem. Find a regular solution of (3.5) for $x=(x_0,\dots,x_{n-1})\in R^n$, $t>0$ with 2^{n-1} initial conditions

$$u_A(x,0)=\varphi_A(x),\quad A\subseteq\{1,\dots,n-1\}. \tag{3.11}$$

Solution. As each u_A is at the same time a solution of the heat equation, by conditions (3.11) all u_A, $A\subseteq\{1,\dots,n-1\}$, are defined by (3.2). The remaining 2^{n-1} unknowns $u_{A\cup\{n\}}$, $A\subseteq\{1,\dots,n-1\}$, are defined from equation (3.9). Then u_A, $u_{A\cup\{n\}}$ satisfy (3.10) too.

If conditions (3.11) are replaced by

$$u_{A\cup\{n\}}(x,0)=\varphi_A(x),\quad A\subseteq\{1,\dots,n-1\},\quad \text{for}\ \ x\in R^n, \tag{3.12}$$

then $u_{A\cup\{n\}}(x,t)$, $t>0$, $A\subseteq\{1,\dots,n-1\}$, is represented by (3.2). Then by equation (3.10) the derivatives $\dfrac{\partial u_A}{\partial t}$ are defined for each $A\subseteq\{1,\dots,n-1\}$, i.e., all u_A, $A\subseteq\{1,\dots,n-1\}$, can be represented as

$$u_A(x,t)=f_A(x,t)+u'_A(x),\quad A\subseteq\{1,\dots,n-1\},$$

where $f_A(x,t)$ are known and $u'_A(x)$, $x\in R^n$, must be defined by equation (3.9). By putting u_A and $u_{A\cup\{n\}}$ into equation (3.9) one will obtain, for $u'_A(x)$, $x\in R^n$, an elliptic equation in $R_{(n-1)}$

$$\overline{\partial}u'=0,\quad u'=\sum_{A\subseteq\{1,\dots,n-1\}}u'_Ae_A. \tag{3.13}$$

To obtain this equation we have used the fact that $f_A(x,t)$ are defined from (3.10) by integration with respect to t. Then taking into consideration that $u_{A\cup\{n\}}$ are solutions of the heat equation we have

$$\sum_{\substack{j=0\\A\subseteq\{1,\dots,n-1\}}}^{n-1}\frac{\partial f_A}{\partial x_j}e_je_A=-\sum_{A\subseteq\{1,\dots,n-1\}}(-1)^{|A|}u_{A\cup\{n\}}e_A.$$

Thus $u'(x)$ with values in $R_{(n-1)}$ is a solution of the regular elliptic equation (3.13) in all R^n, and so by Liouville's theorem it is zero. Hence u_A, $u_{A\cup\{n\}}$ are uniquely defined by (3.12).

Let $u(x,t)$ be a solution of the nonhomogeneous equation in $R^0_{(n)}$

$$\overline{\partial}u-P_nu=f(x,t), \tag{3.14}$$

where

$$f(x,t)=\sum_{A\subseteq\{1,\dots,n-1\}}f_A(x,t)e_A \tag{3.15}$$

or

$$f(x,t)=\sum_{A\subseteq\{1,\dots,n-1\}}f_{A\cup\{n\}}(x,t)e_Ae_n.$$

In this case, $u(x,t)$ is at the same time a solution of the nonhomogenous heat equation

$$\Delta_{(n)}u-\frac{\partial u}{\partial t}=\partial f,\quad \partial f=\sum_{A\subseteq\{1,\dots,n-1\}}F_Ae_A. \tag{3.16}$$

Problem. Define a regular solution of (3.14) subject to the 2^{n-1} conditions

$$u_A(x,0)=0,\quad A\subseteq\{1,\dots,n-1\},\quad x=(x_0,\dots,x_{n-1}). \tag{3.17}$$

Since by (3.16) u_A is a solution of the equation

$$\Delta_{(n)} u_A - \frac{\partial u_A}{\partial t} = F_A, \quad A \subseteq \{1, \dots, n-1\},$$

by condition (3.17) u_A can be represented by (3.4), where $f(x,t)$ is the right-hand side of the last equation. Then the remaining unknowns $u_{A\cup\{n\}}$ are defined by equation (3.9) with the right-hand side of (3.15). Hence, the solution of the problem will be defined completely.

3.1. Pluriparabolic systems and polyheat equations. Let $u(x,t) : R^{n+1} \to R^0_{(n)}$ and consider high order equations

$$(\overline{\partial} - P_n)^m u = 0, \quad m \geq 1. \tag{3.18}$$

By force of (3.5), (3.6), $u(x,t)$ is also a solution of the polyheat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right)^m u = 0, \quad x = (x_0, \dots, x_{n-1}). \tag{3.19}$$

In the case $m = 2$ it will be called the biheat equation. The Cauchy problem for (3.19) will be formulated as follows:

Define a solution $u(x,t)$ of (3.19), $t > 0$, by the conditions

$$u(x,0) = \varphi_0(x), \ \ \frac{\partial u}{\partial t} = \varphi_1(x), \dots, \frac{\partial^{m-1} u}{\partial t^{m-1}} = \varphi_{m-1}(x), \ \ t = 0.$$

First solve this problem for $m = 2$, i.e., for the equation

$$\left(\Delta - \frac{\partial}{\partial t}\right)^2 u = 0, \tag{3.20}$$

with

$$u(x,0) = \varphi_0(x), \quad \frac{\partial u}{\partial t} = \varphi_1(x), \quad t = 0. \tag{3.21}$$

Solution. From (3.20) follows

$$\Delta u - \frac{\partial u}{\partial t} = F(x,t), \tag{3.22}$$

$$\Delta F - \frac{\partial F}{\partial t} = 0, \tag{3.23}$$

where F is defined for $t = 0$ from (3.21), (3.22). Thus $F(x,t)$ is represented as (3.2). Then the solution of (3.22) is represented as

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

where u_1, u_2 are solutions of the equations

$$\Delta u_1 - \frac{\partial u_1}{\partial t} = 0, \ \text{ with the condition } \ u_1(x,0) = \varphi_0(x),$$

$$\Delta u_2 - \frac{\partial u_2}{\partial t} = F(x,t), \ \text{ with the condition } \ u_2(x,0) = 0.$$

Thus $u_1(x,t)$ is represented as (3.2), and u_2 is defined by (3.4). Then for any $m \geq 2$ the problem can be solved by the induction method.

Now for equation (3.18) in the case $m = 2$ consider

Cauchy problem. Define a solution of (3.18) for $t > 0$ by the conditions

$$u_A(x,0) = \varphi_A(x), \quad u_{A\cup\{n\}}(x,0) = \psi_A(x), \quad A \subseteq \{1,\dots,n-1\}, \tag{3.24}$$

i.e., all components of u are given.

Solution. Let

$$\overline{\partial} u - P_n u = F, \tag{3.25}$$

$$\overline{\partial} F - P_n F = 0. \tag{3.26}$$

In this case the right-hand side of (3.9) is F_A which by force of (3.24) is defined for $t = 0$. Thus by force of (3.26) it will be represented as a solution of (3.18) in quadratures. Then by the first conditions (3.24) the solution of (3.25) is defined as one of corresponding problems for a nonhomogeneous equation.

3.2. Elliptic-parabolic, hyperbolic-parabolic and elliptic-hyperbolic-parabolic equations.

Consider the equations

$$\overline{\partial}(\overline{\partial} u + P_n u) = 0, \tag{3.27}$$

$$\Big(\overline{\partial} + e_{n-1}\,\frac{\partial}{\partial x_{n-1}}\Big)(\overline{\partial} u + P_n u) = 0, \tag{3.28}$$

$$\overline{\partial}\Big(\overline{\partial} + e_{n-1}\,\frac{\partial}{\partial x_{n-1}}\Big)(\overline{\partial} u + P_n u) = 0, \tag{3.29}$$

where in (3.27)

$$\overline{\partial} = \sum_{k=0}^{n} \frac{\partial}{\partial x_k}\, e_k, \quad e_k^2 = -e_0, \quad k = 1,\dots,n-1, \quad e_n^2 = 0, \tag{3.30}$$

while in (3.28) and (3.29)

$$\begin{gathered} \overline{\partial} = \sum_{0}^{n-2} \frac{\partial}{\partial x_k}\, e_k + \frac{\partial}{\partial x_n}\, e_n, \\ e_n^2 = 0, \quad e_k^2 = -e_0, \quad k = 1,\dots,n-2, \quad e_{n-1}^2 = e_0. \end{gathered} \tag{3.31}$$

It is obvious that $\partial\overline{\partial} = \Delta$ where Δ is the Laplace operator with respect to variables $x_0,\dots,x_{n-1}$ in case of (3.30) and to variables $x_0,\dots,x_{n-2}$ in case of (3.31), $P_n u$ is defined by (3.6). Then one can see that the solutions of (3.27), (3.28), (3.29) are also the solutions of the equations correspondingly

$$\Delta\Big(\Delta-\frac{\partial}{\partial t}\Big)u(x,t)=0,\quad x=(x_0,\dots,x_{n-1}),\quad x_n\equiv t>0, \tag{3.32}$$

$$\begin{gathered}\Big(\Delta-\frac{\partial^2}{\partial\tau^2}\Big)\Big(\Delta-\frac{\partial}{\partial t}\Big)u(x,\tau,t)=0,\\ x=(x_0,\dots,x_{n-2}),\ \ x_{n-1}\equiv\tau,\ \ x_n\equiv t>0,\end{gathered} \tag{3.33}$$

$$\Delta\Big(\Delta-\frac{\partial^2}{\partial\tau^2}\Big)\Big(\Delta-\frac{\partial}{\partial t}\Big)u(x,\tau,t)=0, \tag{3.34}$$

respectively. Equations (3.32), (3.33), (3.34) are called harmonic-heat, wave-heat and harmonic-wave-heat equations, respectively.

First we will consider the boundary-initial value problems for (3.32).

Dirichlet–Cauchy and Neumann–Cauchy problems. Define a regular solution of (3.32) for $x_{n-1}>0$, $t>0$, $(x_0,x_1,\dots,x_{n-2})\in R^{n-1}$, vanishing at infinity, by the conditions

$$u(x,0)=\varphi(x),\quad x_{n-1}>0,\quad x=(x_0,\dots,x_{n-1}), \tag{3.35}$$

$$u(x,t)=\psi(x_0,x_1,\dots,x_{n-2},t),\quad x_{n-1}=0,\quad t>0, \tag{3.36}$$

or (3.35) and

$$\frac{\partial u}{\partial x_{n-1}}=\psi(x_0,\dots,x_{n-2},t),\quad x_{n-1}=0,\quad t>0. \tag{3.37}$$

Solution. Let

$$\Delta u(x,t)=F(x,t), \tag{3.38}$$

$$\Delta F-\frac{\partial F}{\partial t}=0, \tag{3.39}$$

then by force of (3.35) the unknown function $F(x,t)$ satisfies

$$F(x,0)=\Delta\varphi(x)\equiv f(x)$$

and for (3.39) one has the Cauchy initial value problem which is represented by (3.2). To define $u(x,t)$, we have the Dirichlet problem (3.36) or the Neumann problem (3.37) for equation (3.38), and thus solutions are given above. It is clear that all problems which are solved for harmonic functions can be solved correspondingly for equation (3.32).

Cauchy problem for equation (3.33).

Define solutions of equation (3.33) for $t>0$, $\tau>0$, $x\in R^{n-1}$, by the conditions

$$u(x,0,t)=\varphi_1(x,t),\quad \frac{\partial u}{\partial\tau}=\varphi_2(x,t),\quad \tau=0, \tag{3.40}$$

$$u(x,\tau,0)=\psi(x,\tau). \tag{3.41}$$

Solution. Let

$$\Delta u - \frac{\partial u}{\partial t} = F(x, \tau, t), \tag{3.42}$$

$$\Delta F - \frac{\partial^2 F}{\partial \tau^2} = 0. \tag{3.43}$$

Then by force of (3.40) the unknown function $F(x, \tau, t)$ satisfies

$$F(x, 0, t) = \Delta \varphi_1(x, t) - \frac{\partial \varphi_1(x, t)}{\partial t} \equiv f_1(x, t), \quad \tau = 0,$$
$$\frac{\partial F}{\partial \tau} = \Delta \varphi_2(x, t) - \frac{\partial \varphi_2(x, t)}{\partial t} \equiv f_2(x, t), \quad \tau = 0.$$

Thus for the wave equation (3.43) we have the Cauchy problem whose solution is given above. After defining $u(x, \tau, t)$, we have the Cauchy problem for the nonhomogeneous heat equation (3.42) with condition (3.41). The solution is represented in the form

$$u(x, \tau, t) = u_1(x, \tau, t) + u_2(x, \tau, t),$$

where $u_1(x, \tau, t)$ is a solution of the homogeneous heat equation with the condition $u_1(x, \tau, 0) = \psi_1(x, \tau)$ and $u_2(x, \tau, t)$ is a solution of (3.42) with the condition $u_2(x, \tau, 0) = 0$. Thus using (3.2) and (3.4) the solution can be represented in quadratures.

It is obvious that one can consider the heat-Klein–Gordon equation

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\Big(\Delta - k^2 - \frac{\partial^2}{\partial \tau^2}\Big)u(x, \tau, t) = 0$$

with conditions (3.40), (3.41) and the solution will be represented in quadratures.

In the same way one can consider the Helmholtz-heat equation

$$(\Delta - k^2)\Big(\Delta - \frac{\partial}{\partial t}\Big)u(x, t) = 0$$

with conditions (3.35), (3.36) or (3.35), (3.37). The solutions can be represented in quadratures too.

Now consider the problem for the harmonic-wave-heat equation (3.34) for $t > 0$, $\tau > 0$, $x_{n-2} > 0$ with the conditions:

$$u(x, 0, t) = f_1(x, t), \quad \frac{\partial u}{\partial \tau} = f_2(x, t), \quad \tau = 0, \tag{3.44}$$

$$u(x, \tau, 0) = \varphi(x, \tau), \tag{3.45}$$

$$u(x, \tau, t) = \psi(x_0, \ldots, x_{n-3}, \tau, t), \quad x_{n-2} = 0. \tag{3.46}$$

Solution. Let

$$\Delta u = F(x, \tau, t), \tag{3.47}$$

$$\Big(\Delta - \frac{\partial^2}{\partial \tau^2}\Big)\Big(\Delta - \frac{\partial}{\partial t}\Big)F = 0, \tag{3.48}$$

then by force of (3.44), (3.45) F satisfis conditions (3.40), (3.41), i.e., F as a solution of (3.48) is constructed effectively. Hence u as a solution of (3.47), by condition (3.46) or the condition of the Neumann problem can be represented in quadratures too.

Note that if we consider the problem for the equation

$$\Big(\Delta - \frac{\partial^2}{\partial t^2}\Big)\Big(\Delta - \frac{\partial}{\partial t}\Big)u(x,t) = 0, \quad t > 0, \quad x = (x_0, \dots, x_{n-1}), \tag{3.49}$$

with the conditions

$$u(x,0) = f_1(x), \quad \frac{\partial u}{\partial t} = f_2(x), \quad \frac{\partial^2 u}{\partial t^2} = f_3(x), \quad t = 0,$$
$$u(x,t) = f(x_0, \dots, x_{n-2}, t), \quad x_{n-1} = 0,$$

the solution is defined in an analoguus way to the equation (3.33) with conditions (3.40), (3.41).

We think equations (3.33), (3.34) are more interesting than (3.49) because first of all they are related to equations (3.28), (3.29), i.e., they are suggested by Clifford analysis, and, secondly, it is natural that the time in wave processes and the time in heat processes are different. That is why equations (3.27), (3.28), (3.29), (3.32), (3.33), (3.34) may have important applications in physics.

Now it is clear that to formulate the boundary-initial value problems for equations (3.27), (3.28), (3.29) all conditions considered for each multiplier operator, must be given. Thus using a solution of each problem one can obtain the corresponding solutions in quadratures.

The boundary-initial value problems for nonhomogeneous equations which correspond to the above considered homogenous equations will be solved too. It is obvious that in this case the boundary-initial conditions can be supposed homogeneous. We will consider only one of them as others can be solved in the same way.

Problem. Define a solution of the equation

$$\Delta\Big(\Delta - \frac{\partial^2}{\partial \tau^2}\Big)\Big(\Delta - \frac{\partial}{\partial t}\Big)u(x,\tau,t) = F(x,\tau,t),$$
$$x = (x_0, \dots, x_{n-2}) \in R^{n-1}, \quad \tau > 0, \quad t > 0, \quad x_{n-2} > 0,$$

which vanishes at infinity, and satisfies the conditions:

$$u(x,0,t) = 0, \quad \frac{\partial u}{\partial \tau} = 0, \quad \tau = 0, \tag{3.50}$$
$$u(x,\tau,0) = 0, \tag{3.51}$$
$$u(x,\tau,t) = 0, \quad x_{n-2} = 0. \tag{3.52}$$

Solution. Let

$$
\begin{aligned}
\Delta u &= F_1(x,\tau,t),\\
\Delta F_1 - \frac{\partial^2 F_1}{\partial \tau^2} &= F_2(x,\tau,t),\\
\Delta F_2 - \frac{\partial F_2}{\partial \tau} &= F(x,\tau,t),
\end{aligned}
\tag{3.53}
$$

then by force of (3.50), (3.51) for F_1, F_2 one has the Cauchy homogeneous conditions for nonhomogeneous wave and heat equations, thus the solution are given in quadratures.

Hence $u(x,\tau,t)$ is defined as a solution of (3.53) with condition (3.52).

It seems to me that these equations are beautiful and, as Paul Dirac said about beautiful formulas, their success in applications is ensured.

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