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ENTROPY NUMBERS OF DIAGONAL OPERATORS OF LOGARITHMIC TYPE

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Dedicated to Professor Nicholas Vakhania
on his 70th birthday

Abstract. We determine the asymptotic behaviour (as $k \to \infty$, up to multiplicative constants not depending on k) of the entropy numbers e_k $(D_\sigma : l_p \to l_q)$, $1 \leq p \leq q \leq \infty$, of diagonal operators generated by logarithmically decreasing sequences $\sigma = (\sigma_n)$. This complements earlier results by Carl [2] who investigated the case of power-like decay of the diagonal.

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1. Introduction

In the last few years there has appeared quite a number of research papers (see [17], [18], [7] and [8]) and several monographs (see [9] and [25]) on compact embeddings of function spaces in so-called limiting situations, where the spaces have the same 'differential dimension'. The 'degree' of compactness of a (bounded linear) operator can be measured by the rate of decay of its entropy numbers. Standard tools in the treatment of such embedding problems are discretization techniques, which allow to reduce an infinite dimensional problem to a finite dimensional one, or which reduce function space embeddings to embeddings of weighted vector-valued sequence spaces. This approach requires sharp estimates for the entropy numbers of the latter embeddings. Equivalently, by shifting the weights from spaces to operators, one can consider diagonal operators in sequence spaces without weights.

For classical non-limiting embeddings these weights are of power-like decay. In the above-mentioned limiting situations, however, the resulting weights decrease only logarithmically. This was one of the motivations for the present work.

A further motivation came from the paper by Carl [2] who determined the entropy behaviour of diagonal operators $D_\sigma : l_p \to l_q$, $1 \leq p, q \leq \infty$, in the case where the diagonal σ belongs to some Lorentz space $l_{r,t}$. Typically, for

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$t = \infty$, this means $\sigma_n \preceq n^{-\alpha}$ with $\alpha = 1/r$. We give in this paper sharp entropy estimates for diagonal operators D_σ in the following two cases:

(P) $\sigma_n \sim n^{-\alpha}(1+\log n)^{\delta}$, $1 \le p, q \le \infty$, $\alpha > \max(1/q - 1/p, 0)$, $\delta \in \mathbb{R}$;

(L) $\sigma_n \sim (1+\log n)^{-\delta}$, $1 \le p \le q \le \infty$, $\delta > 0$.

The first case (P), of power-like decay of the diagonal, is a variant of Carl's result [2], and the proof also relies on the same, by now standard, technique of splitting the operator D_σ in dyadic parts. The logarithmic case (L), however, extends the results of [2] to the limiting situation $r = \infty$ or, equivalently, $\alpha = 0$. Here the dyadic splitting does not give satisfactory estimates and so a new technique is needed.

Our new approach (recently introduced in two joint papers of the author with Cobos [5] and Schonbek [15]) is essentially based on a close relation between Gaussian processes (resp. Gaussian measures) and entropy. More precisely, we use Sudakov's famous minoration principle – in the form of an inequality between entropy numbers and the l-norm of an operator – in combination with certain interpolation techniques.

The organization of the paper is as follows. In Section 2 we fix the notation, and give a few preliminary results which will be needed in the sequel. In Section 3 the main results are formulated, while Section 4 contains their proofs.

2. Notation and Preliminary Results

a. Entropy numbers and ideals. The k^{th} (dyadic) entropy number of a bounded linear operator $T : X \to Y$ from a Banach space X into another Banach space Y is defined as

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls of radius } \varepsilon \text{ in } Y\},$$

where B_X denotes the closed unit ball in X. Entropy numbers are closely related to Kolmogorov's concept of metric entropy which dates back to the 1930's and which since then has been successfully applied in various fields of mathematics. For basic properties of entropy numbers and applications to eigenvalue and compactness problems we refer to the monographs by Pietsch [21], König [12], Carl and Stephani [4], Edmunds and Triebel [9], and Triebel [25], and the references given therein; for recent applications see also Edmunds and Haroske [7, 8], and Leopold [17, 18].

We use the common notation l_p^n for the space $\mathbb{R}^n$ (or $\mathbb{C}^n$) equipped with the quasi-norm (which is even a norm for $p \ge 1$)

$$\|x\|_p = \begin{cases} \left(\sum\limits_{k=1}^n |x_k|^p\right)^{1/p} & \text{for} \quad 0 < p < \infty \\ \sup\limits_{1\le k\le n} |x_k| & \text{for} \quad p = \infty. \end{cases}$$

Given two sequences $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ of positive real numbers, we shall write $a_k \preceq b_k$ if there is a constant $c > 0$ such that the inequality $a_k \le c \cdot b_k$ holds for all $k \in \mathbb{N}$, and $a_k \sim b_k$ if both $a_k \preceq b_k$ and $b_k \preceq a_k$ are fulfilled. Logarithms are always taken to base 2, $\log = \log_2$.

The following basic result is due to Schütt [22] who determined the exact asymptotic behaviour of entropy numbers of the formal identity between finite dimensional l_p-spaces.

Lemma 1. *Let* $1 \leq p \leq q \leq \infty$*, then* (*in the real case*)

$$e_k\left(id : l_p^n \to l_q^n\right) \sim \begin{cases} 1 & \text{if } 1 \leq k \leq \log n \\ \left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\right)^{1/p-1/q} & \text{if } \log n \leq k \leq n \\ 2^{-\frac{k-1}{n}} n^{1/q-1/p} & \text{if } k \geq n. \end{cases} \tag{1}$$

Remark 1. Formula (1) remains valid for complex spaces if on the right-hand side n is replaced by $2n$. Roughly speaking, this follows from volume arguments, regarding the n-dimensional complex spaces l_p^n as $2n$-dimensional real spaces.

Remark 2. A consequence of the special case $k = n$ and elementary properties of entropy numbers is the relation

$$e_n\left(id : l_p^n \to l_q^n\right) \sim n^{1/q-1/p} \text{ for all } 1 \leq p, q \leq \infty. \tag{2}$$

The asymptotic relations (1) are fundamental for numerous applications, as explained in the Introduction. In passing, we mention that (1) has recently been extended to the quasi-Banach case $0 < p \leq q \leq \infty$, see [9, 25, 14]. This was again motivated by applications, mainly by the increasing interest during the last decade in quasi-Banach function spaces, for instance in (both classical and generalized) Besov, Sobolev, and Lipschitz spaces.

Another very useful result is due to Gordon, König and Schütt [11, Proposition 1.7.].

Lemma 2. *Let* X *be a real Banach space with* 1*-unconditional basis* $\{x_j\}_{j=1}^{\infty}$ (*for example a symmetric Banach sequence space, or any of the spaces* l_p *with* $1 \leq p < \infty$), *and let* $D_\sigma : X \to X$ *be the diagonal operator induced by* $x_j \mapsto \sigma_j x_j$*, where* $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$*. Then*

$$\sup_{n\in\mathbb{N}} 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n} \leq e_{k+1}(D_\sigma) \leq 6 \cdot \sup_{n\in\mathbb{N}} 2^{-k/n}(\sigma_1 \cdots \sigma_n)^{1/n} \tag{3}$$

for all $k \in \mathbb{N}$.

Remark 3. Inequality (3) remains valid for complex spaces if the factor $2^{-k/n}$ is replaced by $2^{-k/2n}$.

As usual, we denote by $l_{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$, the Lorentz sequence space consisting of all (real or complex) null sequences $(x_k)_{k=1}^{\infty}$ such that the quasi-norm

$$\|x\|_{p,q} := \begin{cases} \left(\sum\limits_{k=1}^{\infty} k^{q/p-1} |x_k^*|^q\right)^{1/q} & \text{if } 0 < q < \infty \\ \sup\limits_{1\leq k<\infty} k^{1/p} |x_k^*| & \text{if } q = \infty \end{cases}$$

is finite. Here the non-increasing rearrangement (x_k^*) of the sequence (x_k) is defined by $x_k^* = x_{\pi(k)}$, where $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation with $\left|x_{\pi(1)}\right| \geq \left|x_{\pi(2)}\right| \geq \cdots \geq 0$. All Lorentz spaces are quasi-Banach spaces, and for $1 < p < \infty$, $1 \leq q \leq \infty$ and $p = q = 1$ the quasi-norms $\|\cdot\|_{p,q}$ are even equivalent to norms. In particular we have $l_{p,p} = l_p$ with equality of (quasi)-norms.

Moreover, for the same range of indices p, q, we shall use the entropy ideals $\mathcal{L}_{p,q}^{(e)}$, formed by all bounded linear operators T between Banach spaces such that the quasi-norm

$$L_{p,q}^{(e)}(T) := \|(e_n(T))\|_{p,q}$$

is finite. The classes $\left(\mathcal{L}_{p,q}^{(e)}, L_{p,q}^{(e)}\right)$ are quasi-normed operator ideals in the sense of Pietsch [21, Definition 6.1.1].

Given any sequence $\sigma = (\sigma_k)_{k=1}^{\infty}$, the diagonal operator D_σ is defined by

$$D_\sigma(x_k) := (\sigma_k x_k). \tag{4}$$

Throughout the paper we shall assume that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. We state now the result of Carl [2] on the entropy numbers of diagonal operators.

Lemma 3. *Let $1 \leq p, q \leq \infty$, $0 < r, s < \infty$ and $0 < t \leq \infty$ such that $1/s := 1/r + 1/p - 1/q > 0$. Then one has the equivalence*

$$\sigma = (\sigma_k) \in l_{r,t} \iff D_\sigma \in \mathcal{L}_{s,t}^{(e)}(l_p, l_q).$$

Remark 4. In the proof the following estimate, for $s > 0$ with $1/s > 1/p - 1/q > 0$,

$$L_{s,\infty}^{(e)}\left(id : l_p^n \to l_q^n\right) \sim n^{1/s+1/q-1/p} \tag{5}$$

was used, which in turn follows from (1) and the elementary properties of entropy numbers.

b. Gaussian measures and entropy. In this subsection all spaces are assumed to be *real*. The l-norm (introduced by Linde and Pietsch [19] under the name of π_γ-norm) of an operator T from a separable Hilbert space H into a Banach space X is defined as

$$l(T) = \sup_{1 \leq i \leq n} \left(\mathbb{E}\left\|\sum_{i=1}^{n} T e_i g_i\right\|^2\right)^{1/2},$$

where (e_i) is an arbitrary orthonormal basis in H and (g_i) a sequence of i.i.d. standard normal random variables. The l-norm is closely related to Gaussian measures on Banach spaces (see, e.g., the monograph by Ledoux and Talagrand [16]) and is an important tool in Banach space geometry (see the book by Pisier [20]). We only need two basic and well-known facts, the first one can be found, e.g., in [19, Theorem 9].

Lemma 4. *There are constants* $0 < c < C < \infty$ *such that the estimate*

$$c \cdot \sup_{n\in\mathbb{N}} \sigma_n \sqrt{1+\log n} \le l(D_\sigma) \le C \cdot \sup_{n\in\mathbb{N}} \sigma_n \sqrt{1+\log n}$$

holds for every diagonal operator $D_\sigma : l_2 \to l_\infty$, *with* $\sigma_1 \ge \sigma_2 \ge \cdots \ge 0$.

The next result is known as Sudakov's minoration principle, see [23] for the original formulation, [16, Theorem 3.18] for a variant in the language of Gaussian processes, or [13] for the version given below.

Lemma 5. *There is a constant* $c > 0$ *such that the inequality*

$$\sup_{k\in\mathbb{N}} k^{1/2} e_k(T^*) \le c \cdot l(T)$$

holds for all operators T *from a Hilbert space* H *into a Banach space* X. (*Here* T^* *denotes the dual of* T.)

The two preceding lemmas immediately imply

Lemma 6. *There is a constant* $c > 0$ *such that for the diagonal operator* $D_\sigma : l_1 \to l_2$, *generated by* $\sigma_k = (1+\log k)^{-1/2}$, *the inequality*

$$e_k(D_\sigma) \le c \cdot k^{-1/2}$$

holds for all $k \in \mathbb{N}$.

Remark 5. By standard considerations one can easily show that this result remains true for complex spaces as well.

c. Complex interpolation and entropy. In this subsection we shall consider only *complex* Banach spaces since the complex interpolation method does not make sense for *real* spaces. Our general references to interpolation theory, in particular to the complex method, are the monographs by Bergh and Löfström [1], and Triebel [26]; we shall use the same notation and terminology, and state some known facts for later applications.

First we fix some notation. If $0 < \theta < 1$ and (A_0, A_1) is an interpolation couple (i.e., a pair of Banach spaces which both embed linearly and continously into a common Hausdorff vector space), then $[A_0, A_1]_\theta$ denotes as usual the complex interpolation space of the couple.

If A is a Banach space and $\lambda > 0$ a positive real number, then λA means the same Banach space A, but equipped with the norm

$$\|x\|_{\lambda A} := \lambda \cdot \|x\|_A .$$

Given $1 \le p \le \infty$ and Banach spaces A_n, $n \in \mathbb{N}$, we denote by $l_p(A_n)$ the set of all sequences $x = (x_n)_n^\infty$ such that $x_n \in A_n$ for all $n \in \mathbb{N}$ and

$$(\|x_n\|_{A_n})_{n=1}^\infty \in l_p .$$

The vector-valued spaces $l_p(A_n)$ are Banach spaces with respect to the norm

$$\|x\| := \left\| (\|x_n\|_{A_n})_{n=1}^\infty \right\|_p .$$

In the particular case $A_n = w_n\mathbb{C}$, with $w_n > 0$ for all $n \in \mathbb{N}$, we obtain weighted l_p-spaces, simply denoted by $l_p(w)$, where $w = (w_n)_{n=1}^{\infty}$ is the weight sequence. The following known interpolation formula for weighted l_p-spaces will be needed later.

Lemma 7. *Let* $1 \leq p_0,\ p_1 \leq \infty$*, but not* $p_0 = p_1 = \infty$*, let* $0 < \theta < 1$*, and let* w_0 *and* w_1 *be arbitrary weight sequences. Then one has with equality of norms*

$$[l_{p_0}(w_0), l_{p_1}(w_1)]_\theta = l_p({w_0}^{1-\theta}{w_1}^{\theta}), \ \ \textit{where} \ \ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{6}$$

Proof. From the very definition of the complex method it follows that for every Banach space A and arbitrary real constants $\lambda_0,\ \lambda_1 > 0$ one has equality of norms

$$[\lambda_0 A, \lambda_1 A]_\theta = \lambda A \ , \text{ where } \lambda = {\lambda_0}^{1-\theta}{\lambda_1}^{\theta}. \tag{7}$$

Moreover, given an arbitrary sequence of Banach couples (A_n, B_n), $n \in \mathbb{N}$, the following interpolation formula for vector-valued l_p-spaces

$$[l_{p_0}(A_n), l_{p_1}(B_n)]_\theta = l_p([A_n, B_n]_\theta), \ \text{ where } \ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{8}$$

holds, also with equality of norms, see, e.g., [26, Theorem 1.18.1. and Remarks 1 and 2, p. 122]. From (8) combined with (7), assertion (6) follows. □

Finally let us recall the nice behaviour of entropy numbers under interpolation, specified to the complex method, which is sufficient for our present purposes. The general result can be found in [21, Theorems 12.1.11 and 12.1.12]. Here $\mathcal{L}(X, Y)$ stands for the class of all bounded linear operators from X into Y.

Lemma 8. *Let* A *and* B *be Banach spaces,* (A_0, A_1) *and* (B_0, B_1) *interpolation couples,* $0 < \theta < 1$ *and* $m, n \in \mathbb{N}$*. Then the following holds:*

(i) *If* $T \in \mathcal{L}(A_0 + A_1, B)$*, then*

$$e_{m+n-1}(T : [A_0, A_1]_\theta \to B) \leq 2e_m(T : A_0 \to B)^{1-\theta} e_n(T : A_1 \to B)^{\theta}.$$

(ii) *If* $T \in \mathcal{L}(A, B_0 \cap B_1)$*, then*

$$e_{m+n-1}(T : A \to [B_0, B_1]_\theta) \leq 2e_m(T : A \to B_0)^{1-\theta} e_n(T : A \to B_1)^{\theta}.$$

3. Main Results

Now we state our main results on the entropy numbers of diagonal operators. The first two theorems deal with the case, where the diagonal decreases essentially as a power.

Theorem 1. *Let* $1 \leq p, q \leq \infty$*,* $\alpha > \max(1/q - 1/p, 0)$ *and* $\delta \in \mathbb{R}$*. Then one has the implication*

$$\sigma_n \sim n^{-\alpha}(1 + \log n)^{\delta} \Longrightarrow e_n\left(D_\sigma : l_p \to l_q\right) \sim n^{1/q-1/p-\alpha}(1 + \log n)^{\delta}.$$

Theorem 2. *Under the assumptions of Theorem* 1 *one has the equivalence*

$$\sigma_n \preceq n^{-\alpha}(1+\log n)^{\delta} \Longleftrightarrow e_n\left(D_\sigma : l_p \to l_q\right) \preceq n^{1/q-1/p-\alpha}(1+\log n)^{\delta}.$$

It seems appropriate to give here some brief comments on the relation of our results (Theorems 1 and 2) and Carl's result [2] (stated in Lemma 3). The main difference lies in the way of measuring the rate of decay of a positive decreasing sequence $\sigma = (\sigma_n)$ of real numbers. While in [2] membership to a Lorentz space $l_{r,t}$ was used, we prefer to work instead with the more direct conditions $\sigma_n \preceq n^{-\alpha}(1+\log n)^{\delta}$. Both ways provide a two-parameter scale of conditions with the 'main' parameter (r, resp. α) and the 'fine' parameter (t, resp. δ). Although quite close to each other, the two scales are different, their only 'intersection point' being the case $(\sigma_n) \in l_{r,\infty} \Longleftrightarrow \sigma_n \preceq n^{-1/r}$. In a certain sense our scale is more sensible. For example, the Lorentz scale cannot distuingish between the conditions $\sigma_n \preceq n^{-\alpha}(1+\log n)^{\delta}$, where $\alpha > 0$ is fixed and $\delta > 0$ is arbitrary.

Now we turn to the case of logarithmic decay of the diagonal σ, which corresponds to the limiting situation $\alpha = 0$ in the preceding theorem. Of course, in order to obtain a bounded operator from l_p into l_q, we have to require $p \le q$. For $p = q$ the entropy behaviour of D_σ is known, see Remark 6 below.

Theorem 3. *Let* $1 \le p < q \le \infty$ *and* $\delta > 0$, *and set* $\delta^* := 1/p - 1/q$. *Then the condition* $\sigma_n \sim (1+\log n)^{-\delta}$ *implies*

$$e_n(D_\sigma : l_p \to l_q) \sim \begin{cases} n^{1/q-1/p}(1+\log n)^{1/p-1/q-\delta} & \text{if} \quad \delta \ge \delta^* \\ n^{-\delta} & \text{if} \quad \delta \le \delta^*. \end{cases}$$

Theorem 4. *Under the assumptions of Theorem* 3 *the condition* $\sigma_n \preceq (1+\log n)^{-\delta}$ *is equivalent to*

$$e_n(D_\sigma : l_p \to l_q) \preceq \begin{cases} n^{1/q-1/p}(1+\log n)^{1/p-1/q-\delta} & \text{if} \quad \delta \ge \delta^* \\ n^{-\delta} & \text{if} \quad \delta \le \delta^*. \end{cases}$$

Remark 6. Note that in the case $p = q$, $\delta > 0$, $\sigma_n \sim (1+\log n)^{-\delta}$ the result of Gordon, König and Schütt [11], stated here as Lemma 2, implies

$$e_n(D_\sigma : l_p \to l_p) \sim (1+\log n)^{-\delta},$$

which in contrast to Theorem 2 means that we have only logarithmic decay of the entropy numbers. This shows that the embeddding $l_p \hookrightarrow l_q$ contributes essentially to the power-like decay of the entropy numbers.

Another new feature of the entropy behaviour of diagonal operators with logarithmic diagonals, compared to power-like diagonals, is the existence of the 'critical' value δ^*, where the asymptotic entropy behaviour changes drastically. For $\delta \ge \delta^*$ the entropy numbers keep almost stable, up to logarithmic factors. A similar phenomenon has occured recently in the study of entropy numbers of convex hulls, see the papers by Carl, Kyrezi and Pajor [3], Gao [10], and Creutzig and Steinwart [6].

4. Proofs

Now we give the proofs of our main results.

Proof of Theorem 1. *Upper estimate:*

For all integers $n \geq 0$ consider the dyadic parts D_n of the operator D_σ, defined by

$$D_n x = (0, \dots, 0, \sigma_{2^n} x_{2^n}, \dots, \sigma_{2^{n+1}-1} x_{2^{n+1}-1}, 0, \dots).$$

Then, clearly,

$$D_\sigma = \sum_{n=0}^{\infty} D_n. \tag{9}$$

Choose $s > 0$ with $1/s > \alpha + 1/p - 1/q$, and observe that the quasi-norm $L^{(e)}_{s,\infty}$ is equivalent to an r-norm, for appropriate $0 < r < 1$. Now, for any fixed $N \in \mathbb{N}$, it follows from (5) and the multiplicativity of entropy numbers that

$$L^{(e)}_{s,\infty}\left(\sum_{n=0}^{N} D_n\right)^r \leq c \sum_{n=0}^{N} L^{(e)}_{s,\infty}(D_n)^r \leq c \sum_{n=0}^{N} \sigma^r_{2^n} L^{(e)}_{s,\infty}(id : l_p^{2^n} \to l_q^{2^n})^r$$
$$\sim \sum_{n=0}^{N} \left(2^{-\alpha n} n^\delta \cdot 2^{(1/s+1/q-1/p)n}\right)^r \sim \left(N^\delta 2^{(1/s+1/q-1/p-\alpha)N}\right)^r.$$

This implies in particular

$$e_{2^N}\left(\sum_{n=0}^{N} D_n\right) \preceq N^\delta 2^{(1/q-1/p-\alpha)N}. \tag{10}$$

Now we estimate the entropy numbers of the tail of the sum (9). To this end, we choose now $s > 0$ with $1/p - 1/q < 1/s < \alpha + 1/p - 1/q$. Again, we use the fact that the quasi-norm $L^{(e)}_{s,\infty}$ is equivalent to an r-norm, and so we can estimate it as above (that we have now different r and s does not matter, this influences only the constants):

$$L^{(e)}_{s,\infty}\left(\sum_{n=N+1}^{\infty} D_n\right)^r \leq c \sum_{n=N+1}^{\infty} L^{(e)}_{s,\infty}(D_n)^r$$
$$\leq c \sum_{n=N+1}^{\infty} \sigma^r_{2^n} L^{(e)}_{s,\infty}(id : l_p^{2^n} \to l_q^{2^n})^r$$
$$\sim \sum_{n=N+1}^{\infty} \left(2^{-\alpha n} n^\delta \cdot 2^{(1/s+1/q-1/p)n}\right)^r \sim \left(N^\delta 2^{(1/s+1/q-1/p-\alpha)N}\right)^r.$$

It follows that

$$e_{2^N}\left(\sum_{n=N+1}^{\infty} D_n\right) \preceq N^\delta 2^{(1/q-1/p-\alpha)N}, \tag{11}$$

and by the additivity of entropy numbers we conclude from (10) and(11)

$$e_{2\cdot 2^N}(D_\sigma) \le e_{2^N}\left(\sum_{n=0}^{N} D_n\right) + e_{2^N}\left(\sum_{n=N+1}^{\infty} D_n\right) \preceq N^\delta 2^{(1/q-1/p-\alpha)N},$$

which proves the upper estimate since $N \in \mathbb{N}$ was arbitrary.

Lower estimate. For arbitrary $n \in \mathbb{N}$, let R_n denote the restriction of D_σ to the coordinates $1, \dots, n$. Then the multiplicativity of entropy numbers and (1) imply the desired relation

$$e_n(D_\sigma) \ge e_n(R_n) \ge \sigma_n \cdot e_n(id : l_p^n \to l_q^n) \sim n^{1/q-1/p-\alpha}(1+\log n)^\delta \ . \qquad \square$$

Proof of Theorem 2. The implication $\Rightarrow$ is covered by Theorem 1. For the converse implication $\Leftarrow$ we proceed similarly to the lower estimate in the proof of Theorem 1. We now have

$$\begin{aligned} n^{1/q-1/p-\alpha}(1+\log n)^\delta &\succeq e_n(D_\sigma) \ge e_n(R_n) \\ &\ge \sigma_n \cdot e_n(id : l_p^n \to l_q^n) \sim \sigma_n \cdot n^{1/q-1/p}, \end{aligned}$$

whence $\sigma_n \preceq n^{-\alpha}(1+\log n)^\delta$. $\square$

Before passing to the proof of Theorem 3 we give an auxiliary result.

Lemma 9. *Let $w = (w_n)$ be a non-increasing weight sequence, let $1 \le p < q \le \infty$ and set $\lambda := 1/p - 1/q$. Then for all $n \in \mathbb{N}$*

$$e_n(D_{w^\lambda} : l_p \to l_q) \le 4 e_n(D_w : l_1 \to l_\infty)^\lambda.$$

Proof. Let us start with the following simple but useful observation. It is obvious that the Banach spaces l_p and $l_p(w)$, $1 \le p \le \infty$, $w = (w_n)$ an arbitrary weight sequence, are isometrically isomorphic, where the isomorphism is given by the diagonal operator D_w, $D_w(x_n) = (w_n x_n)$. For the entropy numbers this implies

$$e_n(id : l_p \to l_q(w)) = e_n(D_w : l_p \to l_q) = e_n(id : l_p(w^{-1}) \to l_q),$$

which means the weights can be shifted from the spaces to the operators.

Let us assume for the moment that $1 < p < q < \infty$.

Step 1. Setting $\theta := 1/p$, we clearly have $\theta \in (0,1)$ and $\frac{1-\theta}{\infty} + \frac{\theta}{1} = \frac{1}{p}$ whence by Lemma 7

$$[l_\infty, l_1(w^{-1})]_\theta = l_p(w^{-\theta}).$$

Moreover, by shifting the weights Lemma 8 implies

$$\begin{aligned} e_n(id : l_p \to l_\infty(w^\theta)) &= e_n(id : l_p(w^{-\theta}) \to l_\infty) \\ &\le 2\|id : l_\infty \to l_\infty\|^{1-\theta} e_n(id : l_1(w^{-1}) \to l_\infty)^\theta \\ &= 2 e_n(D_w : l_1 \to l_\infty)^\theta. \end{aligned}$$

Step 2. Next we set $\eta := p/q$. Then $\eta \in (0,1)$ and $\frac{1-\eta}{\infty} + \frac{\eta}{p} = \frac{1}{q}$ whence Lemma 7 now gives

$$[l_\infty(w^\theta), l_p]_\eta = l_q(w^{\theta(1-\eta)}) = l_q(w^\lambda).$$

Therefore Lemma 8 and Step 1 yield

$$\begin{aligned} & e_n(D_{w^\lambda} : l_p \to l_q) = e_n(id : l_p \to l_q(w^\lambda)) \\ & \leq 2e_n(id : l_p \to l_\infty(w^\theta))^{1-\eta} \|id : l_p \to l_p\|^\eta \\ & = 2e_n(id : l_p \to l_\infty(w^\theta))^{1-\eta} \leq 4e_n(D_w : l_1 \to l_\infty)^\lambda. \end{aligned}$$

If $p = 1$ (resp. $q = \infty$), then the first (resp. second) step can be omitted, and so we have the desired result in all cases. □

Proof of Theorem 3. We adopt the notation L_δ for the diagonal operator D_σ generated by the logarithmic sequence $\sigma_n = (1 + \log n)^{-\delta}$ with $\delta > 0$, i.e.,

$$L_\delta(x_n) = ((1 + \log n)^{-\delta} x_n).$$

Upper estimate: We divide the proof into several steps.

Step 1. First we consider the special case $p = 1$, $q = \infty$, $\delta = 1$. By Lemma 6 we have

$$e_n(L_{1/2} : l_1 \to l_2) \preceq n^{-1/2},$$

and by Tomczak–Jaegermann's results [24] on the duality of entropy numbers this implies

$$e_n(L_{1/2} : l_2 \to l_\infty) \preceq n^{-1/2},$$

therefore the multiplicativity of entropy numbers yields

$$e_{2n}(L_1 : l_1 \to l_\infty) \leq e_n(L_{1/2} : l_1 \to l_2) \cdot e_n(L_{1/2} : l_2 \to l_\infty) \preceq n^{-1},$$

which is the desired upper estimate in our special case.

Step 2. Next we consider the case $p = 1$, $q = \infty$, $\delta > 1$. Here we write $\delta = \alpha + 1$; clearly, $\alpha > 0$. By the result of Gordon, König and Schütt [11], see our Remark 6, we have

$$e_n(L_\alpha : l_1 \to l_1) \sim (1 + \log n)^{-\alpha},$$

which leads, together with Step 1 and the multiplicativity of the entropy numbers, to the estimate

$$e_{2n}(L_\delta : l_1 \to l_\infty) \leq e_n(L_\alpha : l_1 \to l_1) e_n(L_1 : l_1 \to l_\infty) \preceq n^{-1} \cdot (1 + \log n)^{-\alpha}.$$

This is the assertion since $-\alpha = 1/p - 1/q - \delta$ in the present case.

Step 3. In the case $p = 1$, $q = \infty$, $0 < \delta < 1$ we use the interpolation formula

$$[l_1, l_1(w^{-1})]_\delta = l_1(w^{-\delta})$$

with the special weight sequence $w = (w_n)$, $w_n = 1 + \log n$. From the result in Step 1 for $\delta = 1$ we conclude analogously to the proof of Lemma 9 that

$$\begin{aligned} & e_n(L_\delta : l_1 \to l_\infty) = e_n(id : l_1(w^{-\delta}) \to l_\infty) \\ & \leq 2\|id : l_1 \to l_\infty\|^{1-\delta} e_n(id : l_1((w^{-1}) \to l_\infty)^\delta \\ & = 2e_n(L_1 : l_1 \to l_\infty)^\delta \preceq n^{-\delta}. \end{aligned}$$

Step 4. Finally, in the general case $1 \le p < q \le \infty$, $\delta > 0$ the assertion follows from Interpolation Lemma 9 and the results in Steps 1–3.

Lower estimate: For arbitrary $n \in \mathbb{N}$, let R_n denote the restriction of D_σ to the coordinates $1, \dots, n^2$. Then relation (1) and the multiplicativity of entropy numbers imply

$$\begin{aligned} e_n(D_\sigma) &\ge e_n(R_n) \ge \sigma_{n^2} \cdot e_n(id : l_p^{n^2} \to l_q^{n^2}) \\ &\sim (1+\log n)^{-\delta} \cdot \left(\frac{1+\log n}{n}\right)^{1/p-1/q}, \end{aligned}$$

and this shows the lower estimate for $\delta \ge \delta^*$. In the case $0 < \delta < \delta^*$ we consider the restriction of D_σ to the coordinates $1, \dots, 2^n$. Then $\sigma_{2^n} \sim n^{-\delta}$ and, similarly to the above we obtain the desired lower estimate

$$e_n(D_\sigma) \ge \sigma_{2^n} \cdot e_n(id : l_p^{2^n} \to l_q^{2^n}) \sim n^{-\delta}. \qquad \square$$

Proof of Theorem 4. Theorem 4 can be derived from Theorem 3 in the same way as Theorem 2 from Theorem 1 and therefore the details are omitted. $\square$

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