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CONVERGENCE TO ZERO AND BOUNDEDNESS OF OPERATOR-NORMED SUMS OF RANDOM VECTORS WITH APPLICATION TO AUTOREGRESSION PROCESSES

V. V. BULDYGIN, V. A. KOVAL

Abstract. The problems of almost sure convergence to zero and almost sure boundedness of operator-normed sums of different sequences of random vectors are studied. The sequences of independent random vectors, orthogonal random vectors and the sequences of vector-valued martingale-differences are considered. General results are applied to the problem of asymptotic behaviour of multidimensional autoregression processes.

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1. Introduction

The problems of almost sure convergence to zero and almost sure boundedness of sequences of random variables, and the interpretation of these problems from the viewpoint of probability in Banach spaces is comprised within the scope of scientific interests of N. N. Vakhania [1, 2]. This paper deals with almost sure convergence to zero and almost sure boundedness of operator-normed sums of random vectors. In Section 2, a brief survey of the results related to almost sure convergence to zero for these sums is given. In Section 3, integral type conditions for almost sure convergence to zero and almost sure boundedness of operator-normed sums of independent random vectors are obtained. In Section 4, these results are applied to the study of asymptotic behaviour of multidimensional autoregression processes.

We introduce the following notation: $\mathbb{R}^n$ is n-dimensional Euclidean space; $(A_n, n \geq 1) \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$, where $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$ is the class of all linear operators (matrices) mapping $\mathbb{R}^m$ into $\mathbb{R}^d$, $(m, d \geq 1)$; $\|x\|$ and $\langle x, y\rangle$ are the norm of the vector x and the inner product of the vectors x and y correspondingly; $\|A\| = \sup_{\|x\|=1} \|Ax\|$ is the norm of the operator (matrix) A; $A^*(A^\top)$ is the conjugate operator (matrix) of A; $\mathfrak{N}_\infty$ is the class of all strictly monotone infinite sequences of positive integers; $(X_k, k \geq 1)$ is a sequence of random vectors in $\mathbb{R}^m$ defined on a probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$; $S_n = \sum_{k=1}^n X_k$, $n \geq 1$; $1\{\mathcal{D}\}$ is

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the indicator of $\mathcal{D} \in \mathcal{F}$; $\sum_{i=n+1}^{n}(\cdot) = 0$. Recall that a random vector X is called symmetric if X and $-X$ are identically distributed.

2. Almost Sure Convergence to Zero of Operator-Normed Sums of Random Vectors

Prokhorov–Loève type conditions for almost sure convergence to zero of operator-normed sums of independent random vectors. First we consider the case where $(X_k, k \geq 1)$ is a sequence of independent random vectors.

Theorem 1 ([3]). *Let $(X_k,\ k \geq 1)$ be a sequence of independent symmetric random vectors. Assume that $\|A_n S_n\| \underset{n\to\infty}{\longrightarrow} 0$ almost surely. Then:*

(i) *for any $k \geq 1$*

$$\|A_n X_k\| \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{almost surely;}$$

(ii) *for any sequence $(n_j,\ j \geq 1) \in \mathfrak{N}_\infty$*

$$\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\| \underset{j\to\infty}{\longrightarrow} 0 \quad \textit{almost surely.}$$

Theorem 2 ([3]). *Let $(X_k,\ k \geq 1)$ be a sequence of independent symmetric random vectors. Assume that:*

(i) *for any $k \geq 1$*

$$\|A_n X_k\| \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{in probability.} \tag{1}$$

Then there exists a finite class $\mathfrak{N}_f \subset \mathfrak{N}_\infty$ depending on the sequence $(A_n,\ n \geq 1)$ only, such that, given that the condition

(ii)

$$\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\| \underset{j\to\infty}{\longrightarrow} 0 \quad \textit{almost surely}$$

holds for all $(n_j,\ j \geq 1) \in \mathfrak{N}_f$, one has

$$\|A_n S_n\| \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{almost surely.} \tag{2}$$

Remark. Let $(X_k,\ k \geq 1)$ be a sequence of independent random vectors which need not be symmetric. If assumptions (i) and (ii) in Theorem 2 hold and if $\|A_n S_n\| \to 0$ in probability as $n \to \infty$, then (2) holds.

Almost sure convergence to zero of operator-normed sums of orthogonal random vectors. Now consider the case where $(X_k, k \geq 1)$ is a sequence of orthogonal random vectors. By definition, this means that $\mathbf{E}\|X_k\|^2 < \infty$, $k \geq 1$, and for any $a \in \mathbb{R}^m$ and $j \neq k$ $\mathbf{E}(\langle a, X_j\rangle\langle a, X_k\rangle) = 0$.

Theorem 3 ([4]). *Let $(X_k,\ k \geq 1)$ be a sequence of orthogonal random vectors. Assume that condition (1) holds. Then there exists a finite class $\mathfrak{N}_f \subset$*

$\mathfrak{N}_\infty$ *depending on the sequence* $(A_n,\ n \geq 1)$ *only, such that, given that the condition*

$$\sum_{j=1}^{\infty} \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E}\|A_{n_{j+1}} X_k\|^2 (\log 4(n_{j+1} - n_j))^2 < \infty$$

holds for all $(n_j,\ j \geq 1) \in \mathfrak{N}_f$, *one has* (2).

Corollary 1. *Let* $(X_k,\ k \geq 1)$ *be a sequence of orthogonal random vectors. Assume that* $\|A_n\| \to 0$ *as* $n \to \infty$ *and suppose that*

$$\sum_{k=1}^{\infty} \sup_{n \geq k} (\mathbf{E}\|A_n X_k\|^2 \log^2 n) < \infty.$$

Then (2) *holds.*

Almost sure convergence to zero of operator-normed vector-valued martingales. Let $(S_n, \mathcal{F}_n, n \geq 0)$, $S_0 = 0$, be a martingale in $\mathbb{R}^m$, that is $X_k = S_k - S_{k-1}$, $k \geq 1$, is a martingale-difference in $\mathbb{R}^m$.

Theorem 4 ([5]). *Let* $(X_k,\ k \geq 1)$ *be a martingale-difference. Assume that condition* (1) *holds. Then there exists a finite class* $\mathfrak{N}_f \subset \mathfrak{N}_\infty$ *depending on the sequence* $(A_n,\ n \geq 1)$ *only, such that, given that the condition*

$$\sum_{j=1}^{\infty} \mathbf{E}(\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\| 1\{\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\| > \varepsilon\}) < \infty,$$

or equivalently, given the condition

$$\sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \mathbf{P}\{\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\| > t\} dt < \infty,$$

holds for all $(n_j,\ j \geq 1) \in \mathfrak{N}_f$ *and any* $\varepsilon > 0$, *one has* (2).

Theorem 5 ([5, 6]). *Let* $(X_k,\ k \geq 1)$ *be a martingale-difference;* $p > 1$, *and* $\mathbf{E}\|X_k\|^p < \infty$, $k \geq 1$. *Assume that condition* (1) *holds. Then there exists a finite class* $\mathfrak{N}_f \subset \mathfrak{N}_\infty$ *depending on the sequence* $(A_n,\ n \geq 1)$ *only, such that, given that the condition*

$$\sum_{j=1}^{\infty} \mathbf{E}\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\|^p < \infty,$$

holds for all $(n_j,\ j \geq 1) \in \mathfrak{N}_f$, *one has* (2).

Corollary 2. *Let* $(X_k,\ k \geq 1)$ *be a martingale-difference;* $p \in (1, 2]$, *and* $\mathbf{E}\|X_k\|^p < \infty$, $k \geq 1$. *Assume that condition* (1) *holds. Then there exists a finite class* $\mathfrak{N}_f \subset \mathfrak{N}_\infty$ *depending on the sequence* $(A_n,\ n \geq 1)$ *only, such that, given that the condition*

$$\sum_{j=1}^{\infty} \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E}\|A_{n_{j+1}} X_k\|^p < \infty,$$

holds for all $(n_j,\ j \geq 1) \in \mathfrak{N}_f$, *one has* (2).

Corollary 3. *Let $(X_k,\ k \geq 1)$ be a martingale-difference; $p \in (1, 2]$, and $\mathbf{E}\|X_k\|^p < \infty$, $k \geq 1$. Assume that condition* (1) *holds. If*

$$\sum_{k=1}^{\infty} \sup_{n \geq k} \|A_n X_k\|^p < \infty,$$

then (2) *holds.*

Corollary 3 implies a result due to Kaufmann [7].

Corollary 4. *Let $(X_k,\ k \geq 1)$ be a martingale-difference; $p \in (1, 2]$, and $\mathbf{E}\|X_k\|^p < \infty$, $k \geq 1$. Assume that $\|A_n\| \to 0$ as $n \to \infty$ and $\|A_n x\| \geq \|A_{n+1} x\|$ for all $x \in \mathbb{R}^m$, $n \geq 1$. If*

$$\sum_{k=1}^{\infty} \|A_k X_k\|^p < \infty,$$

then (2) *holds.*

Example. Consider the assumption leading to strong consistency of the least squares estimator $\hat{\theta}_n$, $n \geq 1$, of an unknown parameter $\theta \in \mathbb{R}^m$ in the multivariate linear regression model $Y_k = B_k \theta + Z_k$, $k \geq 1$. Here $(Z_k,\ k \geq 1)$ is a martingale-difference in $\mathbb{R}^d$; $(B_k,\ k \geq 1) \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$. Since

$$\hat{\theta}_n = \theta + \Big(\sum_{k=1}^{n} B_k^\top B_k \Big)^{-1} \sum_{k=1}^{n} B_k^\top Z_k, \quad n \geq 1,$$

taking $p = 2$ in Theorem 5 implies a result due to Lai [8]:

If $\sup_{k \geq 1} \mathbf{E}\|Z_k\|^2 < \infty$ *and if* $\Big\| \Big(\sum_{k=1}^{n} B_k^\top B_k \Big)^{-1} \Big\| \to 0$ *as* $n \to \infty$, *then* $\|\hat{\theta}_n - \theta\| \to 0$ *almost surely as* $n \to \infty$.

Almost sure convergence to zero of operator-normed sub-Gaussian vector-valued martingales. A stochastic sequence $(Z_k, k \geq 1) = (Z_k, \mathcal{F}_k,\ k \geq 1)$ is called a sub-Gaussian martingale-difference in $\mathbb{R}^m$, if: 1) $(Z_k, \mathcal{F}_k,\ k \geq 1)$ is a martingale-difference in $\mathbb{R}^m$; 2) for any $k \geq 1$ the random vector $Z_k = (Z_{k1}, \ldots, Z_{km})^\top$ is conditionally sub-Gaussian with respect to the σ-algebra $\mathcal{F}_{k-1}$. Assumption 2) means that $\tau(Z_{kj}) < \infty$ for any $j = 1, \ldots, m$ and any $k \geq 1$. Here

$$\tau(Z_{kj}) = \inf\{a \geq 0 :\ \mathbf{E}_{\mathcal{F}_{k-1}} e^{u Z_{kj}} \leq e^{a^2 u^2 / 2} \quad \text{almost surely, } u \in \mathbb{R}\}.$$

We write $\tau(Z_k) = \max_{j=1,\ldots,m} \tau(Z_{kj})$, $k \geq 1$.

Theorem 6. *Let $X_k = B_k Z_k$, $k \geq 1$, where $(Z_k,\ k \geq 1)$ is a sub-Gaussian martingale-difference in $\mathbb{R}^m$, $\sup_{k \geq 1} \tau(Z_k) < \infty$ and let $(B_k, k \geq 1) \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$. Assume that condition* (1) *holds. Then there exists a finite class $\mathfrak{N}_f \subset \mathfrak{N}_\infty$ depending on the sequence $(A_n,\ n \geq 1)$ only, such that, given that the condition*

$$\sum_{j=1}^{\infty} \exp\Big(-\varepsilon \Big(\sum_{k=n_j+1}^{n_{j+1}} \|A_{n_{j+1}} B_k\|^2 \Big)^{-1} \Big) < \infty \tag{3}$$

holds for all $(n_j,\ j \geq 1) \in \mathfrak{N}_f$ and any $\varepsilon > 0$, one has (2).

Remark. If $(Z_k,\ k \geq 1)$ is a sequence of independent Gaussian vectors, then conditions (1) and (3) are necessary for (2), see [3].

3. Almost Sure Convergence to Zero and Almost Sure Boundedness of Operator-Normed Sums of Independent Random Vectors

Theorem 7. *Let $(X_k, k \geq 1)$ be a sequence of independent zero-mean random vectors. Assume that*

$$\|A_n\| \underset{n\to\infty}{\longrightarrow} 0 \tag{4}$$

and for any sequence $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ there exists $\delta \in (0,1]$ such that

$$\sum_{j=1}^{\infty} \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E}\|A_{n_{j+1}} X_k\|^{2+\delta} < \infty. \tag{5}$$

If for any sequence $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ and all $\varepsilon > 0$

$$\sum_{j=1}^{\infty} \exp\Big(-\varepsilon\Big(\sum_{k=n_j+1}^{n_{j+1}} \mathbf{E}\|A_{n_{j+1}} X_k\|^2\Big)^{-1}\Big) < \infty, \tag{6}$$

then (2) holds.

If $X_k, k \geq 1$, are symmetric random vectors, then this theorem follows from Theorem 4.1.1 in [3]. Now we consider an independent proof in general case. To prove the theorem, we need the following lemma [9].

Lemma 1. *Let $(\xi_1, \ldots, \xi_n)$ be independent random variables such that $\mathbf{E}|\xi_i|^{2+\delta} < \infty$, $1 \leq i \leq n$, for some $\delta \in (0,1]$. Let η be a zero-mean Gaussian random variable such that $\mathbf{E}\eta^2 = \sum_{i=1}^n \mathbf{E}\xi_i^2$. Then for any $t > 0$*

$$\left|\mathbf{P}\Big(\Big|\sum_{i=1}^n \xi_i\Big| > t\Big) - \mathbf{P}\Big(|\eta| > t\Big)\right| \leq \frac{c}{t^{2+\delta}} \sum_{i=1}^n \mathbf{E}|\xi_i|^{2+\delta}$$

where c is an absolute constant.

Proof of Theorem 7. Let $(e_1, \ldots, e_d)$ be an orthonormal basis in $\mathbb{R}^d$. Then

$$A_n S_n = \sum_{k=1}^d \langle A_n^* e_k, S_n\rangle e_k, \qquad n \geq 1.$$

Therefore the theorem will be proved if we show that

$$\langle A_n^* e, S_n\rangle \underset{n\to\infty}{\longrightarrow} 0 \quad \text{almost surely}$$

for an arbitrary fixed vector $e \in \mathbf{R}^d$, $\|e\| = 1$. Since by (4)

$$\|A_n^* e\| \underset{n\to\infty}{\longrightarrow} 0$$

and since $(S_n, n \geq 1)$ is a martingale, it is sufficient to show, by Theorem 4, that for all $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ and any $\varepsilon > 0$

$$\sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \mathbf{P}\Big(\Big| \sum_{k=n_j+1}^{n_{j+1}} \langle A^*_{n_{j+1}} e, X_k \rangle \Big| > t \Big)\, dt < \infty. \tag{7}$$

By Lemma 1, for any $j \geq 1$

$$\begin{aligned} \mathbf{P}\Big(\Big| \sum_{k=n_j+1}^{n_{j+1}} \langle A^*_{n_{j+1}} e, X_k \rangle \Big| > t \Big) & \\ \leq \mathbf{P}\left(|\eta_j| > t\right) + \frac{c}{t^{2+\delta}} \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E} |\langle A^*_{n_{j+1}} e, X_k \rangle|^{2+\delta} & \\ \leq \mathbf{P}\left(|\eta_j| > t\right) + \frac{c}{t^{2+\delta}} \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E} \|A_{n_{j+1}} X_k\|^{2+\delta}, & \qquad (8) \end{aligned}$$

where η_j is a normally distributed random variable with parameters

$$\mathbf{E}\eta_j = 0, \qquad \mathbf{E}\eta_j^2 = \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E} \langle A^*_{n_{j+1}} e, X_k \rangle^2.$$

Since for all $t > 0$

$$\mathbf{P}\left(|\eta_j| > t\right) \leq 2 \exp\left(-\frac{t^2}{2\mathbf{E}\eta_j^2} \right)$$

and

$$\mathbf{E}\eta_j^2 \leq \sum_{k=n_j+1}^{n_{j+1}} \mathbf{E} \|A_{n_{j+1}} X_k\|^2,$$

condition (6) implies

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \mathbf{P}\left(|\eta_j| > t\right) dt &\leq 4\sqrt{2} \sum_{j=1}^{\infty} \left(\mathbf{E}\eta_j^2 \right)^{\frac{1}{2}} \exp\left(-\frac{\varepsilon^2}{2\mathbf{E}\eta_j^2} \right) \\ &\leq 4\sqrt{2} \sup_{j \geq 1} \left(\mathbf{E}\eta_j^2 \right)^{\frac{1}{2}} \sum_{j=1}^{\infty} \exp\left(-\frac{\varepsilon^2}{2\mathbf{E}\eta_j^2} \right) < \infty. \end{aligned}$$

Taking into account the inequalities (8) and (5) we conclude that (7) holds. □

Theorem 8. *Let $(X_k, k \geq 1)$ be a sequence of independent zero-mean random vectors. Assume that $\sup_{n \geq 1} \|A_n\| < \infty$ and that for any $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ there exists $\delta \in (0, 1]$ such that (5) holds. If for any $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ there exists $\varepsilon > 0$ such that (6) holds, then*

$$\sup_{n \geq 1} \|A_n S_n\| < \infty \quad \textit{almost surely.}$$

Proof. Theorem 8 follows from Theorem 7 and Lemma 3 in [10]. We refer the reader to [3] for more details. □

Theorem 9. *Let $(X_k, k \geq 1)$ be a sequence of independent zero-mean random vectors. Assume that $\|A_n\| \underset{n\to\infty}{\longrightarrow} 0$ and for any $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ there exists $\delta \in (0,1]$ such that (5) holds. If for any $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ there exists $\varepsilon > 0$ such that (6) holds, then there exists a nonrandom constant $L \in [0,\infty)$ such that*

$$\limsup_{n\to\infty} \|A_n S_n\| = L \quad \textit{almost surely.} \tag{9}$$

Proof. Theorem 9 follows from Theorem 8 and Theorem 2.3.2 in [3]. We refer the reader to [3] for more details. $\square$

The next corollary of Theorem 9 can be useful in applications.

Corollary 5. *Let $(X_i, i \geq 1)$ be a sequence of independent zero-mean random vectors. Assume that $\|A_n\| \underset{n\to\infty}{\longrightarrow} 0$ and suppose that there exists $\delta \in (0,1]$ such that the following condition holds*

$$\sum_{i=1}^{\infty} \sup_{n\geq i} \mathbf{E}\|A_n X_i\|^{2+\delta} < \infty. \tag{10}$$

Assume also that there are two sequences of positive numbers $(\varphi_n, n \geq 1)$, $(f_n, n \geq 1)$, such that $(f_n, n \geq 1)$ is monotonically nondecreasing and the inequality

$$\sum_{i=k+1}^{n} \mathbf{E}\|A_n X_i\|^2 \leq \varphi_n \left[1 - \left(\frac{f_k}{f_n}\right)^2\right] \tag{11}$$

holds for all $n > k \geq 1$. If for some $M > 0$

$$\sup_{n\geq 2} \varphi_n \ln\left(2 + \sum_{k=1}^{n-1} \min\left\{M, \ln\left(\frac{f_{k+1}}{f_k}\right)\right\}\right) < \infty, \tag{12}$$

then there exists a nonrandom constant $L \in [0,\infty)$ such that (9) holds.

Proof. Let us show that (5) follows from (10):

$$\begin{aligned}\sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \mathbf{E}\|A_{n_{j+1}} X_i\|^{2+\delta} &\leq \sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \sup_{n\geq i} \mathbf{E}\|A_n X_i\|^{2+\delta} \\ &\leq \sum_{i=1}^{\infty} \sup_{n\geq i} \mathbf{E}\|A_n X_i\|^{2+\delta} < \infty.\end{aligned}$$

In addition, for all $(n_j, j \geq 1) \in \mathfrak{N}_\infty$ and some $\varepsilon > 0$ the condition (6) follows from (11) and (12), see [5]. $\square$

4. Asymptotic Behaviour of Multidimensional Autoregression Processes

Let us apply Corollary 5 for obtaining an analog of the bounded law of iterated logarithm for autoregressive processes. Let $\mathbb{R}^m$ be a space of column vectors. Consider an autoregression equation in $\mathbb{R}^m$ written as follows:

$$Y_n = AY_{n-1} + V_n, \quad n \geq 1,\, Y_0 \in \mathbb{R}^m. \tag{13}$$

Here A is an arbitrary $m \times m$ matrix and $(V_n, n \geq 1)$ is a sequence of independent zero-mean random vectors in $\mathbb{R}^m$. Denote by r the spectral radius of the matrix A and by p the maximal multiplicity of the roots of the minimal polynomial of the matrix A whose the absolute value is r. Introduce the following notation:

$$f_n = \sum_{i=1}^{n} \mathbf{E}\|V_i\|^2 r^{-2i}\Big(1 - \frac{i-1}{n}\Big)^{2(p-1)}, \qquad n \geq 1;$$

$$L_n\{f\} = \ln\left(2 + \sum_{k=1}^{n-1} \min\left\{M, \ln\left(\frac{f_{k+1}}{f_k}\right)\right\}\right), \qquad n \geq 2,$$

where M is an arbitrary positive fixed constant and where

$$\chi_n = r^n n^{p-1}\sqrt{f_n L_n\{f\}}, \qquad n \geq 2.$$

Observe that the sequence $\{f\} = (f_n, n \geq 1)$ is monotone nondecreasing. The bounded law of the iterated logarithm for solutions of equation (13) in the case $\mathbf{E}\|V_n\|^2 \equiv \text{const}$ was studied in [11]. Let us consider a general case.

Theorem 10. *Let $\lim_{n\to\infty} f_n = \infty$. Assume that for some $\delta \in (0,1]$ the condition*

$$\sum_{i=1}^{\infty} \sup_{n \geq i} \Big[\chi_n^{-1}(n+1-i)^{p-1} r^{n-i}\Big]^{2+\delta} \mathbf{E}\|V_i\|^{2+\delta} < \infty \tag{14}$$

holds. Then there exists a nonrandom constant $L \in [0, \infty)$ such that

$$\limsup_{n\to\infty} \frac{\|Y_n\|}{\chi_n} = L \qquad \textit{almost surely.} \tag{15}$$

Proof. Assume that $\det A \neq 0$. Then (13) gives

$$Y_n = A^n Y_0 + A^n \sum_{i=1}^{n} A^{-i} V_i\,, \quad n \geq 1.$$

Hence

$$\chi_n^{-1}\|Y_n\| \leq \chi_n^{-1}\|A_n\| \cdot \|Y_0\| + \Big\|\chi_n^{-1} A^n \sum_{i=1}^{n} A^{-i} V_i\Big\|, \quad n \geq 1. \tag{16}$$

Observe that

$$\|A^n\| \leq c_1 (n+1)^{p-1} r^n, \quad n \geq 0, \tag{17}$$

where c_1 is a constant independent of n, see [3]. By (17) one has

$$\chi_n^{-1}\|A^n\| \cdot \|Y_0\| \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore (15) will be proved if we show that there exists a nonrandom constant $L \in [0,\infty)$ such that

$$\limsup_{n\to\infty} \left\| \chi_n^{-1} A^n \sum_{i=1}^{n} A^{-i} V_i \right\| = L \qquad \text{almost surely.} \tag{18}$$

To prove (18), we use Corollary 5 with

$$A_n = \chi_n^{-1} A^n, n \ge 1; \quad X_i = A^{-i} V_i, i \ge 1.$$

Condition (10) follows from (14) by (16). Since (see [5] for more details)

$$\begin{aligned}\sum_{i=k+1}^{n} \|\chi_n^{-1} A^n A^{-i} V_i\|^2 &\le c_1^2 \chi_n^{-2} \sum_{i=k+1}^{n} r^{2(n-i)} (n+1-i)^{2(p-1)} \mathbf{E}\|V_i\|^2 \\ &\le c_1^2 \left(\chi_n^{-1} r^n n^{p-1} \sqrt{f_n}\right)^2 \left[1 - \left(\frac{f_k}{f_n}\right)^2\right],\end{aligned}$$

condition (11) holds. Condition (12) also holds for

$$\varphi_n = c_1^2 \left(\chi_n^{-1} r^n n^{p-1} \sqrt{f_n}\right)^2, \ n \ge 1.$$

Since

$$\chi_n^{-1}\|A^n\| \underset{n\to\infty}{\longrightarrow} 0,$$

Corollary 5 implies that there exists a nonrandom constant $L \in [0,\infty)$ such that

$$\limsup_{n\to\infty} \left\| \chi_n^{-1} A^n \sum_{i=1}^{n} A^{-i} V_i \right\| = L \qquad \text{almost surely.}$$

The theorem is proved in the case of $\det A \ne 0$. The proof of the theorem in the case of $\det A = 0$ follows similar lines [12]. $\square$

Corollary 6. *Let $r = 1$ and $\mathbf{E}\|V_n\|^2 \equiv \text{const}$. Assume that*

$$\sup_{n\ge1} \mathbf{E}\|V_n\|^{2+\delta} < \infty$$

for some $\delta \in (0,1]$. Then there exists a nonrandom constant $L \in [0,\infty)$ such that

$$\limsup_{n\to\infty} \frac{\|Y_n\|}{\sqrt{n^{2p-1}\ln\ln n}} = L \qquad \textit{almost surely.}$$

Proof. Since $r = 1$ and since $\mathbf{E}\|V_n\|^2 \equiv \text{const}$, one has

$$\chi_n \sim c_2 \sqrt{n^{2p-1}\ln\ln n} \qquad (n \to \infty),$$

where c_2 is some positive constant, see [5]. Since for any $i \ge 3$

$$\sup_{n\ge i} \left[(n^{2p-1}\ln\ln n)^{-\frac{1}{2}} (n+1-i)^{p-1}\right]^{2+\delta} \le \frac{1}{(\ln\ln 3)^{\frac{2+\delta}{2}}} \cdot \frac{1}{i^{1+\frac{\delta}{2}}},$$

one has

$$\sum_{i=3}^{\infty} \sup_{n\geq i} \left[(n^{2p-1} \ln\ln n)^{-\frac{1}{2}} (n+1-i)^{p-1}\right]^{2+\delta} < \infty.$$

Thus the condition (14) holds and Corollary 6 is proved. □

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Authors' address:
Department of Higher Mathematics No. 1
National Technical university of Ukraine
Kyiv Polytechnic Institute
prosp. Peremohy 37, 02056 Kyiv
Ukraine