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TRACTABILITY OF TENSOR PRODUCT LINEAR OPERATORS IN WEIGHTED HILBERT SPACES

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Dedicated to N. Vakhania
on the occasion of his 70th birthday

Abstract. We study tractability in the worst case setting of tensor product linear operators defined over weighted tensor product Hilbert spaces. Tractability means that the minimal number of evaluations needed to reduce the initial error by a factor of ε in the d-dimensional case has a polynomial bound in both ε^{-1} and d. By one evaluation we mean the computation of an arbitrary continuous linear functional, and the initial error is the norm of the linear operator S_d specifying the d-dimensional problem.

We prove that nontrivial problems are tractable iff the dimension of the image under S_1 (the one-dimensional version of S_d) of the unweighted part of the Hilbert space is one, and the weights of the Hilbert spaces, as well as the singular values of the linear operator S_1, go to zero polynomially fast with their indices.

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1. Introduction

I am pleased to dedicate this paper to Professor Nicholas N. Vakhania on the occasion of his 70th birthday. I enjoyed many meetings with Professor Vakhania in Dagstuhl, Tbilisi, Warsaw, and New York. I have learned a lot from his books "Probability Distributions on Linear Spaces" [10], and "Probability Distributions on Banach Spaces" [11] (the second book written jointly with V. I. Tarieladze and S. A. Chobanyan). These books have been very helpful in my work and in the work of many colleagues working in the average case setting of information-based complexity, and are always cited in papers dealing with this subject.

I also wish to add that Professor Vakhania solved an important problem in the average case complexity. His paper [12] and the paper [3] prove that every ill-posed problem specified by a measurable unbounded linear operator is well-posed on the average for any Gaussian measure, and its average case complexity is finite for any positive error demand.

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In recent years, my research interests has shifted to multivariate problems and tractability issues. That is why I have decided to send a paper on this subject in token of my appreciation of Professor N. N. Vakhania.

Tractability of multivariate problems has recently become a popular research subject. The reader is referred to [1, 2, 4, 6, 7, 13, 14, 15, 16, 17], and to the surveys [5, 9]. Tractability means that a minimal number of evaluations needed to reduce the initial error by a factor of ε in the d-dimensional case has a polynomial bound in both ε^{-1} and d. Strong tractability means that this bound is independent of d, and is polynomially dependent only on ε^{-1}. Tractability is studied in various settings. Here, we only consider the worst case setting, in which an approximation error is defined as a maximal error over the unit ball of a Hilbert space.

In this paper, we study tractability of linear operators between tensor product Hilbert spaces. In the d-dimensional case, a linear operator is defined as a tensor product of d copies of the same linear operator. The domain of a linear operator is assumed to be a *weighted* tensor product Hilbert space with weights $\gamma_1 \geq \gamma_2 \geq \ldots > 0$, where γ_j corresponds to the jth component of the tensor product.

We comment on the role of the weights γ_j. For the jth component, we consider a Hilbert space that is a direct sum of two Hilbert spaces H_1 and H_2, where $H_1 \cap H_2 = \{0\}$. The space H_1 corresponds to the unweighted part, whereas the space H_2 corresponds to the weighted part with the weight γ_j. That is, for $f = f_1 + f_2$ with $f_i \in H_i$ we have

$$\|f\|^2 = \|f_1\|^2_{H_1} + \gamma_j^{-1} \|f_2\|^2_{H_2}.$$

Hence, if $\|f\| \leq 1$ and γ_j is small, then f_2 must be small too. In this way, the weight γ_j controls the size of the weighted components.

We approximate linear operators evaluating finitely many arbitrary (continuous) linear functionals. The case of a restricted choice of such functionals is studied, e.g., in [14], and leads to different results. For example, we shall prove that there is no difference between strong tractability and tractability. This is *not* the case if we use only function evaluations instead of arbitrary linear functionals.

We want to reduce the initial error which is defined as the norm of the linear operator S_d defining the d-dimensional problem. The initial error is the error, which can be obtained without any evaluation and which formally corresponds to the error of the zero approximation.

The main result of this paper is a full characterization of tractable linear operators. Obviously, some linear operators are trivially tractable. This certainly holds for linear functionals, which can be recovered exactly by just one evaluation. It turns out that the class of tractable linear operators is exactly equal to the class of linear functionals iff the weights γ_j of the Hilbert spaces have the sum-exponent equal to infinity. Here, the *sum-exponent* p_γ is defined as the infimum of nonnegative β for which

$$\sum_{j=1}^{\infty} \gamma_j^{\beta} < \infty.$$

Thus, $p_\gamma = \infty$ means that the last series is always divergent. This holds, in particular, for the unweighted case $\gamma_j = 1$, for which there is no difference between the components from H_1 and H_2.

It is easy to see that $p_\gamma < \infty$ iff the weights γ_j go to zero polynomially fast with j^{-1}. If $p_\gamma < \infty$, then there are non-trivial strongly tractable linear operators. They are fully characterized by two conditions. The first condition is that the dimension of the image of S_1 (the one-dimensional version of S_d) of the unweighted part is one. The second condition is that the sum-exponent p_λ of the singular values $\sqrt{\lambda_i}$ of S_1 is finite. We stress that the requirement $p_\lambda < \infty$ is a stronger condition than mere compactness of S_1.

For strongly tractable problems, the strong exponent is defined as the infimum of p for which a minimal number of evaluations needed to reduce the initial error by a factor ε is of order ε^{-p} for all d. We find that the strong exponent is $2\max(p_\lambda, p_\gamma)$. We stress that the strong exponent is large if either p_λ or p_γ is large.

As mentioned before, we use arbitrary linear functionals as tools for approximating linear operators in this paper. We plan to analyze the natural restriction of arbitrary linear functionals to function evaluations in the near future. Preliminary results indicate that a full characterization of strongly tractable and tractable linear operators is more complex, depending on the specific Hilbert space used when $d = 1$.

2. Tractability and Strong Tractability

Let H_1 and H_2 be two Hilbert spaces such that $H_1 \cap H_2 = \{0\}$. Their inner products are denoted by $\langle \cdot, \cdot \rangle_{H_i}$. For $\gamma \in (0, 1]$, consider the Hilbert space $F_{1,\gamma} = H_1 \oplus H_2$ with the inner product

$$\langle f, g \rangle_{F_{1,\gamma}} = \langle f_1, g_1 \rangle_{H_1} + \gamma^{-1} \langle f_2, g_2 \rangle_{H_2},$$

where $f, g \in F_{1,\gamma}$ have the unique representation $f = f_1 + f_2$, $g = g_1 + g_2$ with $f_1, g_1 \in H_1$ and $f_2, g_2 \in H_2$. Observe that the components in H_1 are unweighted, whereas the components in H_2 are weighted with the parameter γ. If $\|f\|_{F_{1,\gamma}} \le 1$ for small γ, then the component f_2 is negligible. For $\gamma = 1$, we let $F_1 = F_{1,1}$.

Let G_1 be a Hilbert space, and let $S_1 : F_1 \to G_1$ be a (continuous) linear operator. Note that S_1 is also well defined on $F_{1,\gamma}$ for all $\gamma \in (0, 1]$.

For $d \ge 2$, define $F_d = F_1 \otimes \cdots \otimes F_1$ (d times) as the tensor product of d copies of the space F_1. That is, F_d is the completion of linear combinations of tensor products $f_1 \otimes \cdots \otimes f_d$, with $f_i \in F_1$, which we write, for simplicity, as $f_1 f_2 \cdots f_d$. Recall that if f_i are numbers, then their tensor product is just the product of these numbers, and if f_i are univariate functions, then their tensor product is the d-variate function $f(t_1,\dots,t_d)=\prod_{j=1}^d f_j(t_j)$.

The space F_d is a Hilbert space with the tensor product inner product defined as

$$\langle f,g\rangle_{F_d} \;=\; \prod_{j=1}^d \langle f_j,g_j\rangle_{F_1}\,.$$

for $f=f_1\dots f_d\in F_d$ and $g=g_1\cdots g_d\in F_d$ with $f_j,g_j\in F_1$. We define the Hilbert space $G_d=G_1\otimes\cdots\otimes G_1$ (d times) with the inner product $\langle\cdot,\cdot\rangle_{G_d}$ similarly.

For $d\ge 2$, let $S_d=S_1\otimes\cdots\otimes S_1 : F_d\to G_d$ denote the tensor product linear operator consisting of d copies of S_1. Thus, for $f=f_1\cdots f_d$ with $f_j\in F_1$, we have

$$S_d f \;=\; S_1f_1\,\cdots\, S_1f_d\,\in\, G_d.$$

Take now a sequence of weights

$$\gamma_1\;\ge\;\gamma_2\;\ge\;\cdots\;\ge\;\gamma_d\;\ge\;\cdots\;>\;0,$$

and consider the tensor product

$$F_{d,\gamma}\;=\;F_{1,\gamma_1}\,\otimes\,F_{1,\gamma_2}\,\otimes\,\cdots\,\otimes\,F_{1,\gamma_d}.$$

The space $F_{d,\gamma}$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{F_{d,\gamma}}$. To see how the weights γ_j affect the norm, take $f=f_1\cdots f_d$ with $f_j=f_{j,1}+f_{j,2}$, where $f_{j,1}\in H_1$, $f_{j,2}\in H_2$. Then

$$\|f\|^2_{F_{d,\gamma}}\;=\;\prod_{j=1}^d\left(\|f_{j,1}\|^2_{H_1}+\gamma_j^{-1}\|f_{j,2}\|^2_{H_2}\right).$$

Again, if $\|f\|_{F_{d,\gamma}}\le 1$ and the weights γ_j go to zero, then the components $f_{j,2}$ must approach zero.

Observe that the linear operator S_d is well defined on $F_{d,\gamma}$ independently of the weights γ_j. Although the values of S_df do not depend on γ_j, its adjoint S_d^* and its norm $\|S_d\|_{F_{d,\gamma}}$ *do* depend on γ_j. Indeed, for $d=1$, it is easy to check that $S_1^*:G_1\to F_{1,\gamma}$ is given by

$$S_1^*\;=\;S_1\Big|^*_{H_1}\;+\;\gamma\,S_1\Big|^*_{H_2},$$

where $S_1\Big|_{H_i}$ denotes the operator S_1 restricted to H_i and $S_1\Big|^*_{H_i}:G_1\to H_i$ is the adjoint operator of $S_1\Big|_{H_i}$. For $d\ge 2$, we have

$$S_d^*\;=\;\left(S_1\Big|^*_{H_1}+\gamma_1S_1\Big|^*_{H_2}\right)\,\otimes\,\cdots\,\otimes\,\left(S_1\Big|^*_{H_1}+\gamma_dS_1\Big|^*_{H_2}\right).$$

To obtain the norm of S_d, define the non-negative self-adjoint operator $W_{d,\gamma}=S_d^*S_d:F_{d,\gamma}\to F_{d,\gamma}$ and observe that

$$\|S_df\|^2_{G_d}\;=\;\langle S_df,S_df\rangle_{G_d}\;=\;\langle W_{d,\gamma}f,f\rangle_{F_{d,\gamma}}\,.$$

This implies that the norm $\|S_d\|_{F_{d,\gamma}}$ is equal to the square root of the largest eigenvalue of $W_{d,\gamma}$.

We want to approximate $S_d f$ for f from the unit ball of $F_{d,\gamma}$. Our approximations will be of the form[1]

$$U_{n,d}(f) = \sum_{i=1}^{n} L_i(f)\, g_i$$

for certain continuous linear functionals $L_i \in F^*_{d,\gamma}$, and for certain $g_i \in G_d$.

We define the error of $U_{n,d}$ in the worst case sense as the maximal distance between $S_d(f)$ and $U_{n,d}(f)$ over the unit ball of $F_{d,\gamma}$,

$$e(U_{n,d}, F_{d,\gamma}) = \sup_{f \in F_{d,\gamma}, \|f\|_{F_{d,\gamma}} \le 1} \|S_d(f) - U(f)\|_{G_d}.$$

For $n = 0$, we formally set $U_{0,d}(f) = 0$. Then $e(0, F_{d,\gamma}) = \|S_d\|_{F_{d,\gamma}}$ is the initial error, which is the a priori error without sampling the element f. Our goal is to reduce the initial error by a factor $\varepsilon \in (0,1)$. That is, we want to find $U_{n,d}$, or equivalently we want to find $L_i \in F^*_{d,\gamma}$ and $g_i \in G_d$ for $i = 1, \dots, n$, such that

$$e(U_{n,d}, F_{d,\gamma}) \le \varepsilon \, \|S_d\|_{F_{d,\gamma}}.$$

We are ready to recall the concepts of tractability and strong tractability, see [1, 2, 4, 5, 6, 13, 14, 15, 16, 17]. In some of these papers, tractability is defined for absolute errors. In this paper we define tractability for normalized errors. Let

$$n(\varepsilon, S_d) = \min\{\, n \,:\, \exists\, U_{n,d} \text{ with } e(U_{n,d}, F_{d,\gamma}) \le \varepsilon \, \|S_d\|_{F_{d,\gamma}} \,\}$$

be a minimal number of evaluations needed to reduce the initial error by a factor ε. Obviously, the minimal number $n(\varepsilon, S_d)$ also depends on the spaces $F_{d,\gamma}$ and G_d and therefore it depends on the weight sequence γ_j. In fact, as we shall see, the dependence on $\{\gamma_j\}$ will be crucial.

We say that the problem $\{S_d\}$ is *tractable* iff there exist nonnegative numbers C, q and p such that

$$n(\varepsilon, S_d) \le C\, d^q\, \varepsilon^{-p} \qquad \forall\, \varepsilon \in (0,1),\ \forall\, d = 1, 2, \dots . \tag{1}$$

The problem $\{S_d\}$ is *strongly* tractable if $q = 0$ in estimate (1) of $n(\varepsilon, S_d)$.

The essence of tractability is that the minimal number of evaluations is bounded by a polynomial in both d and ε^{-1}, and the essence of strong tractability is that this number has a bound independent of d and polynomial in ε^{-1}. The *exponent* of strong tractability is defined as the infimum of p satisfying (1).

Obviously, some linear operators S_d are trivially strongly tractable independently of the weights γ_j. This holds for $\dim(S_1(F_1)) = 0$, i.e., $S_1 = 0$,

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[1] It is known that neither nonadaptive information nor nonlinear approximations help in approximating linear operators over Hilbert spaces, see e.g., [8]. Hence, it is enough to study linear $U_{n,d}$.

since then $S_d = 0$ and $n(\varepsilon, S_d) = 0$ for all $\varepsilon \in (0,1)$ and $d \geq 1$. Furthermore, if $\dim(S_1(F_1)) = 1$ then S_1 is a continuous linear operator of rank 1, $S(f) = \langle f, h \rangle_{F_1} g$ for a nonzero $h \in F_1$ and a nonzero $g \in G_1$. Then

$$S_d(f) = \langle f, h_d \rangle_{F_d} g_d$$

with $h_d = h^d$ and $g_d = g^d$. This means that setting $L_1(f) = \langle f, h_d \rangle_{F_d}$ and $g_1 = g^d$ we get $e(U_{1,d}, F_{d,\gamma}) = 0$ for all d, and $n(\varepsilon, S_d) \leq 1$.

We define the set of such trivial strongly tractable operators as

$$\text{TRIV}(F_1) = \{ S_1 : \dim(S_1(F_1)) \leq 1 \}.$$

Let

$$\begin{aligned} \text{TRAC}(F_1, \gamma) &= \{ S_1 : \{S_d\} \text{ is tractable } \} \\ \text{STRONG-TRAC}(F_1, \gamma) &= \{ S_1 : \{S_d\} \text{ is strongly tractable } \} \end{aligned}$$

denote the sets of tractable and strongly tractable operators. Clearly,

$$\text{TRIV}(F_1) \subset \text{STRONG-TRAC}(F_1, \gamma) \subset \text{TRAC}(F_1, \gamma) \qquad \forall \gamma = \{\gamma_j\}.$$

The main purpose of this paper is to find the sets STRONG-TRAC(F_1, γ) and TRAC(F_1, γ) and to check whether, or when, they differ from the set TRIV(F_1). As we shall see the answer will depend on the weight sequence γ.

As we shall see in the next section, the exponent of strong tractability depends, in particular, on the sum-exponent of the weight sequence γ, see [14]. By the *sum-exponent* of any nonnegative non-increasing sequence $\{a_j\}$, we mean

$$p_a = \inf \left\{ \beta \geq 0 : \sum_{j=1}^{\infty} a_j^{\beta} \leq \infty \right\}$$

with the convention that $\inf \varnothing = \infty$. In particular, for $a_j = j^{-k}$ we have $p_a = 1/k$ for $k > 0$, and $p_a = \infty$ for $k \leq 0$.

It is easy to check that p_a is finite iff a_j goes to zero polynomially fast in j^{-1}. Indeed, if $\sum_{j=1}^{\infty} a_j^{\beta} = M < \infty$, then $a_j^{\beta} \leq M/j$ and $a_j = O(j^{-1/\beta})$ goes polynomially to zero. The other implication is trivial.

3. Tractability Results

We are ready to prove the main result of this paper.

Theorem 3.1. (1) *If* $p_\gamma = \infty$, *then*

$$\text{TRAC}(F_1, \gamma) = \text{STRONG-TRAC}(F_1, \gamma) = \text{TRIV}(F_1).$$

(2) *If* $p_\gamma < \infty$, *then*

$$\text{TRAC}(F_1, \gamma) = \text{STRONG-TRAC}(F_1, \gamma) = \text{TRIV}(F_1) \cup A(F_1, \gamma),$$

where

$$A(F_1, \gamma) = \{ S_1 : \dim(S_1(H_1)) = 1 \text{ and } p_\lambda < \infty \},$$

where p_λ *is the sum-exponent*[2] *of the ordered eigenvalues* λ_i *of the self-adjoint operator* $S_1\big|_{H_2}^{*} S_1\big|_{H_2}$. *Furthermore, if* $S_1 \in A(F_1, \gamma)$*, then the strong exponent is zero if* $\lambda_2 = 0$*, and* $2\max(p_\gamma, p_\lambda)$ *if* $\lambda_2 > 0$.

Before we prove Theorem 3.1, we comment on its statements. The first part assumes that the sum-exponent of the weights is infinity. This covers the unweighted case $\gamma_j = 1$, for which there is no difference in treating the components from the spaces H_1 and H_2. In this case the result is negative. The only tractable problems are trivial and given by linear operators of rank at most one. As already noticed, such problems can be solved exactly with at most one evaluation, and therefore the strong exponent is zero.

The second case assumes that the sum-exponent of the weights is finite. This means that γ_j goes to zero polynomially fast as j goes to infinity. Then trivial problems are not only strongly tractable problems since strong tractability also holds for problems from $A(F_1, \gamma)$. For such problems we must have that S_1 reduced to H_1 has rank one, and S_1 reduced to H_2 has singular values[3] that go to zero polynomially fast. This, of course, implies that S_1 is a compact operator. However, the converse is, in general, not true. That is, a compact S_1 with $\dim(S_1(H_1)) = 1$ does not necessarily have finite p_λ.

We stress that although we have strong tractability, the strong exponent can be very large. Indeed, if the singular values of $S_1\big|_{H_2}$ or if the weights γ_j go slowly to zero, then at least one of the sum-exponents is large, which implies a large strong exponent. For example, assume that $\lambda_j = j^{-k_1}$, and $\gamma_j = j^{-k_2}$ with positive k_i. Then $p = 2\max(1/k_1, 1/k_2)$ so that if either k_1 or k_2 is small, then p is large.

We stress that for both cases in Theorem 3.1, there is no difference between strong tractability and tractability. That is, the minimal number of evaluations needed to reduce the initial error by a factor of ε has either a bound independent of d or more than polynomially dependent on d. This is a consequence of two assumptions that we have made. The first assumption is that we use arbitrary continuous linear functionals. However, if we restrict ourselves to only function evaluations, then there is a difference between tractability and strong tractability, as proven in [1, 2, 6, 7]. The second assumption is that the weights are non-increasing and nested. That is, for the $(d+1)$-dimensional case we use the same weights as for the d-dimensional case plus γ_{d+1}. For more general weights, as shown in [14], there is a difference between tractability and strong tractability even if arbitrary continuous linear functionals are used.

We now turn to the proof of Theorem 3.1. It will consist of two parts. The first part will be to prove that $\dim(S_1(H_1)) \geq 2$ implies intractability of $\{S_d\}$ independently of the weights γ_j. That is, if S_1 has rank at least two in the unweighted part of the space F_1, then the behavior of S_1 in H_2, as well as

[2]If $\dim(H_2) < \infty$, then we extend the finite sequence λ_j of eigenvalues by setting $\lambda_j = 0$ for $j > \dim(H_2)$. Then, obviously, $p_\lambda = 0$.

[3]A singular value of a linear operator S is the square root of an eigenvalue of S^*S.

the weights γ_j, are irrelevant, and we have intractability. This shows that the unweighted part of the space F_1 allows only at most rank one operators to get tractability. The second part of the proof assumes that $\dim(S_1(H_1)) \le 1$. This case easily reduces to the problem studied in [14], and a slight modification of the proof from [14] allows us to complete the proof of Theorem 3.1.

Lemma 3.1. *If* $\dim(S_1(H_1)) \ge 2$*, then* $\{S_d\}$ *is intractable.*

Proof. For $d = 1$, consider the operator $W_{1,\gamma} = S_1^* S_1 : F_{1,\gamma} \to F_{1,\gamma}$. Let $\lambda_{i,\gamma}$ denote the ordered eigenvalues of $W_{1,\gamma}$, $\lambda_{1,\gamma} \ge \lambda_{2,\gamma} \ge \cdots \ge 0$. The largest $\lambda_{1,\gamma}$ is also equal to the square of the norm $\|S_1\|_{F_{1,\gamma}}$. For $f = f_1 + f_2$, $f_i \in H_i$, we have $\|f\|^2_{F_{1,\gamma}} = \|f_1\|^2_{H_1} + \gamma^{-1}\|f_2\|^2_{H_2}$ and

$$\begin{aligned}\|S_1 f\|_{G_1} &\le \left\| S_1\big|_{H_1} \right\|_{H_1} \|f_1\|_{H_1} + \gamma^{1/2} \left\| S_1\big|_{H_2} \right\|_{H_2} \gamma^{-1/2}\|f_2\|_{H_2} \\ &\le \left(\left\| S_1\big|_{H_1} \right\|^2_{H_1} + \gamma \left\| S_1\big|_{H_2} \right\|^2_{H_2} \right)^{1/2} \|f\|_{F_{1,\gamma}}.\end{aligned}$$

This proves that

$$\lambda_{1,\gamma} \le \left(\left\| S_1\big|_{H_1} \right\|^2_{H_1} + \gamma \left\| S_1\big|_{H_2} \right\|^2_{H_2} \right)^{1/2} \le \left(\left\| S_1\big|_{H_1} \right\|^2_{H_1} + \left\| S_1\big|_{H_2} \right\|^2_{H_2} \right)^{1/2}$$

since $\gamma \in (0, 1]$.

We need a lower bound estimate of $\lambda_{2,\gamma}$. Recall that

$$\lambda_{2,\gamma} = \inf_{h \in F_{1,\gamma}} \sup_{f \in F_{1,\gamma},\, \langle f,h \rangle_{F_{1,\gamma}} = 0} \frac{\langle W_{1,\gamma} f, f \rangle_{F_{1,\gamma}}}{\langle f, f \rangle_{F_{1,\gamma}}}.$$

If we replace $F_{1,\gamma}$ in the supremum by H_1 then we obtain a lower bound on $\lambda_{2,\gamma}$. For $f \in H_1$, we have

$$\langle W_{1,\gamma} f, f \rangle_{F_{1,\gamma}} = \|S_1 f\|^2_{G_1} = \langle V f, f \rangle_{H_1},$$

where $V = S_1\big|^*_{H_1} S_1\big|_{H_1} : H_1 \to H_1$. Since $\langle f, h \rangle_{F_{1,\gamma}} = \langle f, h_1 \rangle_{H_1}$, the last infimum over $h \in F_{1,\gamma}$ is the same as the infimum over $h \in H_1$. Therefore we have

$$\lambda_{2,\gamma} \ge \inf_{h \in H_1} \sup_{f \in H_1,\, \langle f,h \rangle_{H_1} = 0} \frac{\langle V f, f \rangle_{H_1}}{\langle f, f \rangle_{H_1}} = \lambda_2(V),$$

where $\lambda_2(V)$ is the second largest eigenvalue of V. Since $\dim(V(H_1)) = \dim(S_1(H_1))$ is at least 2 by hypothesis, we conclude that $\lambda_2(V)$ is positive. Hence,

$$0 < \lambda_2(V) \le \lambda_1(V) = \left\| S_1\big|_{H_1} \right\|^2_{H_1}.$$

We now turn to $d \geq 2$. Let $\lambda_{i,d,\gamma}$ denote the ordered eigenvalues of $W_{d,\gamma} = S_d^* S_d : F_{d,\gamma} \to F_{d,\gamma}$. From the tensor product construction, we have

$$\{\, \lambda_{i,d,\gamma} \,\} \;=\; \left\{\; \prod_{j=1}^{d} \lambda_{i_j,\gamma_j} \;:\; i_j = 1, 2, \dots \;\right\}.$$

The square of the norm of S_d is the largest eigenvalue $\lambda_{1,d,\gamma}$, and so

$$\|S_d\|_{F_{d,\gamma}}^2 \;=\; \lambda_{1,d,\gamma} \;=\; \prod_{j=1}^{d} \lambda_{1,\gamma_j} \;\leq\; \prod_{j=1}^{d} \left(\left\| S_1\Big|_{H_1} \right\|_{H_1}^2 + \left\| S_1\Big|_{H_2} \right\|_{H_2}^2 \right).$$

It is known, see e.g., [8], that

$$n(\varepsilon, S_d) \;=\; \min\{\, n \;:\; \lambda_{n+1,d,\gamma} \;\leq\; \varepsilon^2 \lambda_{1,d,\gamma} \,\}.$$

Take an arbitrary integer k, and fix ε as

$$\varepsilon \;=\; \frac{1}{2} \left(\frac{\lambda_2(V)}{\left\| S_1\Big|_{H_1} \right\|_{H_1}^2 + \left\| S_1\Big|_{H_2} \right\|_{H_2}^2} \right)^{k/2} \;\in\; (0,1).$$

For $d > k$, consider the vectors $\vec{i} = [i_1, i_2, \dots, i_d]$ with $i_j \in \{1,2\}$. Take k indices i_j equal to 2 and $d-k$ indices i_j equal to 1. We have $\binom{d}{k}$ vectors such that the eigenvalues satisfy

$$\prod_{j=1}^{d} \lambda_{i_j,\gamma_j} \;=\; \prod_{j:\, i_j = 1} \lambda_{1,\gamma_j} \prod_{j:\, i_j = 2} \lambda_{2,\gamma_j} \;=\; \prod_{j:\, i_j = 2} \frac{\lambda_{2,\gamma_j}}{\lambda_{1,\gamma_j}} \prod_{j=1}^{d} \lambda_{1,\gamma_j}.$$

Since

$$\frac{\lambda_{2,\gamma_j}}{\lambda_{1,\gamma_j}} \;\geq\; \frac{\lambda_2(V)}{\|S_1\Big|_{H_1}\|_{H_1}^2 + \|S_1\Big|_{H_2}\|_{H_2}^2},$$

we conclude that

$$\prod_{j=1}^{d} \lambda_{i_j,\gamma_j} \;\geq\; 4\,\varepsilon^2\, \lambda_{1,d,\gamma}.$$

This proves that

$$n(\varepsilon, S_d) \;\geq\; \binom{d}{k} \;=\; \Theta(d^k) \;\;\text{as}\;\; d \to \infty.$$

Since k can be arbitrarily large, this means that $\{S_d\}$ is intractable, as claimed. $\square$

We now turn to

Proof of Theorem 3.1. We can assume that $\dim(S_1(H_1)) \leq 1$ from Lemma 3.1. Consider first the case $\dim(S_1(H_1)) = 0$, i.e., $S_1\Big|_{H_1} = 0$ and $\|S_1\|_{F_{1,\gamma}} =$

$\gamma^{1/2}\|S_1\|_{H_2}$. Similarly, $\|S_d\|_{F_{d,\gamma}} = \left(\prod_{j=1}^d \gamma_j^{1/2}\right)\|S_d\|_{H_2^d}$, where $H_2^d = H_2\otimes\cdots\otimes H_2$ (d times). It is also easy to see that

$$e(U_{n,d}, F_{d,\gamma}) \;=\; \left(\prod_{j=1}^{d} \gamma_j^{1/2}\right) e(U_{n,d}, H_2^d)$$

for any optimal linear $U_{n,d}$. Hence, the weights γ_j do not play any role and the problem reduces to the space H_2^d. It is proven in [14] that $\{S_d\}$ is tractable iff $\dim(S_1(H_2)) \le 1$. Since $S_1(F_1) = S_1(H_2)$, we have tractability iff $S_1 \in \textsc{triv}(F_1)$.

Assume now that $\dim(S_1(H_1)) = 1$. That is, S_1 has the form

$$S_1 f \;=\; \langle f_1, h_1\rangle_{H_1} g_1 \;+\; S f_2$$

for $f_i \in H_i$ and nonzero $h_1 \in H_1$ and $g_1 \in G_1$.

Operators having a similar form were considered in [14]. The only difference is that instead of the inner product $\langle f, h_1\rangle_{H_1}$ a more specific linear functional was considered. This is not important and we can modify $U_{n,d}$ considered in [14] by taking for $n = d = 1$,

$$U_{1,1}(f) \;=\; \langle f, h_1\rangle_{H_1} g_1,$$

and for $n \ge 2$,

$$U_{n,1}(f) \;=\; U_{1,1}(f) + B_{n-1,1}(f_2),$$

where $B_{n-1,1}$ is a sequence of approximation of $S_1\Big|_{H_2}$ in the space H_2 defined as in [14]. The rest of Theorem 3.1 follows from Theorem 1 in [14]. □

4. Examples

We illustrate Theorem 3.1 by two examples of Sobolev spaces of d-variate functions defined over $[0,1]^d$. This will be done for the approximation problem $S_d f = f \in G_d = L_2([0,1]^d)$.

Example 4.1. Let $H_1 = \mathrm{span}(1, x, \dots, x^{r-1})$ be the r-dimensional space of polynomials of degree at most $r-1$ reduced to the interval $[0,1]$ with the inner product

$$\langle f, g\rangle_{H_1} \;=\; \sum_{j=0}^{r-1} f^{(j)}(0) g^{(j)}(0).$$

Let H_2 be the space of functions f defined over $[0,1]$ for which $f^{(r-1)}$ is absolutely continuous, $f^{(r)}$ belongs to $L_2([0,1])$, and $f^{(j)}(0) = 0$ for $j = 0, 1, \dots, r-1$. The inner product in H_2 is

$$\langle f, g\rangle_{H_2} \;=\; \int_0^1 f^{(r)}(t) g^{(r)}(t)\, dt.$$

Obviously, $H_1 \cap H_2 = \{0\}$, as required in our analysis. We thus have

$$F_{1,\gamma} \;=\; \{\, f : [0,1] \to \mathbb{R} \;:\; f^{(r-1)} \text{ abs. cont.}, \;\; f^{(r)} \in L_2([0,1]) \,\}$$

with the inner product

$$\langle f, g\rangle_{F_{1,\gamma}} = \sum_{j=0}^{r-1} f^{(j)}(0)g^{(j)}(0) + \gamma^{-1}\int_0^1 f^{(r)}(t)g^{(r)}(t)\,dt.$$

It is well known that the Hilbert space $F_{1,\gamma}$ has a reproducing kernel of the form

$$K_{1,\gamma}(x,t) = \sum_{j=0}^{r-1}\frac{x^j}{j!}\frac{t^j}{j!} + \gamma\int_0^1 \frac{(x-u)_+^{r-1}}{(r-1)!}\frac{(t-u)_+^{r-1}}{(r-1)!}\,dt,$$

where $u_+ = \max(u,0)$, see, e.g., [5].

For the approximation problem, we have $\dim(S_1(H_1)) = r$. For $r \geq 2$, Lemma 3.1 states that the approximation problem is intractable.

For $r \geq 1$, it it known that the eigenvalues λ_j of the operator $S_1\big|_{H_2}^{*} S_1\big|_{H_2}$ are proportional to j^{-2r}. Hence, for $r = 1$, the approximation problem is strongly tractable iff $p_\gamma < \infty$, and the strong exponent is $\max(2p_\gamma, 1)$.

Example 4.2. We consider spaces similar to those in Example 1, but with a different split between the unweighted and weighted parts. We take $\bar{H}_1 = \mathrm{span}(1)$, the space of constant functions, and

$$\bar{H}_2 = \mathrm{span}(x, x^2, \ldots, x^{r-1}) \oplus H_2,$$

with H_2 as in Example 1. Then we have $\bar{F}_{1,\gamma}$ with the kernel

$$\bar{K}_{1,\gamma}(x,t) = 1 + \gamma\left(\sum_{j=1}^{r-1}\frac{x^j}{j!}\frac{t^j}{j!} + \int_0^1 \frac{(x-u)_+^{r-1}}{(r-1)!}\frac{(t-u)_+^{r-1}}{(r-1)!}\,dt\right).$$

This corresponds to the inner product

$$\langle f, g\rangle_{\bar{F}_{1,\gamma}} = f(0)g(0) + \gamma^{-1}\left(\sum_{j=1}^{r-1} f^{(j)}(0)g^{(j)}(0) + \int_0^1 f^{(r)}(t)g^{(r)}(t)\,dt\right).$$

For this problem, we have $\dim(S_1(H_1)) = 1$. It is known that the eigenvalues λ_j of $S_1\big|_{H_2}^{*} S_1\big|_{H_2}$ are still of order j^{-2r}. Hence, the approximation problem is strongly tractable iff $p_\gamma < \infty$, with the strong exponent $\max(2p_\gamma, r^{-1})$.

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