Georgian Mathematical Journal
Volume 8 (2001), Number 2, 377–388

ON CONVERGENCE OF SERIES OF RANDOM ELEMENTS VIA MAXIMAL MOMENT RELATIONS WITH APPLICATIONS TO MARTINGALE CONVERGENCE AND TO CONVERGENCE OF SERIES WITH p-ORTHOGONAL SUMMANDS

ANDREW ROSALSKY AND ANDREI I. VOLODIN

Dedicated to Professor N.N. Vakhania on the occasion of his 70th birthday

Abstract. The rate of convergence for an almost surely convergent series of Banach space valued random elements is studied in this paper. As special cases of the main result, known results are obtained for a sequence of independent random elements in a Rademacher type p Banach space, and new results are obtained for a martingale difference sequence of random elements in a martingale type p Banach space and for a p-orthogonal sequence of random elements in a Rademacher type p Banach space. The current work generalizes, simplifies, and unifies some of the recent results of Nam and Rosalsky [16] and Rosalsky and Rosenblatt [23, 24].

2000 Mathematics Subject Classification: 60F05, 60B12, 60B11.
Key words and phrases: Almost surely convergent series of Banach space valued random elements, tail series, rate of convergence, martingale difference sequence, p-orthogonal sequence.

1. Introduction

It is a great pleasure for us to contribute to this issue of *Georgian Mathematical Journal* in honor of Professor N. N. Vakhania on the occasion of his 70th birthday. Let $\{V_n, n \geq 1\}$ be a sequence of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space. As usual, their partial sums are denoted by $S_n = \sum_{j=1}^{n} V_j$, $n \geq 1$. We refer the reader to the very detailed and careful discussion in Chapter V of Vakhania, Tarieladze, and Chobanyan [27] concerning the conditions under which S_n converges almost surely (a.s.) to a random element. If S_n converges a.s. to a random element S, then (set $S_0 = 0$)

$$T_n \equiv S - S_{n-1} = \sum_{j=n}^{\infty} V_j, \quad n \geq 1,$$

ISSN 1072-947X / \$8.00 /

is a well-defined sequence of random elements (referred to as the *tail series*) with

$$T_n \to 0 \quad \text{a.s.} \tag{1.1}$$

In this paper, we shall be concerned with the rate of convergence of S_n to S or, equivalently, that of the tail series T_n to 0. More specifically, recalling that (1.1) is equivalent to

$$\sup_{k\geq n} \|T_k\| \xrightarrow{P} 0,$$

we will provide the conditions in Theorem 1 and in its four corollaries for each of

$$\sup_{k\geq n} \|T_k\| = \mathcal{O}_P(b_n)$$

and

$$\frac{\sup_{k\geq n} \|T_k\|}{b_n} \xrightarrow{P} 0 \tag{1.2}$$

to hold, where $\{b_n, n \geq 1\}$ is a sequence of positive constants. These results are, of course, of greatest interest when $b_n = o(1)$. Nam and Rosalsky [16] provided an example showing *inter alia* that a.s. convergence to 0 does not necessarily hold for the expression in (1.2). Moreover, Nam, Rosalsky, and Volodin [19] showed that the limit law (1.2) and the tail series weak law of large numbers

$$\frac{T_n}{b_n} \xrightarrow{P} 0 \tag{1.3}$$

are equivalent when the summands $\{V_n, n \geq 1\}$ are independent and $b_n \downarrow 0$. An example of Nam and Rosalsky [18] reveals that (1.3) does not necessarily imply (1.2) if the monotonicity proviso on $\{b_n, n \geq 1\}$ is dispensed with.

Theorem 1 is a very general result and we will see that some previously obtained results as well as some new results are its immediate corollaries. In Theorem 1, a condition is imposed in general on the joint distributions of random elements $\{V_n, n \geq 1\}$, but no conditions are imposed on the geometry of the underlying Banach space. However, in Corollary 1, $\{V_n, n \geq 1\}$ is a sequence of independent random elements with mean (or Pettis integral) 0 in a Rademacher type p Banach space, in Corollary 2, $\{V_n, n \geq 1\}$ is a martingale difference sequence in a martingale type p Banach space, and in Corollary 3, $\{V_n, n \geq 1\}$ is a p-orthogonal sequence of random elements in a Rademacher type p Banach space. In Corollaries 1, 2, and 3, the moment condition on $\|V_n\|$ and the limiting behavior of b_n are directly connected to the geometric condition imposed on the Banach space. Nevertheless, in Corollary 4, no conditions are imposed on the joint distributions of random elements or on the geometry of the underlying Banach space.

The limit law (1.2) for the tail series was apparently first investigated by Nam and Rosalsky [16] wherein a special case of Corollary 1(ii) was obtained for the (real-valued) random variable case. The Nam and Rosalsky [16] result was generalized by Rosalsky and Rosenblatt [23] to a Banach space setting and the argument was substantially simplified. More specifically, Rosalsky and Rosenblatt [23] established Corollary 1. Furthermore, one of the main results of Rosalsky and Rosenblatt [24] is a special case of Corollary 2; Rosalsky and Rosenblatt [24] in their Corollary 1 established our Corollary 2 when $\{V_n, n \geq 1\}$ is a martingale difference sequence of random variables. Corollary 3 appears to be an entirely new result and it concerns a p-orthogonal sequence of random elements in a Rademacher type p Banach space. Corollary 4 contains both known and new results and these will be discussed in more detail below. The current work generalizes, simplifies, and unifies some of these recent results of Nam and Rosalsky [16] and Rosalsky and Rosenblatt [23,24].

There has been substantial literature on the limiting behavior of tail series following the ground breaking work of Chow and Teicher [3] wherein a tail series law of the iterated logarithm (LIL) was obtained. Barbour [1] then established a tail series analogue of the Lindeberg-Feller version of the central limit theorem. Numerous other investigations on the tail series LIL problem have followed; see Heyde [8], Wellner [28], Kesten [10], Budianu [2], Chow, Teicher, Wei, and Yu [5], Klesov [11], Rosalsky [22], and Mikosch [14] for such work. Klesov [11,12], Mikosch [14], and Nam and Rosalsky [17] studied the tail series strong law of large numbers problem. The only work that the authors are aware of on the limiting behavior of tail series with Banach space valued summands is that of Dianliang [6,7] on the tail series LIL and that of Rosalsky and Rosenblatt [23] and Nam, Rosalsky, and Volodin [19] on the limit law (1.2).

Throughout this paper, $\{V_n, n \geq 1\}$ is a sequence of random elements taking values in a real separable Banach space $(\mathcal{X}, \|\cdot\|)$ and $b = \{b_n, n \geq 1\}$ is a sequence of positive constants.

Let us introduce the following class of functions. We will say that a function $f : R^+ \to R^+$ belongs to the class $I(b)$ if:

(i) f is continuous and nondecreasing,
(ii) f is *semiadditive*, that is, there exists a constant $C < \infty$ such that $f(s+t) \leq C(f(s) + f(t))$ for all $s, t \in R^+$,
(iii) there exists a sequence of positive constants $\{b_n^f, n \geq 1\}$ (depending only on f and b) such that

$$B^f(\varepsilon) \equiv \sup_{n \geq 1} \frac{b_n^f}{f(\varepsilon b_n)} < \infty \text{ for all } \varepsilon > 0.$$

For example, for any sequence $b = \{b_n, n \geq 1\}$ of positive constants, the function $f(t) = t^p, t \geq 0$ where $p \geq 0$ is in $I(b)$ with $C = 2^p, b_n^f = b_n^p, n \geq 1$, and $B^f(\varepsilon) = \varepsilon^{-p}, \varepsilon > 0$.

Let us recall a few well-known definitions. The Banach space $(\mathcal{X}, \|\cdot\|)$ is said to be of *Rademacher type* p $(1 \leq p \leq 2)$ if there exists a constant $C < \infty$ such that

$$E\Big\| \sum_{j=1}^n V_j \Big\|^p \leq C \sum_{j=1}^n E\|V_j\|^p$$

for all independent $\mathcal{X}$-valued random elements $V_1, \cdots, V_n$ with mean 0. We refer the reader to Pisier [21] and Woyczyński [31] for a detailed discussion of this notion.

The Banach space $(\mathcal{X}, \|\cdot\|)$ is said to be of *martingale type* p $(1 \le p \le 2)$ if there exists a constant $C < \infty$ such that for all martingales $\{S_n, n \ge 1\}$ with values in $\mathcal{X}$

$$\sup_{n\ge 1} E\|S_n\|^p \le C \sum_{n=1}^{\infty} E\|S_n - S_{n-1}\|^p$$

where $S_0 \equiv 0$. It can be shown using the classical methods from martingale theory that if $\mathcal{X}$ is of martingale type p, then there exists a constant $C < \infty$ such that

$$E\Big\{ \max_{n\le k\le m} \|S_k - S_{n-1}\|^p \Big\} \le C \sum_{j=n}^{m} E\|V_j\|^p \tag{1.4}$$

for all $\mathcal{X}$-valued martingales $\{S_n = \sum_{j=1}^{n} V_j, n \ge 1\}$ and all $m \ge n \ge 1$. A detailed discussion concerning martingale type p Banach spaces can be found in Pisier [20,21], Woyczyński [29,30], and Schwartz [25].

Of course, every real separable Banach space is of both Rademacher and martingale type 1.

A sequence of random elements $\{V_n, n \ge 1\}$ is said to be *p-orthogonal* $(1 \le p < \infty)$ if $E\|V_n\|^p < \infty$ for all $n \ge 1$ and

$$E\Big\| \sum_{j=1}^{n} a_{\pi(j)} V_{\pi(j)} \Big\|^p \le E\Big\| \sum_{j=1}^{m} a_{\pi(j)} V_{\pi(j)} \Big\|^p$$

for all sequences of constants $\{a_n, n \ge 1\}$, for all choices of $m > n \ge 1$, and for all permutations π of the integers $\{1, \cdots, m\}$. We refer to Howell and Taylor [9] and Móricz, Su, and Taylor [15] for details.

Theorem 1 will be established in Section 2 and its Corollaries in Section 3. Throughout, the symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

2. The Main Result

The main result, Theorem 1, may now be established.

Theorem 1. *Let $\{V_n, n \ge 1\}$ be a sequence of random elements in a real separable Banach space and suppose that there exists a continuous nondecreasing function $f : R^+ \to R^+$ with*

$$\lim_{m\to\infty} E\bigg\{ \max_{n\le k\le m} f\Big(\Big\| \sum_{j=n}^{k} V_j \Big\|\Big) \bigg\} = o(1) \quad as \quad n \to \infty. \tag{2.1}$$

Then the series $\sum_{n=1}^{\infty} V_n$ converges a.s. and the tail series $\{T_n = \sum_{j=n}^{\infty} V_j, n \ge 1\}$ is a well-defined sequence of random elements. Moreover:

(i) *If there exists a function* $f \in I(b)$ *such that*

$$\lim_{m\to\infty} E\Big\{\max_{n\le k\le m} f\Big(\Big\|\sum_{j=n}^{k} V_j\Big\|\Big)\Big\} = O(b_n^f) \text{ as } n\to\infty \text{ and } \lim_{\varepsilon\to\infty} B^f(\varepsilon) = 0,$$

then the tail series satisfies the relation

$$\sup_{k\ge n} \|T_k\| = \mathcal{O}_P(b_n).$$

(ii) *If there exists a function* $f \in I(b)$ *such that*

$$\lim_{m\to\infty} E\Big\{\max_{n\le k\le m} f\Big(\Big\|\sum_{j=n}^{k} V_j\Big\|\Big)\Big\} = o(b_n^f) \text{ as } n\to\infty, \tag{2.2}$$

then the tail series obeys the limit law

$$\frac{\sup_{k\ge n}\|T_k\|}{b_n} \xrightarrow{P} 0.$$

Proof. For arbitrary $\varepsilon > 0$ and $n \ge 1$

$$\begin{aligned}
&P\Big\{\sup_{m>n}\Big\|\sum_{j=1}^{m} V_j - \sum_{j=1}^{n} V_j\Big\| > \varepsilon\Big\}\\
&\le \frac{1}{f(\varepsilon)} E\Big\{f\Big(\sup_{m>n}\Big\|\sum_{j=1}^{m} V_j - \sum_{j=1}^{n} V_j\Big\|\Big)\Big\}\\
&\qquad \text{(by monotonicity of } f \text{ and the Markov inequality)}\\
&= \frac{1}{f(\varepsilon)} E\Big\{\sup_{m>n} f\Big(\Big\|\sum_{j=n+1}^{m} V_j\Big\|\Big)\Big\}\\
&\qquad \text{(since } f \text{ is continuous and nondecreasing)}\\
&= \frac{1}{f(\varepsilon)} \lim_{m\to\infty} E\Big\{\max_{n+1\le k\le m} f\Big(\Big\|\sum_{j=n+1}^{k} V_j\Big\|\Big)\Big\}\\
&\qquad \text{(by the Lebesgue monotone convergence theorem)}\\
&= o(1) \;\; \text{(by (2.1))}.
\end{aligned}$$

Then by Corollary 3.3.4 of Chow and Teicher [4, p. 68] (modified to a Banach space setting), $\sum_{n=1}^{\infty} V_n$ converges a.s. Thus, the tail series $\{T_n = \sum_{j=n}^{\infty} V_j, n \ge 1\}$ is a well-defined sequence of random elements.

Next, for arbitrary $\varepsilon > 0$

$$P\Big\{\frac{\sup_{k\ge n}\|T_k\|}{b_n} > \varepsilon\Big\} \le \frac{1}{f(\varepsilon b_n)} E\Big\{f\Big(\sup_{k\ge n}\|T_k\|\Big)\Big\}$$

(by monotonicity of f and the Markov inequality)

$$= \frac{1}{f(\varepsilon b_n)} E\Big\{\sup_{k\ge n} f(\|T_k\|)\Big\} \text{ (since } f \text{ is continuous and nondecreasing)}$$

$$= \frac{1}{f(\varepsilon b_n)} \lim_{N\to\infty} E\Big\{ \max_{n\le k\le N} f\Big(\Big\| \lim_{m\to\infty} \sum_{j=k}^{m} V_j \Big\|\Big)\Big\}$$

(by the Lebesgue monotone convergence theorem)

$$= \frac{1}{f(\varepsilon b_n)} \lim_{N\to\infty} E\Big\{ \max_{n\le k\le N} \lim_{m\to\infty} f\Big(\Big\| \sum_{j=k}^{m} V_j \Big\|\Big)\Big\} \text{ (since } f \text{ is continuous)}$$

$$= \frac{1}{f(\varepsilon b_n)} \lim_{N\to\infty} E\Big\{ \lim_{m\to\infty} \max_{n\le k\le N} f\Big(\Big\| \sum_{j=k}^{m} V_j \Big\|\Big)\Big\}$$

$$\le \frac{1}{f(\varepsilon b_n)} \lim_{N\to\infty} \liminf_{m\to\infty} E\Big\{ \max_{n\le k\le N} f\Big(\Big\| \sum_{j=k}^{m} V_j \Big\|\Big)\Big\} \text{ (by Fatou's lemma)}$$

$$\le \frac{1}{f(\varepsilon b_n)} \liminf_{m\to\infty} E\Big\{ \max_{n\le k\le m} f\Big(\Big\| \sum_{j=k}^{m} V_j \Big\|\Big)\Big\}$$

$$\le \frac{1}{f(\varepsilon b_n)} \liminf_{m\to\infty} E\Big\{ \max_{n+1\le k\le m} f\Big(\Big\| \sum_{j=n}^{m} V_j \Big\| + \Big\| \sum_{j=n}^{k-1} V_j \Big\|\Big)\Big\}$$

(since f is nondecreasing)

$$\le \frac{C}{f(\varepsilon b_n)} \liminf_{m\to\infty} E\Big\{ \max_{n\le k\le m-1} f\Big(\Big\| \sum_{j=n}^{k} V_j \Big\|\Big) + f\Big(\Big\| \sum_{j=n}^{m} V_j \Big\|\Big)\Big\}$$

(since f is semiadditive)

$$\le \frac{C}{f(\varepsilon b_n)} \lim_{m\to\infty} E\Big\{ 2 \max_{n\le k\le m} f\Big(\Big\| \sum_{j=n}^{k} V_j \Big\|\Big)\Big\}$$

$$= \frac{2C}{f(\varepsilon b_n)} \lim_{m\to\infty} E\Big\{ \max_{n\le k\le m} f\Big(\Big\| \sum_{j=n}^{k} V_j \Big\|\Big)\Big\}$$

$$\le 2CB^f(\varepsilon) \frac{\lim_{m\to\infty} E\Big\{ \max_{n\le k\le m} f\Big(\Big\| \sum_{j=n}^{k} V_j \Big\|\Big)\Big\}}{b_n^f}.$$

Parts (i) and (ii) now follow easily. □

3. Applications

We now examine four special cases of Theorem 1 and these will be presented as Corollaries. To prove Corollary 1, the following lemma is used and it follows immediately from Proposition 1.1 of Kwapień and Woyczyński [13, p. 15] and the relation $EX = \int_0^\infty P\{X > t\}dt$ where X is a nonnegative random variable.

Lemma 1. *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent random elements. Then for every continuous strictly increasing function* $f : R^+ \to R$ *with* $f(0)=0$,

$$E\left\{ \max_{n\leq k\leq m} f\left(\left\| \sum_{j=n}^{k} V_j \right\| \right) \right\} \leq 3 \max_{n\leq k\leq m} E\left\{ f\left(3\left\| \sum_{j=n}^{k} V_j \right\| \right) \right\},\ m \geq n \geq 1.$$

Corollary 1(ii) is Theorem 4.2 of Rosalsky and Rosenblatt [23] and Corollary 1(i) is stated in Remark (ii) after the proof of Theorem 4.2 of Rosalsky and Rosenblatt [23].

Corollary 1 (Rosalsky and Rosenblatt [23]). *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent mean* 0 *random elements taking values in a real separable, Rademacher type* p $(1 \leq p \leq 2)$ *Banach space* $\mathcal{X}$.

(i) *If*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = \mathcal{O}(b_n^p), \tag{3.1}$$

then the series $\sum_{n=1}^{\infty} V_n$ *converges a.s. and the tail series satisfies the relation*

$$\sup_{k\geq n} \|T_k\| = \mathcal{O}_P(b_n).$$

(ii) *If*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = o(b_n^p), \tag{3.2}$$

then the tail series obeys the limit law

$$\frac{\sup_{k\geq n} \|T_k\|}{b_n} \xrightarrow{P} 0. \tag{3.3}$$

Proof. In Theorem 1 take $f(t) = t^p, t \geq 0$. Then $f \in I(b)$ with $b_n^f = b_n^p, n \geq 1$ and $B^f(\varepsilon) = \varepsilon^{-p}, \varepsilon > 0$. Now by Lemma 1 and since $\mathcal{X}$ is of Rademacher type p, we have that

$$\lim_{m\to\infty} E\left\{ \max_{n\leq k\leq m} \left\| \sum_{j=n}^{k} V_j \right\|^p \right\} \leq C \sum_{j=n}^{\infty} E\|V_j\|^p = o(1) \text{ as } n \to \infty$$

under either (3.1) or (3.2). The Corollary follows immediately from Theorem 1. □

Remark. Example 5.3 of Rosalsky and Rosenblatt [23] reveals that both parts of Corollary 1 can fail if the Rademacher type p hypothesis is weakened to Rademacher type q where $1 \leq q < p \leq 2$.

The next Corollary was obtained by Rosalsky and Rosenblatt [24] in the random variable case.

Corollary 2. *Let* $\{S_n = \sum_{j=1}^{n} V_j, n \geq 1\}$ *be a martingale taking values in a real separable, martingale type* p $(1 \leq p \leq 2)$ *Banach space* $\mathcal{X}$.

(i) *If*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = \mathcal{O}(b_n^p),$$

then the series $\sum_{n=1}^{\infty} V_n$ *converges a.s. and the tail series satisfies the relation*

$$\sup_{k \geq n} \|T_k\| = \mathcal{O}_P(b_n).$$

(ii) *If*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = o(b_n^p), \tag{3.4}$$

then the tail series obeys the limit law

$$\frac{\sup_{k \geq n} \|T_k\|}{b_n} \xrightarrow{P} 0.$$

Proof. By the hypothesis that $\mathcal{X}$ is of martingale type p, we have in view of (1.4) that for all $m \geq n \geq 1$

$$E\left\{ \max_{n \leq k \leq m} \left\| \sum_{j=n}^{k} V_j \right\|^p \right\} \leq C \sum_{j=n}^{m} E\|V_j\|^p.$$

The rest of the argument follows as in Corollary 1. $\square$

Remark. Rosalsky and Rosenblatt [23] provided an example of a sequence $\{V_n, n \geq 1\}$ of independent mean 0 square integrable random variables such that setting $b_n = M_n/2^n, n \geq 1$, where $\{M_n, n \geq 1\}$ is a sequence of positive constants:

(a) If $M_n \to \infty$ (no matter how slowly), then (3.2) (with $p = 2$) and (3.3) hold.

(b) If $\liminf_{n\to\infty} M_n < \infty$, then (3.2) (with $p = 2$) and (3.3) fail.

This example thus illustrates the sharpness of Theorem 1(ii) and Corollaries 1(ii) and 2(ii) and it also shows that these results can fail if o is replaced by $\mathcal{O}$ in (2.2), (3.2), and (3.4), respectively.

The following two results, which are used in the proof of Corollary 3, appear in the literature and are stated for the convenience of the reader.

Lemma 2 (Howell and Taylor [9]). *If the Banach space* $(\mathcal{X}, \|\cdot\|)$ *is of Rademacher type* p $(1 \leq p \leq 2)$, *then there exists a constant* $C < \infty$ *such that*

$$E\left\| \sum_{j=1}^{n} V_j \right\|^p \leq C \sum_{j=1}^{n} E\|V_j\|^p, n \geq 1,$$

for all p*-orthogonal sequences* $\{V_n, n \geq 1\}$ *of* $\mathcal{X}$*-valued random elements.*

Lemma 3 (Móricz, Su, and Taylor [15]). *Let $\{V_n, n \geq 1\}$ be a p-orthogonal $(1 \leq p < \infty)$ sequence of random elements and suppose that there exists a sequence of nonnegative numbers $\{u_n, n \geq 1\}$ such that*

$$E\Big\|\sum_{j=n}^{m} V_j\Big\|^p \leq \sum_{j=n}^{m} u_j$$

for all $m \geq n \geq 1$. Then

$$E\left\{\max_{n\leq k\leq m}\Big\|\sum_{j=n}^{k} V_j\Big\|^p\right\} \leq (Log(2n))^p \sum_{j=n}^{m} u_j, m \geq n \geq 1,$$

where Log denotes the logarithm to the base 2.

The next Corollary appears to be entirely new.

Corollary 3. *Let $\{V_n, n \geq 1\}$ be a p-orthogonal sequence of random elements taking values in a real separable, Rademacher type p $(1 \leq p \leq 2)$ Banach space.*

(i) *If $b_n = o(1)$ and*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = \mathcal{O}((b_n/\log n)^p),$$

then the series $\sum_{n=1}^{\infty} V_n$ converges a.s. and the tail series satisfies the relation

$$\sup_{k\geq n}\|T_k\| = \mathcal{O}_P(b_n).$$

(ii) *If $b_n = \mathcal{O}(1)$ and*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = o((b_n/\log n)^p),$$

then the series $\sum_{n=1}^{\infty} V_n$ converges a.s. and the tail series obeys the limit law

$$\frac{\sup_{k\geq n}\|T_k\|}{b_n} \xrightarrow{P} 0.$$

Proof. By Lemmas 2 and 3, we have

$$E\left\{\max_{n\leq k\leq m}\Big\|\sum_{j=n}^{k} V_j\Big\|^p\right\} \leq C\,(\text{Log }(2n))^p \sum_{j=n}^{m} E\|V_j\|^p \tag{3.5}$$

for all $m \geq n \geq 1$ and the argument proceeds as in Corollary 1, *mutatis mutandis.* □

Remark. Inequality (3.5) is a generalization of the famous Menchoff maximal inequality for sums of orthogonal random variables (see, e.g., Stout [26, p. 18]).

The last Corollary has no assumptions concerning the joint distributions of random elements and imposes no geometric condition on the Banach space as in Corollaries 1, 2, and 3. Corollary 4(ii) was obtained by Rosalsky and Rosenblatt [23] but Corollary 4(i) is new (although it was obtained by Rosalsky and Rosenblatt [24] in the random variable case). It appears to us that Corollary

4(i) cannot be obtained by using the approach taken by Rosalsky and Rosenblatt [23].

Corollary 4. *Let* $\{V_n, n \geq 1\}$ *be a sequence of random elements in a real separable Banach space and let* $0 < p \leq 1$.

(i) *If*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = \mathcal{O}(b_n^p),$$

then the series $\sum_{n=1}^{\infty} V_n$ *converges a.s. and the tail series satisfies the relation*

$$\sup_{k \geq n} \|T_k\| = \mathcal{O}_P(b_n).$$

(ii) *If*

$$\sum_{j=n}^{\infty} E\|V_j\|^p = o(b_n^p),$$

then the tail series obeys the limit law

$$\frac{\sup_{k \geq n} \|T_k\|}{b_n} \xrightarrow{P} 0.$$

Proof. Since $0 < p \leq 1$, we have

$$E\left\{ \max_{n \leq k \leq m} \left\| \sum_{j=n}^{k} V_j \right\|^p \right\} \leq E\left\{ \sum_{j=n}^{m} \|V_j\|^p \right\} = \sum_{j=n}^{m} E\|V_j\|^p$$

for all $m \geq n \geq 1$. The rest of the argument follows as in Corollary 1. $\square$

Remark. The *proofs* of Theorem 1(ii) and part (ii) of all the Corollaries show that the hypotheses indeed entail the limit law

$$\frac{\sup_{k \geq n} \|T_k\|}{b_n} \xrightarrow{\mathcal{L}_p} 0.$$

References

1. A. D. Barbour, Tail sums of convergent series of independent random variables. *Proc. Cambridge Philos. Soc.* **75**(1974), 361–364.
2. G. Budianu, On the law of the iterated logarithm for tail sums of random variables. (Romanian) *Stud. Cerc. Mat.* **33**(1981), 149–158.
3. Y. S. Chow and H. Teicher, Iterated logarithm laws for weighted averages. *Z. Wahrsch. Verw. Gebiete* **26**(1973), 87–94.
4. Y. S. Chow and H. Teicher, Probability theory: independence, interchangeability, martingales, 3rd ed. *Springer-Verlag, New York*, 1997.
5. Y. S. Chow, H. Teicher, C. Z. Wei, and K. F. Yu, Iterated logarithm laws with random subsequences. *Z. Wahrsch. Verw. Gebiete* **57**(1981), 235–251.
6. D. Dianliang, The law of iterated logarithm of tail sums of Banach space valued random variables. (Chinese) *Acta Sci. Natur. Univ. Jilin.* **2**(1988), 15–24.

7. D. Dianliang, The estimation of the superior limit in the law of the iterated logarithm for tail sums. *Northeast. Math. J.* **7**(1991), 265–274.

8. C. C. Heyde, On central limit and iterated logarithm supplements to the martingale convergence theorem. *J. Appl. Probab.* **14**(1977), 758–775.

9. J. O. Howell and R. L. Taylor, Marcinkiewicz–Zygmund weak laws of large numbers for unconditional random elements in Banach spaces. *Probability in Banach Spaces* III, *Proceedings of the Third International Conference on Probability in Banach Spaces Held at Tufts University, Medford USA, August* 14–16, 1980; *Lecture Notes in Math.* 860, 219–230, *Springer-Verlag, Berlin*, 1981.

10. H. Kesten, The speed of convergence of a martingale. *Israel J. Math.* **32**(1979), 83–96.

11. O. I. Klesov, Rate of convergence of series of random variables. (Russian) *Ukrain. Mat. Zh.* **35**(1983), 309–314; English translation: *Ukrainian Math. J.* **35**(1983), 264–268.

12. O. I. Klesov, Rate of convergence of some random series. (Russian) *Teor. Veroyatnost. Mat. Statist.* **30** (1984), 81–92; English translation: *Theor. Probab. Math. Statist.* **30**(1985), 91–101.

13. S. Kwapień and W.A. Woyczyński, Random series and stochastic integrals: single and multiple. *Birkhäuser, Boston*, 1992.

14. T. Mikosch, Almost sure behavior of tail series in functional spaces. *Anal. Math.* **16**(1990), 123–133.

15. F. Móricz, K.-L. Su, and R. L. Taylor, Strong laws of large numbers for arrays of orthogonal random elements in Banach spaces. *Acta Math. Hungar.* **65**(1994), 1–16.

16. E. Nam and A. Rosalsky, On the rate of convergence of series of random variables. (Ukrainian) *Teor. Īmovīr. ta Mat. Statist.* **52**(1995), 120–131; English translation: *Theor. Probab. Math. Statist.* **52**(1996), 129–140.

17. E. Nam and A. Rosalsky, On the almost certain rate of convergence of series of independent random variables. *J. Korean Statist. Soc.* **24**(1995), 91–109.

18. E. Nam and A. Rosalsky, On the convergence rate of series of independent random variables. *Research Developments in Probability and Statistics: Festschrift in Honor of Madan L. Puri on the Occasion of his* 65*th Birthday* (*E. Brunner and M. Denker, eds.*), 33–44, *VSP, Utrecht,* 1996.

19. E. Nam, A. Rosalsky, and A. I. Volodin, On convergence of series of independent random elements in Banach spaces. *Stochastic Anal. Appl.* **17**(1999), 85–97.

20. G. Pisier, Martingales with values in uniformly convex spaces. *Israel J. Math.* **20**(1975), 326–350.

21. G. Pisier, Probabilistic methods in the geometry of Banach spaces. *Probability and Analysis, Lectures given at the* 1*st* 1985 *Session of the Centro Internazionale Matematico Estivo* (*C.I.M.E.*) *held at Varenna* (*Como*)*, Italy,* 31 *May*-8 *June,* 1985 (*G. Letta and M. Pratelli, eds.*)*, Lecture Notes in Math.* 1206, 167–241, *Springer-Verlag, Berlin*, 1986.

22. A. Rosalsky, Almost certain limiting behavior of the tail series of independent summands. *Bull. Inst. Math. Acad. Sinica* **11**(1983), 185–208.

23. A. Rosalsky and J. Rosenblatt, On the rate of convergence of series of Banach space valued random elements. *Nonlinear Anal.* **30**(1997), 4237–4248.

24. A. Rosalsky and J. Rosenblatt, On convergence of series of random variables with applications to martingale convergence and convergence of series with orthogonal summands. *Stochastic Anal. Appl.* **16**(1998), 553–566.

25. L. Schwartz, Geometry and probability in Banach spaces, Notes by Paul R. Chernoff. *Lecture Notes in Math.* 852, *Springer-Verlag, Berlin*, 1981.

26. W. F. Stout, Almost sure convergence. *Academic Press, New York*, 1974.

27. N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, Probability distributions on Banach spaces. *D. Reidel Publishing Company, Dordrecht,* 1987.

28. J. A. Wellner, A strong invariance theorem for the strong law of large numbers. *Ann. Probab.* **6**(1978), 673–679.

29. W. A. Woyczyński, Geometry and martingales in Banach spaces. *Probability-Winter School, Proceedings of the 4th Winter School on Probability, Karpacz, Poland, January* 1975 (*Z. Ciesielski, K. Urbanik, and W.A. Woyczyński, eds.*); *Lecture Notes in Math.* 472, 229–275, *Springer-Verlag, Berlin*, 1975.

30. W. A. Woyczyński, Asymptotic behavior of martingales in Banach spaces, *Probability in Banach Spaces, Proceedings of the 1st International Conference on Probability in Banach Spaces,* 20–26 *July* 1975, *Oberwolfach* (*A. Beck, ed.*)*; Lecture Notes in Math.* 526, 273–284, *Springer-Verlag, Berlin*, 1976.

31. W. A. Woyczyński, Geometry and martingales in Banach spaces-Part II: independent increments. *Probability on Banach Spaces* (*J. Kuelbs, ed.*); *Advances in Probability and Related Topics* (*P. Ney, ed.*) 4, 267–517, *Marcel Dekker, New York*, 1978.

(Received 3.07.2000)

Authors' addresses:

Andrew Rosalsky
Department of Statistics
University of Florida
Gainesville, Florida 32611
U.S.A.
E-mail: rosalsky@stat.ufl.edu

Andrei I. Volodin
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S 0A2
Canada
E-mail: volodin@math.uregina.ca