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PROBABILITY MEASURES WITH BIG KERNELS

NGUYEN DUY TIEN AND V. TARIELADZE

To Professor Nicholas Vakhania
from two of his first graduate students

Abstract. It is shown that in an infinite-dimensional dually separated second category topological vector space X there does not exist a probability measure μ for which the kernel coincides with X. Moreover, we show that in "good" cases the kernel has the full measure if and only if it is finite-dimensional. Also, the problem posed by S. Chevet [5, p. 69] is solved by proving that the annihilator of the kernel of a measure μ coincides with the annihilator of μ if and only if the topology of μ-convergence in the dual space is essentially dually separated.

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1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space. It is well-known that any convergent in measure μ sequence of measurable functions converges everywhere if and only if μ is *completely atomic* (i.e., X equals to the union of all μ-atoms). Also, it is well-known that any convergent in measure μ sequence of measurable functions converges μ-almost everywhere if and only if μ is *purely atomic* (i.e., the union of all μ-atoms has μ-measure one).

The situation may change if we consider the question not for all measurable functions, but for sequences of measurable functions taken from a given fixed set F.

The purpose of this note is to study the question when X is a dually separated real topological vector space and $F = X^*$ is the dual space. $\mathcal{B}$ will be any σ-algebra of subsets of X with respect to wich all members of X^* are measurable. For a probability measure μ given on $\mathcal{B}$ its *annihilator* $\mathcal{N}_\mu$ is the set of all $x^* \in X^*$ which are zero μ-a.e.; μ will be called *scalarly non-degenerate* if $\mathcal{N}_\mu = \{0\}$. τ_μ will stand for the topology in X^* of convergence in measure μ. The *kernel* $\mathcal{H}_\mu$ of μ is defined as the dual space of (X^*, τ_μ). We also consider the *initial kernel*

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$\mathcal{K}_\mu := \mathcal{H}_\mu \cap X$. Evidently, if $a \in X$ is a fixed element, then $a \in \mathcal{K}_\mu$ if and only if $\lim_{n\to\infty} x_n^*(a) = 0$ for an arbitrary sequence (x_n^*) in X^* which τ_μ-converges to zero.

Our results are related to the question of "the richness" of the kernel of a probability measure. Proposition 3.4 asserts that for a given μ with $\mathcal{K}_\mu = \mathcal{H}_\mu$ the annihilator of $\mathcal{K}_\mu$ into X^* coincides with $\mathcal{N}_\mu$ if and only if (X^*, τ_μ) is an essentially dually separated space (this means that the space $(X^*/\mathcal{N}_\mu, \tau_\mu/\mathcal{N}_\mu)$ is dually separated). This statement can be considered as a solution of the problem posed in [5, p. 69] about the characterization of (cylindrical) measures μ for which the equality ${\mathcal{K}_\mu}^\circ = \mathcal{N}_\mu$ is true. Here we do not deal with cylindrical measures, but the proof works for them, too.

The next statement considers very big kernels. It is easy to see that for any scalarly non-degenerate probability measure μ in a finite-dimensional Hausdorff topological vector space X, the kernel coincides with X. Our Theorem 4.1 says that a probability measure with a similar property does not exist when X is an infinite-dimensional second category dually separated topological vector space. Note that the statement is formulated without any further supposition of local convexity, metrizability or separability. For separable Frechet spaces the statement is known and is not difficult to prove (see Remark 4.2).

The above discussed result shows that for an infinite-dimensional second category dually separated space X and a given probability measure μ the convergence in measure μ of a sequence of even continuous linear functionals may not imply its everywhere convergence. However Theorem 4.1 leaves open a similar question about a.e. convergence. We call a probability measure μ *nearly atomic* if any sequence from X^* that converges to zero in measure μ, converges also μ-almost everywhere. It is evident that if for a given μ we have $\mu(\mathcal{K}_\mu) = 1$, then μ is nearly atomic. Our Theorem 5.2 asserts that, rather unexpectedly, the following converse also is true: if μ is a nearly atomic Radon probability measure such that $\mathcal{K}_\mu = \mathcal{H}_\mu$, then $\mu(\mathcal{K}_\mu) = 1$. Our proof uses heavily the characterization of the nuclear subspaces of L_0 given in [12] and Minlos' theorem. Our formulation of Theorem 5.2 is closely related to (and covers) the following result [10, Theorem 1]: let μ be a scalarly non-degenerate probability measure on a separable Frechet space X, then $\mu(\mathcal{K}_\mu) = 1$ if and only if (X^*, τ_μ) is a nuclear (locally convex) space.

The question of existence in every Banach space of a probability measure μ such that $\mu(\mathcal{K}_\mu) = 1$ and μ does not charge finite-dimensional subspaces, posed in [19, p. 68] has been answered positively in [10, Remark 3]: such a measure exists in any (dually separated) topological vector space which contains at least one infinite-dimensional compact subset. By our Theorem 5.4 a "good" measure with this property does not exist: if μ is a Radon probability measure on a dually separated space with properties $\mu(\mathcal{K}_\mu) = 1$, $\mathcal{K}_\mu = \mathcal{H}_\mu$ and (X^*, τ_μ) is a *locally bounded* space, then $\dim(\mathcal{K}_\mu) < \infty$. This result is applicable, e.g., for p-stable measures with $0 < p \leq 2$, while a similar statement due to [2] covers only the case $1 < p \leq 2$.

2. "Kernels" of Topologies

Let, to begin with, X be a (not necessarily Hausdorff) topological vector space over $\mathbb{R}$. X^* will denote the vector space of all continuous linear functionals defined on X (i.e., X^* is the dual space of X). We denote X^{*a} the algebraic dual space of the vector space X^*, i.e., X^{*a} consists of *all* linear functionals defined on X^*. For non-empty subsets $U \subset X$ and $V \subset X^*$ and $C \subset X^{*a}$ we put

$$U^\circ := \{x^* \in X^* : |x^*(x)| \leq 1 \ \ \forall x \in U\},$$
$$^\circ V := \{x \in X : |x^*(x)| \leq 1 \ \ \forall x^* \in V\}$$

and

$$V \quad := \{f \in X^{*a} : |f(x^*)| \leq 1 \ \ \forall x^* \in V\},$$
$$C := \{x^* \in X^* : |f(x^*)| \leq 1 \ \ \forall f \in C\}.$$

Clearly, if $U \subset X$ and $V \subset X^*$ are vector subspaces, then U° coincides with the annihilator of U into X^* and $^\circ V$ coincides with the annihilator of V into X.

We say that X is *a dually separated space* if X^* separates the points of X. Clearly, X is dually separated if and only if for any non-zero $x \in X$ there exists $x^* \in X^*$ such that $x^*(x) \neq 0$.

Let us also say that X is *an essentially dually separated space* if the quotient $X/cl(\{0\})$ is a dually separated space (here $cl(\{0\})$ is the closure into X of the one-element set $\{0\}$).

We have: X is essentially dually separated if and only if for any non-zero $x \in X \setminus cl(\{0\})$ there exists $x^* \in X^*$ such that $x^*(x) \neq 0$.

Thanks to the Hahn–Banach theorem, any Hausdorff locally convex space is dually separated and any locally convex space is essentially dually separated. Any dually separated space is Hausdorff, but the converse is not true in general: it is known that when $0 \leq p < 1$ and λ is Lebesgue measure in $[0, 1]$, then for the space $X = L_p([0, 1], \lambda)$ one even has that $X^* = \{0\}$. On the other hand, for $0 < p < 1$ the sequence space l_p presents an example of a non-locally convex complete metrizable dually separated space.

When X is dually separated, we shall always identify X with its canonical image into X^{*a} (i.e., any $x \in X$ will be identified with the linear functional $x^* \to x^*(x)$).

The notation $\sigma(X, X^*)$ and $\sigma(X^*, X)$ will have the usual meaning.

Suppose from now on that X is a dually separated space. In X^* we shall consider several topologies. As usual, $\tau(X^*, X)$ will be the Mackey topology (= the topology of uniform convergence on weakly compact convex subsets of X), $k(X^*, X)$ will stand for the topology of uniform convergence on compact subsets of X, $kc(X^*, X)$ for the topology of uniform convergence on compact convex subsets of X and $pr(X^*, X)$ for the topology of uniform convergence on precompact subsets of X.

Let τ be a (not necessarily Hausdorff) vector topology in X^*. Denote by $\mathcal{N}_\tau$ the τ-closure of $\{0\}$ (i.e., $\mathcal{N}_\tau$ is the "nucleo" of τ). Let us put also $\mathcal{E}_\tau := {}^\circ(\mathcal{N}_\tau)$. Observe that $\mathcal{E}_\tau$ is a weakly closed vector subspace of X.

The topological dual space of the topological vector space (X^*, τ) will be denoted $\mathcal{H}_\tau$, i.e., $\mathcal{H}_\tau := (X^*, \tau)^*$. We put $\mathcal{K}_\tau := \mathcal{H}_\tau \cap X$.

Consequently, $\mathcal{H}_\tau$ consists of all linear functionals given on X^* which are τ-continuous, while $\mathcal{K}_\tau$ consists of those $x \in X$ for which the linear functional $x^* \to x^*(x)$ is τ-continuous.

The topologies τ for which $\mathcal{K}_\tau = \mathcal{H}_\tau$ will play an important role in what follows, so let us make for them a name. Recall that a vector topology τ is called *compatible* with the duality (X, X^*) if $\mathcal{H}_\tau = X$.

Let us say that a vector topology τ given in X^* is *subcompatible* with the duality (X, X^*) if $\mathcal{H}_\tau \subset X$.

By the Mackey–Arens theorem a locally convex topology τ given in X^* is compatible with the duality (X, X^*) if and only if $\sigma(X^*, X) \subset \tau \subset \tau(X^*, X)$. In particular, since we have $\sigma(X^*, X) \subset kc(X^*, X) \subset \tau(X^*, X)$, the topology $kc(X^*, X)$ is compatible with the duality (X, X^*), while, if e.g., X is a metrizable non-complete locally convex space, then the topology $k(X^*, X)$ is not compatible with the duality (X, X^*).

From the above comments it is clear that a vector topology τ given in X^* is subcompatible if $\tau \subset \tau(X^*, X)$. This condition is also necessary for the subcompatibility provided τ is a locally convex topology (this follows easily from Lemma 2.3(b) below). A similar characterization of general (sub)compatible vector topologies seems not to be possible (cf. [9]).

Proposition 2.1. *Let X be a dually separated topological vector space and τ be a vector topology in X^*. Then:*

(a) *Always $\mathcal{N}_\tau \subset \mathcal{K}_\tau^\circ$. Equivalently, always $cl_w(\mathcal{K}_\tau) \subset \mathcal{E}_\tau$.*

(b) *Suppose that τ is subcompatible with the duality (X, X^*). Then the equality*

$$\mathcal{N}_\tau = \mathcal{K}_\tau^\circ \textit{ or, equivalently, the equality } cl_w(\mathcal{K}_\tau) = \mathcal{E}_\tau$$

holds if and only if (X^*, τ) is an essentially dually separated topological vector space.

(b′) *Suppose that τ is subcompatible with the duality (X, X^*). Then the equality $cl_w(\mathcal{K}_\tau) = X$ holds if and only if (X^*, τ) is a dually separated topological vector space.*

In these statements, instead of the closure in the weak topology one can take the closure in the original topology of X provided X is locally convex.

Proof. (a) Take $x^* \in \mathcal{N}_\tau$ and let us show that for any fixed $x \in \mathcal{K}_\tau$ we have $|x^*(x)| \leq 1$. Since x, by the definition, is τ-continuous, $V := \{y^* \in X^* : |y^*(x)| \leq 1\}$ is a closed neighborhood of zero in (X^*, τ). Hence $x^* \in \mathcal{N}_\tau \subset V$.

(b) *"If" part.* Thanks to (a) we need to show only that $\mathcal{K}_\tau^\circ \subset \mathcal{N}_\tau$. Take $x^* \notin \mathcal{N}_\tau$. Since τ is subcompatible, we have $(X^*, \tau)^* = \mathcal{K}_\tau$. Since (X^*, τ) is essentially dually separated too, there is $x \in \mathcal{K}_\tau$ such thet $x^*(x) \neq 0$, hence (as $\mathcal{K}_\tau$ is a vector subspace) we get $x^* \notin \mathcal{K}_\tau^\circ$.

"Only if" part. Suppose we have the inclusion $\mathcal{K}_\tau^\circ \subset \mathcal{N}_\tau$, but (X^*, τ) is not essentially dually separated. Since τ is subcompatible, we have $(X^*, \tau)^* = \mathcal{K}_\tau$. Consequently, there exists $x^* \notin \mathcal{N}_\tau$ such that for all $x \in \mathcal{K}_\tau$ we have $x^*(x) = 0$, i.e., $x^* \in \mathcal{K}_\tau^\circ$, which is a contradiction.

(b′) follows from (b). □

Corollary 2.2. [1] *Let X be a dually separated topological vector space and τ be a vector topology in X^*. Then:*

(a) *Always $\mathcal{N}_\tau \subset \mathcal{H}_\tau$. Equivalently, the $\sigma(X^{*a}, X^*)$-closure of $\mathcal{H}_\tau$ is always a subset of $\mathcal{N}_\tau$.*

(b) *The equality $\mathcal{N}_\tau = \mathcal{H}_\tau$ holds or, equivalently, the $\sigma(X^{*a}, X^*)$-closure of $\mathcal{H}_\tau$ is $\mathcal{N}_\tau$ if and only if (X^*, τ) is an essentially dually separated topological vector space.*

(b′) *(X^*, τ) is a dually separated topological vector space if and only if $\mathcal{H}_\tau$ is dense in $(X^{*a}, \sigma(X^{*a}, X^*))$.*

Proof. Since $(X^{*a}, \sigma(X^{*a}, X^*))^* = X^*$, (a) follows from Proposition 2.1(a) applied to the locally convex space $(X^{*a}, \sigma(X^{*a}, X^*))$. (b) and (b′) are true by similar reasons (as τ is always subcompatible with the duality (X^{*a}, X^*)). □

Using the next lemma, we shall clarify the structure of $\mathcal{K}_\tau$.

Lemma 2.3. *Let X be a dually separated topological vector space, $V \subset X^*$ be a subset and τ be a vector topology in X^*. Then:*

(a) *If V is absorbing in X^*, then ${}^\circ V$ is a weakly closed and weakly bounded absolutely convex subset of X.*

(b) *If V is a τ-neighborhood of zero and τ is subcompatible with the duality (X, X^*), then ${}^\circ V$ is a weakly compact absolutely convex subset of X.*

(c) *If V is a $kc(X^*, X)$-neighborhood of zero, then ${}^\circ V$ is a compact absolutely convex subset of X.*

(c′) *If X is **locally convex**, V is a $\tau \cap pr(X^*, X)$-neighborhood of zero and τ is subcompatible with the duality (X, X^*), then ${}^\circ V$ is a compact absolutely convex subset of X.*

Proof. (a) is evident. (b) is simply Alaoglu's theorem applied to (X^*, τ).

(c) Since V is a $kc(X^*, X)$-neighborhood of zero, there is a compact absolutely convex subset $A \subset X$ such that $A^\circ \subset V$. Clearly A is weakly compact and absolutely convex, too. Hence ${}^\circ V \subset {}^\circ(A^\circ) = A$. Since ${}^\circ V$ is closed in X and X is Hausdorff, we get that ${}^\circ V$ is compact in X.

(c′) Since V is a $pr(X^*, X)$-neighborhood of zero, there is a precompact subset $A \subset X$ such that $A^\circ \subset V$. Hence ${}^\circ V \subset {}^\circ(A^\circ)$. Since X is locally convex and A is precompact in X, ${}^\circ(A^\circ)$, the closed absolutely convex hull of A, is also precompact in X. Hence its subset ${}^\circ V$ is precompact in X and is weakly compact too by (b). These two conditions, according to [4, Ch. IV, §1, Prop. 3] imply that ${}^\circ V$ is compact in X. □

[1] Suggested by the referee.

Corollary 2.4. *Let X be a dually separated topological vector space and τ be a pseudometrizable vector topology in X^*. Then:*

(a) *$\mathcal{K}_\tau$ is always a countable union of an increasing sequence of weakly closed weakly bounded absolutely convex subsets of X.*

(b) *If τ is subcompatible with the duality (X, X^*), then $\mathcal{K}_\tau$ is a countable union of an increasing sequence of weakly compact absolutely convex subsets of X.*

(c) *If $\tau \subset kc(X^*, X)$, then τ is subcompatible with the duality (X, X^*) and $\mathcal{K}_\tau$ is a countable union of an increasing sequence of compact absolutely convex subsets of X.*

(c′) *If X is* ***locally convex****, $\tau \subset pr(X^*, X)$ and τ is subcompatible with the duality (X, X^*), then $\mathcal{K}_\tau$ is a countable union of an increasing sequence of compact absolutely convex subsets of X.*

Proof. Let $V_n,\ n \in \mathbb{N}$, be a decreasing fundamental sequence of neighborhood of zero in the topology τ. Denote $D_n = {}^\circ(V_n),\ n \in \mathbb{N}$. Evidently,

$$\mathcal{K}_\tau = \bigcup_{n=1}^{\infty} D_n. \tag{1}$$

Now it is clear that (a) follows from (1) and Lemma 2.3(a), (b) follows from (1) and Lemma 2.3(b).

(c) Since $kc(X^*, X) \subset \tau(X^*, X)$, we get $\tau \subset \tau(X^*, X)$ and τ is subcompatible by the Mackey–Arens theorem. The rest follows from (1) and Lemma 2.3(c).

(c′) follows from (1) and Lemma 2.3(c′). □

3. Kernels of Measures

Hereafter X will be a dually separated topological vector space over $\mathbb{R}$. The expression "μ is a probability measure in X" will mean that μ is a probability measure defined on a σ-algebra $\mathcal{B}$ of subsets of X, with respect to which all continuous linear functionals are measurable. When we speak about (weak) Borel or (weak) Radon measures, they will be supposed to be defined on the (weak) Borel σ-algebra of X.

Fix a probability measure μ in X. We denote by τ_μ the topology in X^* of the convergence in measure μ. Then τ_μ is a pseudometrizable vector topology in X^*.

All notions introduced in the previous section make their sense for τ_μ, too. To simplify the notation, we will put, $\mathcal{N}_\mu := \mathcal{N}_{\tau_\mu}$, etc.

We begin the consideration with $\mathcal{N}_\mu$ and $\mathcal{E}_\mu := {}^\circ(\mathcal{N}_\mu)$.

Clearly, $\mathcal{N}_\mu = \{x^* \in X^* \mid x^* = 0\ \ \mu\text{-a.e.}\}$. The measure μ is called *scalarly non-degenerate* if $\mathcal{N}_\mu = \{0\}$. It turns out that τ_μ is a Hausdorff topology if and only if μ is a scalarly non-degenerate measure.

Proposition 3.1. *Let X be a dually separated topological vector space and μ be a Borel probability measure in X which has a support. Then $supp(\mu) \subset \mathcal{E}_\mu$,* *hence* $\mu(\mathcal{E}_\mu) = 1$ *and the restriction of* μ *on the Borel* σ*-algebra of* $\mathcal{E}_\mu$ *is a scalarly non-degenerate measure in* $\mathcal{E}_\mu$*.*

Proof. Take any $x^* \in \mathcal{N}_\mu$. Then $\mu(\ker x^*) = 1$. Since $ker(x^*)$ is closed in X, we have $supp(\mu) \subset ker(x^*)$. Hence, $supp(\mu) \subset \cap_{x^* \in \mathcal{N}_\mu} ker(x^*) = \mathcal{E}_\mu$. The rest is easy too. □

For the proof of Theorem 4.1 below we shall use a non-trivial property of scalarly non-degenerate measures formulated in the next statement.

Proposition 3.2 ([8]). *Let* X *be a topological vector space and* μ *be a scalarly non-degenerate probability measure in* X*. Let also* $B \subset X^*$ *be a convex subset that is compact in the topology* $\sigma(X^*, X)$*. Then* $\sigma(X^*, X)|_B = \tau_\mu|_B$*.*

In particular, $(B, \tau_\mu|_B)$ *is a compact Hausdorff topological space and* $(B, \sigma(X^*, X)|_B)$ *a metrizable topological space.*

Remark 3.3. (1) In [20, Th. 1.5.3, p.77] a direct proof of Proposition 3.2 is given when X is a normed space and B is the closed unit ball of X^*. The proof works in the above general setting too.

(2) It is easy to see that when X is weakly separable, then in X there exists a scalarly non-degenerate discrete Radon probability measure.

It is easy to see that when X is weakly separable, then in X there exists a discrete Radon probability measure. Also, when X is weakly separable, then Proposition 3.2 is easy to prove since, in such a case, the $\sigma(X^*, X)$-compact subsets of X^* are metrizable. The advantage of Proposition 3.2 is the fact (first noted in [18]) that it allows one to prove the converse: if in a metrizable locally convex spaces X there exists a scalarly non-degenerate probability measure, then X is separable. From this conclusion and Proposition 3.1 it follows at once, that in a complete metrizable locally convex spaces X, any weak-Borel probability measure which has a support extends to a Radon measure in X (see [18], [5] and [20, pp. 79–81], for some more consequences).

The *kernel* of a measure μ in X is defined as the dual space $\mathcal{H}_\mu$ of (X^*, τ_μ) (see [5, 6]). Consequently, we have $\mathcal{H}_\mu = \mathcal{H}_{\tau_\mu}$.

We call the set $\mathcal{K}_\mu := \mathcal{H}_\mu \cap X$ the *initial kernel* of μ. Consequently, $\mathcal{K}_\mu = \mathcal{K}_{\tau_\mu}$.

Observe that, since τ_μ is a pseudometrizable vector topology in X^*, we can say that a functional $f \in (X^*)^a$ belongs to $\mathcal{H}_\mu$ if and only if $\lim\limits_{n\to\infty} f(x_n^*) = 0$ for an arbitrary sequence (x_n^*) in X^* which τ_μ-converges to zero.

In a similar way we have that an element $x \in X$ belongs to $\mathcal{K}_\mu$ if and only if $\lim\limits_{n\to\infty} x_n^*(x) = 0$ for an arbitrary sequence (x_n^*) in X^* which τ_μ-converges to zero.

In general, it is possible that for a given μ the topology τ_μ is Hausdorff, but $\mathcal{H}_\mu = \{0\}$ (see [5], where many such examples are given with the corresponding references). The next statement describes the situation where a similar phenomenon cannot happen and solves the problem posed in [5, p. 69].

Proposition 3.4. *Let* X *be a dually separated topological vector space and* μ *be a probability measure in* X*. Then:*

(a) *Always* $\mathcal{N}_\mu \subset \mathcal{K}_\mu^\circ$*. Equivalently, always* $cl_w(\mathcal{K}_\mu) \subset \mathcal{E}_\mu$*.*

(b) *Suppose that* τ_μ *is subcompatible with the duality* (X, X^*)*. Then the equality*

$$\mathcal{N}_\mu = \mathcal{K}_\mu^\circ \text{ or, equivalently, the equality, } cl_w(\mathcal{K}_\mu) = \mathcal{E}_\mu$$

holds if and only if (X^*, τ_μ) *is an essentially dually separated topological vector space.*

(b′) *Suppose that* μ *is scalarly non-degenerate and* τ_μ *is subcompatible with the duality* (X, X^*)*. Then the equality* $cl_w(\mathcal{K}_\mu) = X$ *holds if and only if* (X^*, τ_μ) *is a dually separated topological vector space.*

In these statements, instead of the closure in the weak topology, we can the closure in the original topology of X *provided* X *is locally convex.*

Proof. Apply Proposition 2.1 to the topology τ_μ. □

Corollary 3.5. *Let* X *be a dually separated topological vector space and* μ *be a probability measure in* X*. Then:*

(a) *Always* $\mathcal{N}_\mu \subset \mathcal{H}_\mu$*. Equivalently, always the* $\sigma(X^{*a}, X^*)$*-closure of* $\mathcal{H}_\mu$ *is a subset of* $\mathcal{N}_\mu$ *.*

(b) *The equality* $\mathcal{N}_\mu = \mathcal{H}_\mu$ *holds, or equivalently, the* $\sigma(X^{*a}, X^*)$*-closure of* $\mathcal{H}_\mu$ *is* $\mathcal{N}_\mu$ *if and only if* (X^*, τ_μ) *is an essentially dually separated topological vector space.*

(b′) (X^*, τ_μ) *is a dually separated topological vector space if and only if* $\mathcal{H}_\mu$ *is dense in* $(X^{*a}, \sigma(X^{*a}, X^*))$*.*

Proof. Apply Corollary 2.2 to the topology τ_μ. □

Remark 3.6. (1) Originally, in [5, p. 69] the following problem was posed: "Find a characterization of cylindrical measures μ on X such that $\mathcal{N}_\mu = \mathcal{H}_\mu$". Since the topology τ_μ, as it is introduced in [5] for a cylindrical measure μ, is a vector topology in X^*, from Corollary 2.2 we can conclude that for a cylindrical measure μ on X we have $\mathcal{N}_\mu = \mathcal{H}_\mu$ if and only if (X^*, τ_μ) is an essentially dually separated topological vector space.

(2) In Proposition 3.4(b) the supposition "τ_μ is subcompatible with duality (X, X^*)" cannot be suppressed in general. Indeed, as it is shown in [7, p.7], there exists a Hausdorff locally convex space X, a functional $f \in (X^*)^a \setminus X$ and a probability measure μ on the cylindrical σ-algebra of X, such that $\hat{\mu}(x^*) = \exp(if(x^*))$, $\forall x^* \in X^*$. Clearly, for this measure we have $\mathcal{N}_\mu = ker(f) \neq X^*$, $\mathcal{H}_\mu = \mathbb{R}\cdot f$ and $\mathcal{K}_\mu = \{0\}$. Note that the considered measure does not admit a Radon extension.

(3) In connection with (2), the following question raised by the referee is of interest: is there a (Radon probability) measure μ in a space X with $\mathcal{H}_\mu \neq \mathcal{K}_\mu$, $\mathcal{K}_\mu^\circ = \mathcal{H}_\mu$? The answer is positive. Indeed, in [11] an example is given of a Radon probability measure in a sigma-compact inner product space X such that $\mathcal{H}_\mu \neq \mathcal{K}_\mu = X$, hence $\mathcal{K}_\mu^\circ = \mathcal{H}_\mu = \{0\}$.

Corollary 3.7. *Let* X *be a dually separated topological vector space and* μ *be a Borel probability measure in* X *which has a support,* τ_μ *is subcompatible* *with the duality* (X, X^*) *and* (X^*, τ_μ) *is essentially dually separated. Suppose further that* $\dim(\mathcal{K}_\mu) < \infty$ *(or merely that* $\mathcal{K}_\mu$ *is weakly closed).*

Then $\mathcal{K}_\mu = \mathcal{E}_\mu$ *and* $\mu(\mathcal{K}_\mu) = 1$.

In particular, the conclusion is true whenever $\dim(X) < \infty$.

Proof. The equality $\mathcal{K}_\mu = \mathcal{E}_\mu$ follows from Proposition 3.4(b). Since, by supposition, μ has a support too, by Proposition 3.1 we get $\mu(\mathcal{K}_\mu) = 1$. □

As Remark 3.6(3) shows in general the kernel even of a Radon probability measure μ in an inner product space may not be contained in the initial space and hence the topology τ_μ may not be subcompatible with the duality (X, X^*). To formulate a natural restriction on a measure which will allow us to avoid such a "pathology", let us say that a Borel probability measure μ in a Hausdorff topological vector space X is *convex-tight* if for any $\varepsilon > 0$ there exists a compact convex $K \subset X$ such that $\mu(K) > 1 - \varepsilon$.

Recall also that a Hausdorff topological vector space X is said to have the *convex compactness property* [14] if the closed convex hull of any compact subset of X is again compact.

Proposition 3.8 (cf. [11, Prop. 3.8]**).** *Let* X *be a dually separated topological vector space and* μ *be a Borel probability measure in* X. *Suppose further that at least one of the following suppositions is satisfied:*

(0) μ *is convex-tight.*

(1) X *has the convex compactness property and* μ *is Radon.*

(2) X *is a quasi-complete locally convex space and* μ *is Radon.*

(3) X *is a complete metrizable locally convex (=Frechet) separable space.*

Then we have:

(4) $\tau_\mu \subset kc(X^*, X)$, *the topology* τ_μ *is subcompatible with the duality* (X, X^*) *and* $\mathcal{K}_\mu$ *is a countable union of an increasing sequence of compact absolutely convex subsets of* X.

Proof. (0) ⇒ (4). The proof of $\tau_\mu \subset kc(X^*, X)$ is standard. The rest follows from Corollary 2.4(c) applied to τ_μ.

(1) ⇒ (4). When X has the convex compactness property, we have that any Radon probability measure in X is convex-tight and (0) ⇒ (4) is applicable.

(2) ⇒ (4) follows from (1) ⇒ (4) since any quasi-complete Hausdorff locally convex space has the convex compactness property.

(3) ⇒ (4) follows from (2) ⇒ (4) since in any Frechet separable space any Borel probability measure is Radon. □

Remark 3.9. (1) The implication (3) ⇒ (4) of Proposition 3.8 is due to W. Smolenski [16, Prop. 2].

(2) Proposition 3.8 does not cover one important case: it is known that for *any* Gaussian Radon measure γ in an arbitrary locally convex space X, the inclusion $\mathcal{H}_\gamma \subset X$ always holds [3] (see also [1, p. 359], where similar statement for the symmetric Gaussian Radon measures was obtained earlier). Note that this result cannot be derived from the implication (0) ⇒ (4) of Proposition 3.8

since it is still the open problem whether any Gaussian Radon measure in any locally convex space is convex-tight (cf. [17]).

(3) The implication (1) $\Rightarrow$ (4) of Proposition 3.8 for the metrizable locally convex spaces can give nothing new since for such a space the presence of the convex compactness property implies its completeness [14, Th. 2.3].

4. Very Big Kernels

In this section we consider the question of the existence of probability measures whose (initial) kernels coincide with the whole space. Plainly, such measures exist in the finite-dimensional Hausdorff topological vector spaces. The next statement shows that a certain converse assertion is also true.

Theorem 4.1. *Let X be a dually separated topological vector space of second category (in the Baire sense) for which there exists a probability measure μ in X with $\mathcal{K}_\mu = X$, then* $\dim X < \infty$.

Proof. Fix a probability measure μ in X such that $\mathcal{K}_\mu = X$. Clearly, τ_μ is Hausdorff. So (X^*, τ_μ) is a metrizable topological vector space. Let us show that this space is complete. Let (x_n^*) be a Cauchy sequence in (X^*, τ_μ). Since the identity mapping $(X^*, \tau_\mu) \to (X^*, \sigma(X^*, X))$ is continuous and linear, it is uniformly continuous. This implies that (x_n^*) is a Cauchy sequence in $(X^*, \sigma(X^*, X)))$. From this (as $\mathbb{R}$ is complete) we get that there is a mapping $l : X \to \mathbb{R}$ such that $\lim_n x_n^*(x) = l(x),\ \forall x \in X$. Evidently, l is linear. Since X is of second category, by the corresponding variant of the Banach–Steinhaus theorem, we can conclude that l is continuous. Then (x_n^*) tends to l in measure μ. Concequently, (X^*, τ_μ) is a complete metrizable space.

By Corollary 2.4(a) (applied to $\tau = \tau_\mu$) we can write $X = \mathcal{K}_\mu = \bigcup_{n=1}^{\infty} D_n$, where (D_n) is a sequence of weakly closed weakly bounded subsets of X. Clearly, $D_n,\ n \in \mathbb{N}$, are closed sets in the topology of X, too. Since X is of second category, we obtain that at least one of $D_n,\ n \in \mathbb{N}$, has a non-empty interior. So we have that in X there exists a $\sigma(X, X^*)$-bounded neighborhood of zero U. Put $B := U^\circ$. Observe:

(1) Since U is a neighborhood of zero in X, clearly, the set B is absolutely convex and, by Alaoglu's theorem, is $\sigma(X^*, X)$-compact too.

(2) Since U is $\sigma(X, X^*)$-bounded, the set B is absorbing in X^* (this is evident).

Now item (1), since μ is scalarly non-degenerate, according to Proposition 3.2, implies:

(3) B is compact in (X^*, τ_μ).

Then by item (2) we have $X^* = \bigcup_{n=1}^{\infty} n{\cdot}B$. Since (X^*, τ_μ) is a complete metrizable space, it is of second category. Hence, at least one of the τ_μ-compact sets $n{\cdot}B$, $n = 1, 2, \ldots$, has a nonempty τ_μ-interior. Consequently, the space (X^*, τ_μ) is locally compact. Therefore, $\dim(X^*) < \infty$ and since X^* separates the points of X, we also have that $\dim X < \infty$. □

Remark 4.2. For a separable Frechet space Theorem 4.1 follows at once from the implication (3) $\Rightarrow$ (4) of Proposition 3.8.

Remark 4.3. (1) If the space X is not of second category, then Theorem 4.1 is not valid. Indeed, let X be any dually separated space with the countable algebraic dimension, $A \subset X$ be an algebraic basis of X, and let μ be any probability measure in X with $\mu\{a\} > 0$ for all $a \in A$. Then $\sigma(X^*, X) = \tau_\mu$ and so $\mathcal{H}_\mu = X$.

(2) In view of (1) it seems very likely that if X is a Hausdorff locally convex space such that for some Radon probability measure μ in X we have $\mathcal{K}_\mu = X$, then X must be of at most countable algebraic dimension (cf. also [19, p. 68]). But this is not so: if F is any nuclear Frechet space and $X := (F, k(F^*, F))$, then in X there exists a Radon probability measure μ for which $\mathcal{K}_\mu = X$ and $\mu(E) = 0$ for any finite-dimensional vector subspace $E \subset X$ [10, p. 199, Lemma 2].

(3) If μ is a probability measure in a dually separated space X such that $\mathcal{K}_\mu = X$, then, clearly, the topologies $\sigma(X^*, X)$ and τ_μ have the same sets of convergent sequences, but $\sigma(X^*, X)$ and τ_μ may not coincide (the coincidence $\sigma(X^*, X) = \tau_\mu$ will imply that $\sigma(X^*, X)$ is a metrizable topology, which in turn will give that the algebraic dimension of X is at most countable).

5. Nearly Atomic Measures

In the previous section we were considering the measures having very big kernels in set-theoretic sense. Here we shall deal with measures, which will have the big kernels in the measure-theoretic sense, i.e., with measures which will have the kernels of full measure.

We shall use the following remarkable result.

Theorem 5.1 ([12, Th.8]**;** [13, p. 206, Th. 6A]**).** *Let (Ω, ν) be a probability space and Y be a vector subspace of $L_0(\Omega, \nu)$. The following statements are equivalent:*

(i) *Any sequence from Y that converges to zero in measure ν, converges also ν-almost everywhere.*

(ii) *Y is a nuclear locally convex space with respect to the topology of the convergence in measure ν.*

Let us say that a probability measure μ in a dually separated space X is *nearly atomic* if any sequence from X^* that converges to zero in measure μ, converges also μ-almost everywhere. The following assertion provides a description of the nearly atomic measures by means of their kernels.

Theorem 5.2. *Let X be a dually separated topological vector space and μ be a Radon probability measure in X such that τ_μ is subcompatible with the duality (X, X^*). Then the following statements are equivalent:*

(i) *μ is nearly atomic.*

(ii) *(X^*, τ_μ) is a nuclear locally convex space.*

(iii) $\mu(\mathcal{K}_\mu) = 1$.

Proof. (i)⇒(ii) is a particular case of the implication (i) ⇒ (ii) of Theorem 5.1.

(ii)⇒(iii). Evidently, the Fourier transform $\hat{\mu}$ considered as a functional on $Y := (X^*, \tau_\mu)$ is continuous and positive definite. Then according to the Minlos' theorem [20, Th. 6.4.3, p.410] there exists a Radon probability measure μ' in $(Y^*, \sigma(Y^*, Y))$, whose Fourier transform coincides with $\hat{\mu}$. Since $Y^* = \mathcal{H}_\mu = \mathcal{K}_\mu \subset X$ (by supposition!), we get that in fact μ' is a Radon probability measure in $(X, \sigma(X, X^*))$. Since μ is Radon in X (again by the supposition!), it is also Radon in $(X, \sigma(X, X^*))$. From this, as $\hat{\mu} = \hat{\mu'}$, according to the uniqueness theorem for the Fourier transform for Radon measures [20, Th. 4.2.2(b), p.200] we get that μ and μ' coincide on the Borel σ-algebra of $(X, \sigma(X, X^*))$. Finally, since $\mathcal{K}_\mu$ is a Borel set in $(X, \sigma(X, X^*))$ (by Corollary 2.4(a)), we get $\mu(\mathcal{K}_\mu) = \mu'(\mathcal{K}_\mu) = 1$.

(iii)⇒(i) is evident (and is true for any weak Borel μ). □

Remark 5.3. In [10, Th. 1] the equivalence (iii)⇔(ii) is proved for the case of scalarly non-degenerate measures in a separable Frechet space; moreover, the proof of the implication (iii)⇒(ii) is direct, i.e., does not use Theorem 5.1 (the paper [12] is not mentioned at all; seemingly its existence was unknown for the authors of [10]). Then from Theorem 1 of [10] a more precise version of our Theorem 5.1 is derived in [10]. Namely, Theorem 2 in [10, p.200] asserts that when Y is separable and (ii) is satisfied, then there exists $\Omega_0 \subset \Omega$, $\nu(\Omega_0) = 1$ such that any sequence from Y that is convergent in measure ν, is also convergent everywhere on Ω_0.

We conclude the section by showing that (as this was expected) the nearly atomic measures in "good" cases are degenerate.

Theorem 5.4. *Let X be a dually separated topological vector space and μ be a probability measure in X. Then the following statements are valid:*

(a) *If μ is nearly atomic and (X^*, τ_μ) is a locally bounded topological vector space, then* $\dim(\mathcal{H}_\mu) < \infty$.

(b) *If $\dim(\mathcal{K}_\mu) < \infty$, μ is a Borel measure with a support such that τ_μ is subcompatible with the duality (X, X^*) and (X^*, τ_μ) is an essentially dually separated space, then $\mu(\mathcal{K}_\mu) = 1$.*

Proof. (a) The initial supposition together with the implication (i)⇒(ii) of Theorem 5.1 implies that $E := (X^*/\mathcal{N}_\mu, \tau_\mu/\mathcal{N}_\mu)$ is a Hausdorff nuclear locally convex space. Since E by supposition is a locally bounded space, we get that E is a normable nuclear space. Consequently, $\dim(E) < \infty$. Therefore, $\mathcal{H}_\mu$, as the dual space of this finite-dimensional space is also finite-dimensional.

(b) follows from Corollary 3.7. □

Remark 5.5. (1) Theorem 5.4(a) is applicable for p-stable measures such that $0 < p \leq 2$, while an analogous assertion from [2] (cf., [5, pp. 62–63]) covers only the case of p-stable measures with $1 < p \leq 2$ (see also [19]).

(2) In Theorem 5.4(a) the supposition "(X^*, τ_μ) is a locally bounded topological vector space" cannot be suppressed, e.g., thanks to Remark 4.3(2).

(3) In Theorem 5.4(b) the supposition "(X^*, τ_μ) is essentially dually separated" cannot be suppressed since, e.g., in $\mathbb{R}^{\mathbb{N}}$ there exists even a scalarly non-degenerate product probability measure μ for which $\mathcal{K}_\mu = \{0\}$ [15].

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Author's addresses:

Nguyen Duy Tien
Faculty of Mathematics
University of Hanoi
Hanoi, Vietnam
E-mail: nduytien@netnam.org.vn

V. Tarieladze
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093
Georgia
E-mail: tar@scien.compmath.acnet.ge
visit01@dma.uvigo.es