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ASYMPTOTIC BEHAVIOR OF SINGULAR AND ENTROPY NUMBERS FOR SOME RIEMANN–LIOUVILLE TYPE OPERATORS

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Abstract. The asymptotic behavior of the singular and entropy numbers is established for the Erdelyi–Köber and Hadamard integral operators (see, e.g., [15]) acting in weighted L^2 spaces. In some cases singular value decompositions are obtained as well for these integral transforms.

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In this paper, we investigate the asymptotic behavior of singular and entropy numbers for the following integral operators:

$$I_{\alpha,\sigma}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x^\sigma-y^\sigma)^{\alpha-1}f(y)\,dy,\quad x>0,\quad \alpha>0,\quad \sigma>0,$$

(Erdelyi–Köber operator) and

$$H_\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_1^x \left(\ln\frac{x}{y}\right)^{\alpha-1}f(y)\,dy,\quad x>1,\quad \alpha>0,$$

(Hadamard operator) in some weighted L^2 spaces. We get singular value decompositions for these integral transforms.

Analogous problems for the Riemann–Liouville operator

$$R_\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}f(y)\,dy,\quad \alpha>0,$$

were studied in [1]–[6]. We refer also to [7]–[8], where some powerful tools were developed for establishing the asymptotics of singular numbers of certain pseudo-differential operators (see also [9] for some properties of singular numbers for the weighted Riemann–Liouville operator $R_{\alpha,v}f(x)\equiv v(x)R_\alpha f(x)$, where $\alpha>1/2$).

Two-sided estimates of singular (approximation) numbers for the weighted Hardy operator $\mathcal{H}_{v,w}f(x)=v(x)\int\limits_0^x f(y)w(y)\,dy$ were given in [10]–[12] (for some related topics concerning the weighted Volterra integral operators see [13], [14]).

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Note that some mapping properties of the operators $I_{\alpha,\sigma}$ and H_α were established in [15].

Let A and B be infinite-dimensional Hilbert spaces. It is known that if $K : A \to B$ is an injective compact linear operator, then there exist:

(a) an orthonormal basis $\{u_j\}_{Z_+}$ in A;

(b) an orthonormal basis $\{v_j\}_{Z_+}$ in B;

(c) a nonincreasing sequence $\{s_j(K)\}_{Z_+}$ of positive numbers with limit 0 as $j \to +\infty$ such that

$$Ku_j = s_j(K)v_j, \quad j \in Z_+.$$

The numbers $s_j(K)$ are known as singular numbers or s-numbers of the operator K, the system $\{s_j(K), u_j, v_j\}_{j\in Z_+}$ is called a singular system of K. For the operator K the singular value decomposition

$$Kf = \sum_{j=0}^{\infty} s_j(K)(f, u_j)_A v_j, \quad f \in A,$$

is valid.

Let w be a measurable a.e. positive function on $\Omega \subset R_+$. We denote by $L^2_w(\Omega)$ the class of all measurable functions $f : \Omega \to R_+$ for which

$$\|f\|_{L^2_w(\Omega)} = \Big(\int_\Omega |f(x)|^2 w(x)\, dx \Big)^{1/2} < \infty.$$

In the sequel by writing $a_n \approx b_n$ for sequences of positive numbers a_n and b_n we mean that there exist positive constants c_1 and c_2 such that $c_1 \le a_n/b_n \le c_2$ for all $n \in \mathbb{N}$.

The following result is well-known (see [5]):

Theorem A. *Let $\alpha > 0$, $\beta > -1$, $\varphi(t) = t^{-\beta}e^{-t}$, $\psi(t) = t^{-(\alpha+\beta)}e^{-t}$. Then the singular system $\{s_j(R_\alpha), u_j, v_j\}_{j\in Z_+}$ of the operator $R_\alpha : L^2_\varphi(R_+) \to L^2_\psi(R_+)$ is given by*

$$\begin{aligned} s_n(R_\alpha) &= \Big(\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \Big)^{1/2}, \qquad (1) \\ u_n(t) &= \Big(\frac{n!}{\Gamma(n+\beta+1)} \Big)^{1/2} t^\beta L_n^{(\beta)}(t), \\ v_n(t) &= \Big(\frac{n!}{\Gamma(n+\alpha+\beta+1)} \Big)^{1/2} t^{\alpha+\beta} L_n^{(\alpha+\beta)}(t), \end{aligned}$$

and $s_n(R_\alpha)/n^{-\alpha/2} \to 1$ as $n \to \infty$, where $L_n^{(\gamma)}$ is the Laguerre polynomial:

$$L_n^{(\gamma)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\gamma}{n-k} \frac{x^k}{k!}, \quad \gamma > -1, \quad n \in Z_+.$$

Theorem B ([4]). *Let $\alpha > 0$, $\lambda > \alpha - 1/2$, $\lambda \neq 0$. Then the operator $R_\alpha : L^2_\varphi(R_+) \to L^2_\psi(R_+)$, where $\varphi(x) = x^{1/2-\lambda}(1+x)^{2\alpha}$, $\psi(x) = x^{1/2-\lambda-\alpha}$, has the following singular system:*

$$\begin{aligned} s_n(R_\alpha) &= \Big(\frac{\Gamma(n+\lambda-\alpha+1/2)}{\Gamma(n+\lambda+\alpha+1/2)}\Big)^{1/2}, \qquad (2)\\ u_n(t) &= 2^\lambda a_n t^{\lambda-1/2}(1+t)^{-\lambda-\alpha-1/2} C_n^\lambda\Big(\frac{1-t}{1+t}\Big),\\ v_n(t) &= 2^\lambda b_n t^{\lambda+\alpha-1/2}(1+t)^{-\lambda-\alpha-3/2} P_n^{(\lambda-\alpha-1/2,\lambda+\alpha-1/2)}\Big(\frac{1-t}{1+t}\Big), \end{aligned}$$

where

$$\begin{aligned} a_n &= \Big(\frac{2^{2\lambda-1}(n+\lambda)n!}{\pi\Gamma(n+2\lambda)}\Big)^{1/2}\Gamma(\lambda),\\ b_n &= \Big(\frac{2^{1-2\lambda}(n+\lambda)n!\Gamma(n+2\lambda)}{\Gamma(n+\lambda-\alpha+1/2)\Gamma(n+\lambda+\alpha+1/2)}\Big)^{1/2}, \end{aligned}$$

$C_n^\lambda(t)$ is the Gegenbauer polynomial

$$C_n^\lambda(t) = \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{[n/2]}(-1)^j\frac{\Gamma(\alpha+n-j)}{j!(n-2j)!}(2t)^{n-2j},$$

and $P_m^{(\alpha,\beta)}$ is the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(t) = 2^{-n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(t-1)^{n-m}(t+1)^m, \quad n \in Z_+.$$

Moreover, $\lim_{n\to\infty} s_n(R_\alpha)/n^{-\alpha} = 1$.

Theorem C ([6]). *The singular values of the operator $R_\alpha : L^2(0,1) \to L^2_{x^{-\gamma}}(0,1)$ have the following asymptotics:*

$$s_n(R_\alpha) \approx n^{-\alpha}, \quad 0 \le \gamma < \alpha.$$

When $\gamma = 0$, the upper estimate in the previous statement was derived in [1], [2], while the lower estimate was given in [2].

The following lemma follows immediately:

Lemma 1. *Let φ, ψ, v and w be measurable a.e. positive functions on $\Omega \subseteq R_+$. Then the operator A is compact from $L^2_\varphi(\Omega)$ to $L^2_\psi(\Omega)$ if and only if the operator $A_1 f(x) = v^{1/2}(x)A(fw^{-1/2})(x)$ is compact from $L^2_{\varphi w^{-1}}(\Omega)$ to $L^2_{\psi v^{-1}}(\Omega)$.*

Taking into account the definition of the singular system of the operator, we easily derive the next statement.

Lemma 2. *Let v and w be a.e. positive measurable functions on $\Omega \subseteq R_+$. A system $\{s_j(A), u_j, v_j\}_{j\in Z_+}$ is a singular system for the operator $A : L^2_\varphi(\Omega) \to L^2_\psi(\Omega)$ if and only if the operator $A_1 : L^2_{\varphi w^{-1}}(\Omega) \to L^2_{\psi v^{-1}}(\Omega)$ has the singular* *system* $\{s_j(A_1), w^{1/2}u_j, v^{1/2}v_j\}_{j\in Z_+}$, *where* $A_1f(x) = v^{1/2}(x)A(fw^{-1/2})(x)$ *and* $s_j(A_1) = s_j(A)$.

Let $\mathcal{I}_{\alpha,\sigma}f(x) = I_{\alpha,\sigma}(f\rho)(x)$, where $\rho(y) = y^{\sigma-1}$, $\alpha > 0$, $\sigma > 0$ and $x > 0$. From the definition of compactness we easily deduce

Lemma 3. *Let* $\alpha > 0$, $\sigma > 0$ *and let* $\Omega = (0,1)$ *or* $\Omega = (0,\infty)$. *Assume that* v *and* w *are measurable a.e. positive functions on* Ω. *Then the operator* $\mathcal{I}_{\alpha,\sigma}$ *is compact from* $L^2_w(\Omega)$ *to* $L^2_v(\Omega)$ *if and only if* R_α *is compact from* $L^2_W(\Omega)$ *to* $L^2_V(\Omega)$, *where* $W(x) = w(x^{1/\sigma})x^{1/\sigma-1}$, $V(x) = v(x^{1/\sigma})x^{1/\sigma-1}$.

Now we prove the following statement:

Lemma 4. *Let* $\alpha > 0$, $\sigma > 0$ *and let* v *and* w *be measurable a.e. positive functions on* Ω, *where* $\Omega = (0,\infty)$ *or* $\Omega = (0,1)$. *Then for the singular system* $\{s_j(\mathcal{I}_{\alpha,\sigma}\}, \overline{u}_j, \overline{v}_j\}_{j\in Z_+}$ *of the operator* $\mathcal{I}_{\alpha,\sigma} : L^2_w(\Omega) \to L^2_v(\Omega)$ *we have* $s_j(\mathcal{I}_{\alpha,\sigma}) = \sigma^{-1}s_j(R_\alpha)$, $\overline{u}_j(x) = \sigma^{1/2}u_j(x^\sigma)$, $\overline{v}_j(x) = \sigma^{1/2}v_j(x^\sigma)$, *where* $\{s_j(R_\alpha), u_j, v_j\}_{j\in Z_+}$ *is a singular system for the operator* $R_\alpha : L^2_W(0,\infty) \to L^2_V(0,\infty)$, *with* $W(x) = w(x^{1/\sigma})x^{1/\sigma-1}$ *and* $V(x) = v(x^{1/\sigma})x^{1/\sigma-1}$.

Proof. Let $\Omega = (0,\infty)$. Using the change of variable $y = t^{1/\sigma}$, we have

$$(\mathcal{I}_{\alpha,\sigma}\overline{u}_j)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x^\sigma - y^\sigma)^{\alpha-1}y^{\sigma-1}\overline{u}_j(y)\,dy$$

$$= \frac{\sigma^{1/2}}{\Gamma(\alpha)}\int_0^x (x^\sigma - y^\sigma)^{\alpha-1}u_j(y^\sigma)y^{\sigma-1}\,dy = \frac{\sigma^{-1/2}}{\Gamma(\alpha)}\int_0^{x^\sigma}(x^\sigma - t)^{\alpha-1}u_j(t)\,dt$$

$$= \sigma^{-1/2}(R_\alpha u_j)(x^\sigma) = s_j(R_\alpha)\sigma^{-1/2}v_j(x^\sigma) = \sigma^{-1}s_j(R_\alpha)\overline{v}_j(x).$$

Further, the change of variable yields

$$\int_0^\infty \overline{v}_j(x)\overline{v}_i(x)v(x)\,dx = \sigma\int_0^\infty v_j(x^\sigma)v_i(x^\sigma)V(x^\sigma)x^{\sigma-1}dx$$

$$= \int_0^\infty v_j(x)v_i(x)V(x)dx = \delta_{ij},$$

where δ_{ij} denotes Kronecker's symbol.

Analogously, we have

$$\int_0^\infty \overline{u}_j(x)\overline{u}_i(x)w(x)\,dx = \int_0^\infty u_j(x)u_i(x)W(x)\,dx = \delta_{ij},$$

Hence $\{\overline{v}_j\}$ and $\{\overline{u}_j\}$ are orthonormal systems in $L^2_v(R_+)$ and $L^2_w(R_+)$, respectively.

The case $\Omega = (0,1)$ follows in a similar way. $\square$

Theorem 1. *Let $\alpha > 0$, $\sigma > 0$ and $0 \le \gamma < \alpha$. Then there exist positive constants c_1 and c_2 depending on α, σ and γ such that for the singular numbers of the operator $I_{\alpha,\sigma} : L^2_{x^{1-\sigma}}(0,1) \to L^2_{x^{\sigma-1-\gamma\sigma}}(0,1)$ we have $s_n(I_{\alpha,\sigma}) \approx n^{-\alpha}$.*

Proof. By Lemma 2 we have that $s_j(I_{\alpha,\sigma}) = s_j(\mathcal{I}_{\alpha,\sigma})$, where $\mathcal{I}_{\alpha,\sigma}$ acts from $L^2_{x^{\sigma-1}}(0,1)$ to $L^2_{x^{\sigma-1-\gamma\sigma}}(0,1)$, while Lemma 4 yields $s_j(\mathcal{I}_{\alpha,\sigma}) = 1/\sigma s_j(R_\alpha)$, where R_α is the Riemann–Liouville operator acting from $L^2(0,1)$ to $L^2_{x^{-\gamma}}(0,1)$. Theorem C completes the proof. $\square$

Theorem 2. *Let $\alpha > 0$, $\sigma > 0$, $\lambda > \alpha - 1/2$ and $\lambda \ne 0$. Assume that $w(x) = x^{1-\sigma/2-\sigma\lambda}(1+x^\sigma)^{2\alpha}$, $v(x) = x^{3\sigma/2-\sigma\lambda-\sigma\alpha-1}$. Then the operator $I_{\alpha,\sigma} : L^2_w(0,\infty) \to L^2_v(0,\infty)$ has a singular system $\{s_n(I_{\alpha,\sigma}), \overline{u}_n, \overline{v}_n\}_{n\in Z_+}$, where*

$$\begin{aligned}
s_n(I_{\alpha,\sigma}) &= 1\big/\sigma\left(\frac{\Gamma(n+\lambda-\alpha+1/2)}{\Gamma(n+\lambda+\alpha+1/2)}\right)^{1/2},\\
\overline{u}_n(x) &= \sigma^{1/2}2^\lambda a_n x^{\sigma(\lambda+1/2)-1}(1+x^\sigma)^{-\lambda-\alpha-1/2}C_n^\lambda\left(\frac{1-x^\sigma}{1+x^\sigma}\right),\\
\overline{v}_n(x) &= \sigma^{1/2}2^\lambda b_n x^{\sigma(\lambda+\alpha-1/2)}(1+x^\sigma)^{-\lambda-\alpha-3/2}P_n^{(\lambda-\alpha-1/2,\lambda+\alpha-1/2)}\left(\frac{1-x^\sigma}{1+x^\sigma}\right),
\end{aligned}$$

$C_n^\lambda(x)$ and $P_n^{(\alpha,\beta)}$ are Gegenbauer and Jacobi polynomials, respevetively (see Theorem B), and a_n, b_n are the constants defined in Theorem B. Moreover,

$$\lim_{n\to\infty} s_n(I_{\alpha,\sigma})/n^{-\alpha} = 1/\sigma.$$

Proof. Lemma 2 implies that the singular system $\{s_m(I_{\alpha,\sigma}), \overline{u}_m, \overline{v}_m\}_{m\in Z_+}$ of the map $I_{\alpha,\sigma} : L^2_w(0,\infty) \to L^2_v(0,\infty)$ coincides with the singular system $\{s_m(\mathcal{I}_{\alpha,\sigma}), \widetilde{u}_m, \widetilde{v}_m\}_{m\in Z_+}$ of the map $\mathcal{I}_{\alpha,\sigma} : L^2_W(0,\infty) \to L^2_V(0,\infty)$, where $W(x) = w(x)x^{2(\sigma-1)}$, $V(x) = v(x)$, $\widetilde{u}_m(x) = x^{1-\sigma}u_m(x)$, $\widetilde{v}_m(x) = \overline{v}_m(x)$. Further, by Lemma 4 we have that the operator $R_\alpha : L^2_\varphi(0,\infty) \to L^2_\psi(0,\infty)$ ($\varphi(x) = x^{1/2-\lambda}(1+x)^{2\alpha}$, $\psi(x) = x^{1/2-\lambda-\alpha}$) has a singular system $\{s_m(R_\alpha), u_m, v_m\}_{m\in Z_+}$, where

$$s_m(R_\alpha) = \sigma s_m(\mathcal{I}_{\alpha\sigma}) \approx m^{-\alpha},\ \ \overline{u}_m(x) = \sigma^{1/2}x^{\sigma-1}u_m(x^\sigma),\ \ \overline{v}_m(x) = \sigma^{1/2}v_m(x^\sigma). \quad \square$$

Analogously, we have

Theorem 3. *Let $\alpha > 0$, $\sigma > 0$, $\beta > -1$, $w(y) = y^{-\sigma\beta-\sigma+1}e^{-y^\sigma}$ and $v(y) = y^{-\sigma(\alpha+\beta)+\sigma-1}e^{-y^\sigma}$. Then the operator $I_{\alpha,\sigma} : L^2_w(0,\infty) \to L^2_v(0,\infty)$ has a singular system $\{s_m(I_{\alpha,\sigma}), \overline{u}_m, \overline{v}_m\}_{m\in Z_+}$ defined by*

$$\begin{aligned}
s_n(I_{\alpha,\sigma}) &= 1\big/\sigma\left(\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\right)^{1/2},\\
\overline{u}_n(x) &= \sigma^{1/2}x^{\sigma-1+\sigma\beta}\left(\frac{n!}{\Gamma(n+\beta+1)}\right)^{1/2}L_n^{(\beta)}(x^\sigma),\\
\overline{v}_n(x) &= \sigma^{1/2}\left(\frac{n!}{\Gamma(n+\alpha+\beta+1)}\right)^{1/2}x^{\sigma(\alpha+\beta)}L_n^{(\alpha+\beta)}(x^\sigma),
\end{aligned}$$

where $L_n^{(\gamma)}(x)$ *is a Laguerre polynomial* (*see Theorem* A). *Moreover,*

$$\lim_{n\to\infty} s_n(I_{\alpha,\sigma})/n^{-\alpha/2} = 1/\sigma.$$

Now we consider the operator of Hadamard's type H_α.

The following lemma holds:

Lemma 5. *Let* $\alpha > 0$ *and* (v, w) *be a pair of weights defined on* $(1,\infty)$. *Then* $\{s_m(L_\alpha), \overline{u}_m, \overline{v}_m\}_{m\in Z_+}$ *is a singular system for the operator* $L_\alpha : L^2_w(1,\infty) \to L^2_v(1,\infty)$, *where*

$$L_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_1^x \left(\ln\frac{x}{y}\right)^{\alpha-1} f(y)\, \frac{dy}{y}\,,$$

if and only if the Riemann–Liouville operator $R_\alpha : L^2_W(0,\infty) \to L^2_V(0,\infty)$ *has a singular system* $\{s_m(R_\alpha), \widetilde{u}_m, \widetilde{v}_m\}_{m\in Z_+}$, *where* $W(x) = w(e^x)e^x$, $V(x) = v(e^x)e^x$, $s_m(R_\alpha) = s_m(L_\alpha)$, $\widetilde{u}_m(x) = \overline{u}_m(e^x)$, $\widetilde{v}_m(x) = \overline{v}_m(e^x)$.

Proof. Using the change of variable $y = e^z$ we have

$$(L_\alpha \overline{u}_m)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_1^x \left(\ln\frac{x}{y}\right)^{\alpha-1} \overline{u}_m(y)\, \frac{dy}{y}$$

$$= \frac{1}{\Gamma(\alpha)} \int\limits_0^{\ln x} (\ln x - z)^{\alpha-1} \widetilde{u}_m(z)\, dz = (R_\alpha \widetilde{u}_m)(\ln x) = \widetilde{v}(\ln x) s_j(R_\alpha).$$

On the other hand,

$$\int\limits_0^\infty \widetilde{u}_i(x)\widetilde{u}_j(x) W(x)\, dx = \int\limits_0^\infty \overline{u}_i(e^x)\overline{u}_j(e^x) w(e^x) e^x\, dx = \delta_{ij},$$

$$\int\limits_0^\infty \widetilde{v}_i(x)\widetilde{v}_j(x) V(x)\, dx = \int\limits_1^\infty \overline{v}_i(y)\overline{v}_j(y) v(y)\, dy = \delta_{ij},$$

where δ_{ij} is Kronecker's symbol. □

Lemmas 2 and 5 yield the following statements:

Theorem 4. *Let* $\alpha > 0$, $\beta > -1$, $w(x) = \ln^{-\beta} x$, $v(x) = x^{-2} \ln^{-(\alpha+\beta)} x$. *Then the operator* $H_\alpha : L^2_w(1,\infty) \to L^2_v(1,\infty)$ *has a singular system* $\{s_n(H_\alpha), \widetilde{u}_n, \widetilde{v}_n\}_{n\in Z_+}$, *where* $s_n(H_\alpha) = s_n(R_\alpha)$ ($s_m(R_\alpha)$ *is defined by* (1)),

$$\widetilde{u}_n(x) = x^{-1}\left(\frac{n!}{\Gamma(n+\beta+1)}\right)^{1/2} L_n^{(\beta)}(\ln x) \ln^\beta x,$$

$$\widetilde{v}_n(x) = \left(\frac{n!}{\Gamma(n+\alpha+\beta+1)}\right)^{1/2} L_n^{(\alpha+\beta)}(\ln x) \ln^{\alpha+\beta} x,$$

and $L_n^{(\gamma)}$ *is the Laguerre polynomial. Moreover,*

$$\lim_{n\to\infty} s_n(H_\alpha)/n^{-\alpha/2} = 1.$$

Theorem 5. *Let $\lambda > \alpha - \frac{1}{2}$, $\lambda \neq 0$. Then the operator $H_\alpha : L^2_w(1,\infty) \to L^2_v(1,\infty)$ has a singular system $\{s_n(H_\alpha), \widetilde{u}_m, \widetilde{v}_n\}_{m \in Z_+}$, where $v(x) = x^{-1} \ln^{1/2-\lambda-\alpha} x$, $w(x) = (1+\ln x)^{2\alpha} x \ln^{1/2-\lambda} x$, $s_n(H_\alpha) = s_n(R_\alpha)$ ($s_n(R_\alpha)$ is defined by (2)),*

$$\widetilde{u}_n(x) = 2^\lambda a_n (1+\ln x)^{-\lambda-\alpha-1/2} C_n^\lambda\Big(\frac{1-\ln x}{1+\ln x}\Big) x^{-1} \ln^{\lambda-1/2} x,$$

$$v_n(x) = 2^\lambda b_n (1+\ln x)^{-\lambda-\alpha-3/2} P_n^{(\lambda-\alpha-1/2, \lambda+\alpha-1/2)}\Big(\frac{1-\ln x}{1+\ln x}\Big) \ln^{\lambda+\alpha-1/2} x.$$

Moreover,

$$\lim_{n\to\infty} s_n(H_\alpha)/n^{-\alpha} = 1.$$

Definition 1. Let X and Y be Banach spaces and let T be a bounded linear map from X to Y. Then for all $k \in N$, the k^{th} entropy number $e_k(T)$ of T is defined by

$$e_k(T) = \inf\Big\{\varepsilon > 0 : \ T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} (b_i + \varepsilon U_Y) \ \text{for some} \ b_1, \ldots, b_{2^{k-1}} \in Y\Big\},$$

where U_X and U_Y are the closed unit balls in X and Y, respectively.

It is easy to verify that $\|T\| = e_1(T) \geq e_2(T) \geq \cdots \geq 0$.

For other properties of the entropy numbers see, e.g., [16].

It is known (see, e.g., [15]), that if T is a compact linear map of a Hilbert space X into a Hilbert space Y, then $s_n(T) \approx n^{-\lambda}$ if and only if $e_n(T) \approx n^{-\lambda}$. Hence we can get asymptotics of the entropy numbers for the operators $I_{\alpha,\sigma}$ and H_α. In particular, Theorems 1, 2 and 3 yield

Proposition 1. *Let $\alpha > 0$ and $\sigma > 0$. Then the following statements are valid:*

(a) *If $0 \leq \gamma < \alpha$, then the asymptotic formula*

$$e_n(I_{\alpha,\sigma}) \approx n^{-\alpha} \tag{3}$$

holds for the operator $I_{\alpha,\sigma} : L^2_{x^{1-\sigma}}(0,1) \to L^2_{x^{\sigma-1-\gamma\sigma}}(0,1)$.

(b) *Assume that $\lambda > \alpha - 1/2$ and $\lambda \neq 0$. Then the asymptotic formula* (3) *is valid for the map $I_{\alpha,\sigma} : L^2_w(0,\infty) \to L^2_v(0,\infty)$, where $w(x) = x^{-\sigma/2-\sigma\lambda+1}(1+x^\sigma)^{2\alpha}$ and $v(x) = x^{3\sigma/2-\sigma\lambda-\sigma\alpha-1}$.*

(c) *For the entropy numbers $e_n(I_{\alpha,\sigma})$ of the operator $I_{\alpha,\sigma} : L^2_w(0,\infty) \to L^2_v(0,\infty)$ ($w(y) = y^{-\sigma\beta-\sigma+1} e^{-y^\sigma}$, $v(y) = y^{-\sigma(\alpha+\beta)+\sigma-1} e^{-y^\sigma}$, $\beta > -1$) we have*

$$e_n(I_{\alpha,\sigma}) \approx n^{-\alpha/2}.$$

Let $T : L^2_w \to L^2_v$ be a compact linear operator. We shall denote by $n(t,T)$ the distribution function of singular values for the operator T, i.e.,

$$n(t,T) \equiv \sharp\Big\{k : \ s_k(T) > t\Big\}.$$

Theorem 6. *Let $\alpha > 1/2$ and $\sigma > 0$. Assume that v is a measurable a.e. positive function of $(0, \infty)$ satisfying the condition*

$$\sum_{k\in\mathbb{Z}} \Big(\int_{2^{k/\sigma}}^{2^{(k+1)/\sigma}} v(y) y^{(2\alpha-1)\sigma}\, dy \Big)^{1/(2\alpha)} < \infty. \tag{4}$$

Then for the operator $I_{\alpha,\sigma} : L^2_w(R_+) \to L^2_v(R_+)$, where $w(x) = x^{1-\sigma}$, the asymptotic formula

$$\lim_{t\to 0} t^{1/\alpha} n(t, I_{\alpha,\sigma}) = \frac{\sigma^{-1/\alpha+1}}{\pi} \int_0^\infty v^{1/(2\alpha)}(y) y^{(1-\sigma)(1/(2\alpha)-1)}\, dy$$

holds.

Proof. Condition (4) implies that

$$\sum_{k\in\mathbb{Z}} \Big(\int_{2^k}^{2^{k+1}} \overline{v}^2(y) y^{2\alpha-1}\, dy \Big)^{1/(2\alpha)} < \infty, \tag{5}$$

where $\overline{v}(x) \equiv [v(x^{1/\sigma})x^{1/\sigma-1}]^{1/2}$. By virtue of Theorem 1 from [9] we have that for the operator $R_{\alpha,\overline{v}} : L^2(R_+) \to L^2(R_+)$, where $R_{\alpha,\overline{v}} f(x) \equiv \overline{v}(x) R_\alpha f(x)$, the asymptotic formula

$$\lim_{t\to 0} t^{1/\alpha} n(t, R_{\alpha,\overline{v}}) = \pi^{-1} \int_{R_+} \overline{v}^{1/\alpha}(x)\, dx$$

holds. Further, using Lemmas 1, 2 and 3 we obtain that $s_k(R_{\alpha,\overline{v}}) = \sigma \cdot s_k(I_{\alpha,\sigma})$. Consequently,

$$\lim_{t\to 0} t^{1/\alpha} n(t, I_{\alpha,\sigma}) = \sigma^{-1/\alpha} \lim_{t\to 0} t^{1/\alpha} n(t, R_{\alpha,\overline{v}})$$
$$= \sigma^{-1/\alpha} \frac{1}{\pi} \int_0^\infty (\overline{v}(x))^{1/\alpha}\, dx = \frac{\sigma^{-1/\alpha+1}}{\pi} \int_0^\infty (v(y))^{1/(2\alpha)} y^{(1-\sigma)(1/(2\alpha)-1)}\, dy. \qquad \square$$

Theorem 7. *Let $\alpha > 1/2$ and $\sigma > 0$. Suppose that v is a measurable a.e. positive function on $(0, 1)$ satisfying the condition*

$$\sum_{k\in\mathbb{Z}} \Big(\int_{a_k}^{a_{k+1}} v(x) x^{-\sigma+2\alpha\sigma} (1 - x^\sigma)^{-1}\, dx \Big)^{1/(2\alpha)} < \infty, \quad a_k = (2^k/(2^k + 1))^{1/\sigma}. \tag{6}$$

Then for the operator $I_{\alpha,\sigma}$ acting from $L^2_w(0, 1)$ into $L^2_v(0, 1)$, where $w(x) = (1 - x^\sigma)^{2\alpha} x^{1-\sigma}$, we have

$$\lim_{t\to 0} t^{1/\alpha} n(t, I_{\alpha,\sigma}) = \frac{\sigma^{-1/\alpha+1}}{\pi} \int_0^1 v^{1/(2\alpha)}(x) x^{(1-\sigma)(1/(2\alpha)-1)} (1 - x^\sigma)^{-1}\, dx.$$

Proof. Using Lemmas 1–4 we have that $s_n(I_{\alpha,\sigma}) = 1/\sigma s_n(R_\alpha)$, where R_α is the Riemann–Liouville operator acting from $L^2_{w_1}(0,1)$ into $L^2_{v_1}(0,1)$, with

$$w_1(x) = w(x^{1/\sigma})x^{1-1/\sigma}, \quad v_1(x) = v(x^{1/\sigma})x^{1/\sigma-1}.$$

Further, by the change of variable $x = y/(1-y)$ we obtain that the operator $\overline{R}_\alpha : L^2_{w_2}(R_+) \to L^2_{v_2}(R_+)$ has singular numbers $s_n(\overline{R}_\alpha) = \sigma s_n(I_{\alpha,\sigma})$, where $w_2(x) = w_1(x/(x+1))(x+1)^{-2}$, $v_2(x) = v_1(x/(x+1))(x+1)^{-2}$ and $\overline{R}_\alpha f(x) = \psi(x)R_\alpha(f\varphi)(x)$ with $\psi(x) = (x+1)^{-\alpha+1}$, $\varphi(x) = (x+1)^{-1-\alpha}$. Hence for the singular numbers of the Riemann–Liouville operator $R_\alpha : L^2_{w_3}(R_+) \to L^2_{v_3}(R_+)$ we derive $s_n(R_\alpha) = \sigma s_n(I_{\alpha,\sigma})$, where $w_3(x) = w_2(x)(x+1)^{2\alpha+2} = 1$ and $v_3(x) = v_2(x)(x+1)^{2-2\alpha}$. Further, condition (6) implies (5) with v_3 instead of v. Thus, taking into account Theorem 1 from [9], we arrive at

$$\lim_{t\to 0} t^{1/\alpha} n(t, I_{\alpha,\sigma}) = \sigma^{-1/\alpha} \lim_{t\to 0} t^{1/\alpha} n(t, R_\alpha)$$

$$= \sigma^{-1/\alpha}\frac{1}{\pi}\int_0^\infty v_4^{1/\alpha}(x)\,dx = \frac{\sigma^{-1/\alpha+1}}{\pi}\int_0^1 (v(y))^{1/(2\alpha)} y^{(1-\sigma)(1/(2\alpha)-1)}(1-y^\sigma)^{-1}\,dy.$$

In the last equality we used the change of variable twice. □

Finally, we have

Theorem 8. *Let $\alpha > 1/2$ and let v be a measurable a.e. positive function on $(1,\infty)$ satisfying the condition*

$$\sum_{k\in\mathbb{Z}} \Bigg(\int_{a_k}^{a_{k+1}} v(x)\ln^{2\alpha-1} x\,dx\Bigg)^{1/(2\alpha)} < \infty, \quad a_k = e^{2^k}. \tag{7}$$

Then for the operator $H_\alpha : L^2_w(1,\infty) \to L^2_v(1,\infty)$, where $w(x) = e^x$, the asymptotic formula

$$\lim_{t\to 0} t^{1/\alpha} n(t, H_{\alpha,\sigma}) = \frac{1}{\pi}\int_1^\infty v^{1/(2\alpha)}(x)x^{1/(2\alpha)-1}\,dy \tag{8}$$

holds.

Proof. Taking into account Lemmas 2 and 5 we obtain that $s_n(R_\alpha) = s_n(H_\alpha)$, where R_α is the Riemann–Liouville operator acting from $L^2(R_+)$ into $L^2_{v_1}(R_+)$, $v_1(x) = v(e^x)e^x$. By condition (7), Theorem 1 from [9] and the change of variable $x = e^y$ we conclude that (8) holds. □

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