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ON NORMAL APPROXIMATION OF LARGE PRODUCTS OF FUNCTIONS: A REFINEMENT OF BLACKWELL'S RESULT

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Abstract. An asymptotic expansion for the approximation of standardized products of large numbers of smooth positive functions by $\exp(-x^2/2)$ is given. This result is closely related to the Bernstein–von Mises theorem on the normal approximation of posterior distributions.

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1. Introduction

In [1] D. Blackwell, using a simple elementary way, proved that under some conditions standardized products of large numbers of smooth positive functions are close to $\exp(-x^2/2)$. Moreover, he gave an estimation of this closeness. In the present note we observe that, under additional smoothness conditions, a little more effort provides an asymptotic expansion making this closeness still more precise.

As was pointed out by D. Blackwell, the result he proved is closely related to the Bernstein–von Mises theorem stating the approximability of posterior distributions by normal ones. More precisely, he noted that if one considers the product of smooth positive functions as a likelihood function, one obtains that properly normalized it is often nearly normal, but under wide conditions the likelihood function is close to the posterior density of the parameter. The Bernstein–von Mises theorem was deeply studied by Le Cam [5], [6], Le Cam and Yang [7] (see Historical Remarks therein), De Groot [2] and others, with refinements made by Lindley [8], Johnson [4], Ghost et. al. [3].

2. Approximation of Sums

Let F be a class of real functions defined on an interval (a,b). Assume that the functions in F are $t+3$ times continuously differentiable, where t is an integer ≥ 0 and there are positive constants M_i, $i=3,\dots,t+3$, and m such that

$$\sup\Big\{|f^{(k)}(x)|:\ x\in(a,b),\ f\in F\Big\}\leq M_k,\quad k=3,\dots,t+3,$$
$$\sup\Big\{f^{(2)}(x):\ x\in(a,b),\ f\in F\Big\}\leq -m. \tag{1}$$

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Let a sum S of n functions from F satisfy $S^{(1)}(x_0) = 0$ at a point $x_0 \in (a + d, b - d)$, $d > 0$. Denote $\sigma^2 = -1/S^{(2)}(x_0)$. Then, by the Taylor formula

$$S(x_0 + \sigma y) = S(x_0) - \frac{y^2}{2} + \sum_{k=3}^{t+2} \frac{1}{k!} S^{(k)}(x_0)(\sigma y)^k + \frac{1}{(t+3)!} S^{(t+3)}(x_0 + \theta\sigma y)(\sigma y)^{t+3}, \tag{2}$$

where $0 < \theta < 1$. Obviously condition (1) implies

$$\sigma \le (mn)^{-1/2}, \quad |S^{(k)}(x_0)\sigma^k| \le M_k m^{-k/2} n^{1-k/2}, \quad k = 3, \dots, t+3. \tag{3}$$

Moreover, if y is bounded, say $|y| \le A$, then

$$x_0 + \sigma\theta y \in (a, b), \tag{4}$$

if $\sigma|y| \le (mn)^{-1/2} A < d$, i.e., if

$$n > A^2 m^{-1} d^{-2}. \tag{5}$$

When (4) is satisfied we have as in (3)

$$|S^{(t+3)}(x_0 + \theta\sigma y)\sigma^{t+3}| \le M_{t+3} m^{-(t+3)/2} n^{-(t+1)/2}. \tag{6}$$

Thus we have proved the following

Proposition. *If a sum S of n functions from F has a maximum at a point $x_0 \in (a + d, b - d)$, $d > 0$, and condition* (5) *is satisfied, then expansion* (2) *is true with the terms in it satisfying* (3), (6).

3. Approximation of Products

Now we will use the above proposition to construct an approximation of products.

Consider a product $Q = \prod_{i=1}^{n} g_i$ of n positive functions defined on (a, b), and assume that $Q^{(1)}(x_0) = 0$, where $x_0 \in (a + d, b - d)$, $d > 0$, and that $f_i = \log g_i$, $i = 1, \dots, n$, satisfy condition (1). Furthermore, let

$$S = \log Q, \quad \sigma^2 = -Q(x_0)/Q^{(2)}(x_0) = -1/S^{(2)}(x_0)$$

and denote

$$Q_S = Q(x_0 + \sigma y)/Q(x_0)$$

the standardized form of Q. Then for $|y| \le A$, if condition (5) is satisfied, the above proposition implies

$$\begin{aligned} Q_S(y) &= \exp\Big(\log Q(x_0 + \sigma y) - \log Q(x_0)\Big) \\ &= \exp\Big(S(x_0 + \sigma y) - S(x_0)\Big) = \exp(-y^2/2) I, \end{aligned}$$

where

$$I = \exp\left(\sum_{k=3}^{t+2} \frac{1}{k!} S^{(k)}(x_0)(\sigma y)^k + \frac{1}{(t+3)!} S^{(t+3)}(x_0 + \theta\sigma y)(\sigma y)^{t+3} \right). \tag{7}$$

Next, using the expansion $e^u = 1 + u + \cdots + \frac{u^t}{t!} + \frac{u^{t+1}}{(t+1)!} e^{\lambda u}$, $0 < \lambda < 1$, we can write

$$I = 1 + \sum_{s=3}^{t+2} \sum_{r=1}^{s} \sum{}' (k_1! \dots k_r!)^{-1} S^{(k_1)}(x_0) \dots S^{(k_r)}(x_0)(\sigma y)^{2r+s} + R,$$

where the summation $\sum'$ is over all integers $k_1, \dots, k_r$ such that $k_j \geq 3$, $j = 1, \dots, r$, $k_1 + \cdots k_r = 2r + s$. By (1) we have (3), and hence

$$\left| S^{(k_1)}(x_0) \dots S^{(k_r)}(x_0) \sigma^{2r+3} \right| \leq M_{k_1} \cdots M_{k_r} m^{-(2r+s)/2} n^{-s/2}.$$

It is straightforward to check that when condition (5) is also satisfied

$$|R| \leq c(t, m, M_3, \dots, M_{t+3}, A) n^{-(t+1)/2}.$$

4. Examples

In the examples that follow we take for simplicity $t = 2$.

a. Let $0 < a < b < 1$ and consider on (a, b) the product $Q(x) = x^l (1-x)^{n-l}$, where n is an integer and $0 \leq l \leq n$; Q is a product of n functions each equal to x or $1 - x$.

Considered on $(0,1)$, the function Q attains its maximum at $x_0 = l/n$. We will assume that $a + d \leq l/n \leq b - d$ for some $d > 0$. Now an elementary computation shows that for the derivatives of $S(x) = \log Q(x) = l \log x + (n - l) \log(1 - x)$ we have

$$S^{(2)}(x_0) = -1/\sigma^2 = -\frac{n^3}{l(n-l)},$$

$$S^{(3)}(x_0) |S^{(2)}(x_0)|^{-3/2} = \frac{2(n-2l)}{(l(n-l)n)^{1/2}},$$

$$S^{(4)}(x_0)(S^{(2)}(x_0))^{-2} = -6\, \frac{3l^2 - 3ln + n^2}{l(n-l)n}.$$

Thus in the case we consider

$$\frac{Q(x_0 + \sigma y)}{Q(x_0)} = \exp(-y^2/2)\Sigma + R, \tag{8}$$

where

$$\Sigma = 1 + \frac{1}{3} \frac{n - 2l}{(l(n-l)n)^{1/2}} y^3 - \frac{1}{4} \frac{3l^2 - 3ln + n^2}{l(n-l)n} y^4 + \frac{1}{16} \frac{(n-2l)^2}{l(n-l)n} y^6,$$

and when $|y| \leq A$ and (5) is satisfied

$$|R| \leq c n^{-3/2} \tag{9}$$

with c depending only on a, b, A.

b. Next consider $Q(x) = x^n e^{-x}$, $x > 0$. As is pointed out in [1], the case of this function reduces to the case of $Q_1(x) = (xe^{-x})^n$ since, as is easy to check, they have the same standardized forms:

$$Q_S(x) = Q_{1S}(x) = (1 + n^{-1/2}y)^n \exp(-n^{1/2}y).$$

For Q_1 we have $S_1 = \log Q_1 = nf$, $f = \log x - x$, and $f^{(1)}(x) = (1/x) - 1$, and $f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k}$, $k \geq 2$. Thus $x_0 = 1$, $\sigma = 1$ and the general result of Secion 2 is applicable with $0 < a < 1 < b < \infty$, $d = \min(|a-1|, |b-1|)$, $m = b^{-2}$. Namely, we have

$$Q_S(y) = Q_{1S}(y) = \exp(-y^2/2)\Big(1 + \frac{1}{3}\,y^3 n^{-1/2} - \Big(\frac{1}{4}\,y^4 - \frac{1}{16}\,y^6\Big)n^{-1}\Big) + R,$$

and when $|y| \leq A$ and condition (5) is satisfied $|R| \leq cn^{-3/2}$ with c depending only on a, b, A.

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