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# TOPOLOGICAL SPACES WITH THE STRONG SKOROKHOD PROPERTY

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*We dedicate our work to the 70th birthday of Professor Vakhania whose fruitful research over the last 40 years has considerably influenced the theory of infinite dimensional probability distributions and whose classical books on the subject have become an important source of reference and inspiration.*

**Abstract**. We study topological spaces with the strong Skorokhod property, i.e., spaces on which all Radon probability measures can be simultaneously represented as images of Lebesgue measure on the unit interval under certain Borel mappings so that weakly convergent sequences of measures correspond to almost everywhere convergent sequences of mappings. We construct non-metrizable spaces with such a property and investigate the relations between the Skorokhod and Prokhorov properties. It is also shown that a dyadic compact has the strong Skorokhod property precisely when it is metrizable.

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## Introduction

According to a celebrated result of A. V. Skorokhod [28], for every sequence of Borel probability measures $\mu_n$ on a complete separable metric space $X$ that is weakly convergent to a Borel probability measure $\mu_0$, one can find Borel mappings $\xi_n\colon [0,1] \to X$, $n = 0, 1, \dots$, such that $\lim\limits_{n\to\infty} \xi_n(t) = \xi_0(t)$ for almost all $t \in [0,1]$ and the image of Lebesgue measure $\lambda$ under $\xi_n$ is $\mu_n$ for every $n \geq 0$. Various extensions of this result have been found since then (see, e.g., [3], [5], [6], [10], [13], [18], [26], and the references therein). The most important for us is the extension discovered independently by Blackwell and Dubbins [3] and Fernique [13], according to which all Borel probability measures on $X$ can be parameterized simultaneously by the mappings from $[0,1]$ with the preservation of the above correspondence. More precisely, with every Borel probability measure $\mu$ on $X$ one can associate a Borel mapping $\xi_\mu\colon [0,1] \to X$ such that the image under $\xi_\mu$ of Lebesgue measure equals $\mu$, and if measures $\mu_n$ converge weakly to $\mu$, then $\lim\limits_{n\to\infty} \xi_{\mu_n}(t) = \xi_\mu(t)$ for almost all $t \in [0,1]$. It has been recently shown in [5] that this result can be derived from its simple one-dimensional case and

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certain deep topological selection theorems. In addition, it has been shown in [5] that there are other interesting links between the Skorokhod parameterization of probability measures on topological spaces and topological properties of those spaces. The principal concept in this paper is a space with the strong Skorokhod property for Radon measures defined as a space on which all Radon probability measures admit a simultaneous parameterization $\mu \mapsto \xi_\mu$ by Borel mappings from $[0,1]$ endowed with Lebesgue measure such that one obtains the above mentioned correspondence between weak convergence of measures and almost everywhere convergence of mappings.

In this work, we study the strong Skorokhod property in the nonmetrizable case. In particular, we construct a countable nonmetrizable topological space with the strong Skorokhod property (under the continuum hypothesis, we find even a countable topological group with this property) and prove a theorem which enables one to construct broad classes of spaces with the strong Skorokhod property. A new class of spaces, called almost metrizable, is introduced, and it is shown that an almost metrizable space has the strong Skorokhod property precisely when it is sequentially Prokhorov. Some examples of nonmetrizable compact spaces with or without the strong Skorokhod property are considered, and it is shown that a dyadic compact space with the strong Skorokhod property is metrizable. A large number of open problems are posed.

## 1. Notation and Terminology

We assume throughout the paper that $X$ is a Tychonoff (i.e., completely regular) topological space. Let $C_b(X)$ be the space of all bounded continuous functions on $X$ and let $\mathcal{B}(X)$ be the Borel $\sigma$-field of $X$. The symbol $\mathcal{P}(X)$ denotes the space of all Borel probability measures on $X$. Let $\mathcal{P}_0(X)$ and $\mathcal{P}_r(X)$ denote, respectively, the spaces of all Baire and Radon (i.e., inner compact regular) probability measures on $X$. A probability measure $\mu$ on a space $X$ is called *discrete* if $\mu(X \setminus C) = 0$ for some countable subset $C \subset X$. The Dirac measure at $x$ is denoted by $\delta_x$.

The weak topology on $\mathcal{P}(X)$, $\mathcal{P}_r(X)$ or $\mathcal{P}_0(X)$ is the restriction of the weak topology on the linear space of all bounded Borel (or Baire) measures that is generated by the seminorms

$$p_f(\mu) = \left| \int_X f(x)\,\mu(dx) \right|, \quad f \in C_b(X).$$

Thus, a sequence of measures $\mu_n$ converges weakly to a measure $\mu$ precisely when

$$\lim_{n\to\infty} \int_X f(x)\,\mu_n(dx) = \int_X f(x)\,\mu(dx), \quad \forall\, f \in C_b(X).$$

It is well known that the weak topology is generated by the base of sets $W(\mu, U, a) := \{\nu \in \mathcal{P}_r(X)\colon \nu(U) > \mu(U) - a\}$, where $U$ is open in $X$ and $a > 0$. Weak convergence is denoted by $\mu_n \Rightarrow \mu$. Recall that the weak topology is Hausdorff on $\mathcal{P}_0(X)$ and $\mathcal{P}_r(X)$ and that $\mathcal{P}(X) = \mathcal{P}_0(X) = \mathcal{P}_r(X)$ for any completely regular Souslin space $X$. See [2] and [30] for additional information about weak convergence of probability measures.

If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $f\colon X \to Y$ is a measurable mapping, then the image of a measure $\mu$ on $X$ under the mapping $f$ is denoted by $\mu \circ f^{-1}$ and defined by the formula

$$\mu \circ f^{-1}(B) = \mu\Big(f^{-1}(B)\Big), \quad B \in \mathcal{B}.$$

We recall that a family $\mathcal{M}$ of nonnegative Borel measures on a topological space $X$ is called uniformly tight if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset X$ such that $\mu(X \backslash K_\varepsilon) < \varepsilon$ for all $\mu \in \mathcal{M}$.

We shall call a topological space $X$ *sequentially Prokhorov* if every sequence of Radon probability measures on $X$ that converges weakly to a Radon measure is uniformly tight.

Let us denote by $\mathbb{R}_0^\infty$ the space of all real sequences of the form $(x_1, x_2, \dots, x_n, 0, 0, \dots)$.

**Definition 1.1.** (i) We shall say that a family $\mathcal{M}$ of Borel probability measures on a topological space $X$ has the strong Skorokhod property if, with every measure $\mu \in \mathcal{M}$, one can associate a Borel mapping $\xi_\mu\colon [0,1] \to X$ with $\lambda \circ \xi_\mu^{-1} = \mu$, where $\lambda$ is Lebesgue measure, such that if a sequence of measures $\mu_n \in \mathcal{M}$ converges weakly to a measure $\mu \in \mathcal{M}$, then

$$\lim_{n\to\infty} \xi_{\mu_n}(t) = \xi_\mu(t) \quad \text{for almost all} \quad t \in [0,1]. \tag{1.1}$$

If (1.1) holds under the additional assumption that $\{\mu_n\}$ is uniformly tight, then $\mathcal{M}$ is said to have the uniformly tight strong Skorokhod property.

(ii) We shall say that a topological space $X$ has the strong Skorokhod property for Radon measures if the family $\mathcal{P}_r(X)$ of all Radon measures has that property. If the family of all discrete probability measures on $X$ has the strong Skorokhod property, then $X$ is said to have that property for discrete measures.

The *uniformly tight Skorokhod property* for $X$ is defined analogously. In a similar manner we define also the strong and uniformly tight strong Skorokhod properties for probability measures with finite supports and two-point supports.

We shall use the terms *Skorokhod parameterization* and *Skorokhod representation* for mappings $\mu \mapsto \xi_\mu$ of the type described in this definition.

It is clear that a sequentially Prokhorov space has the strong Skorokhod property for Radon measures if and only if it has the uniformly tight strong Skorokhod property for Radon measures.

An advantage of dealing with Radon measures is that the strong Skorokhod property for them is inherited by arbitrary subspaces (see [5, Lemma 3.1]).

It has been proved in [5] that every metrizable space has the strong Skorokhod property for Radon measures. On the other hand, there exist non-metrizable Souslin spaces that fail to have the strong Skorokhod property for Radon measures. In particular, according to [5], the space $\mathbb{R}_0^\infty$ of all finite real sequences with its natural topology of the inductive limit of the spaces $\mathbb{R}^n$ does not even the uniformly tight strong Skorokhod property; moreover, one can find a weakly convergent uniformly tight sequence of probability measures on $\mathbb{R}_0^\infty$ that does not admit a Skorokhod parameterization by mappings. It is worth noting that $\mathbb{R}_0^\infty$ has the weak Skorokhod property, i.e., admits a parameterization of all probability measures such that every weakly convergent uniformly tight sequence has a subsequence satisfying (1.1). This kind of Skorokhod's property motivated by [18] has been considered in [5] and will be the subject of a separate paper of the authors.

## 2. The strong Skorokhod Property of Almost Metrizable Spaces

It has been proved in [5] that the class of topological spaces with the strong Skorokhod property for Radon measures includes all metrizable spaces. In this section, we show that this class is even wider and includes all almost metrizable sequentially Prokhorov spaces. In particular, we construct a class of nonmetrizable topological spaces with the strong Skorokhod property for Radon measures. Our simplest example is a countable set which is the set of natural numbers with an extra point from its Stone–Čech compactification.

First we show that the uniformly tight strong Skorokhod property for Radon measures is preserved by bijective continuous proper mappings. In particular, the strong Skorokhod property for Radon measures is preserved by bijective continuous proper mappings onto sequentially Prokhorov spaces. We recall that a mapping $f\colon X \to Y$ between topological spaces is called *proper* if $f^{-1}(K)$ is compact for every compact subspace $K \subset Y$.

**Theorem 2.1.** *Let $X$ and $Y$ be two topological spaces such that there exists a bijective continuous proper mapping $F\colon X \to Y$. Assume that $X$ has the uniformly tight strong Skorokhod property for Radon measures. Then $Y$ possesses this property as well. In particular, if $Y$ is sequentially Prokhorov, then $Y$ has the strong Skorokhod property for Radon measures.*

*Proof.* Since $F$ is proper and continuous, for every Radon probability measure $\mu$ on $Y$ there exists a unique Radon probability measure $\widehat{\mu}$ on $X$ such that $\widehat{\mu} \circ F^{-1} = \mu$ (see, e.g., [4, §6.1]). Let us take a Skorokhod parameterization $\nu \mapsto \xi_\nu$ of Radon probability measures on $X$ by Borel mappings from $[0,1]$, which exists by our hypothesis. Then $\mu \mapsto F\circ\xi_{\widehat{\mu}}$ is the desired parameterization on $Y$. Indeed, assume that Radon probability measures $\mu_n$ converge weakly to a Radon measure $\mu$ on $Y$ and that the sequence $\{\mu_n\}$ is uniformly tight. Since $F$ is proper, the sequence of measures $\widehat{\mu_n}$ is uniformly tight as well. In addition, for every compact set $Q \subset X$, one has

$$\limsup_{n\to\infty} \widehat{\mu_n}(Q) = \limsup_{n\to\infty} \mu_n\Bigl(F(Q)\Bigr) \le \mu\Bigl(F(Q)\Bigr) = \widehat{\mu}(Q).$$

Together with the uniform tightness this yields that $\limsup\limits_{n\to\infty} \widehat{\mu_n}(Z) \le \widehat{\mu}(Z)$ for every closed set $Z \subset X$, which shows that $\widehat{\mu_n} \Rightarrow \widehat{\mu}$. Therefore, $\xi_{\widehat{\mu_n}}(t) \to \xi_{\widehat{\mu}}(t)$ for almost every $t \in [0,1]$. For such $t$, we also have $\xi_{\mu_n}(t) \to \xi_\mu(t)$ by the continuity of $F$. $\square$

**Remark 2.2.** It follows from the above proof that for a bijective continuous proper mapping $f\colon X \to Y$ and a uniformly tight weakly convergent sequence $\mu_n \Rightarrow \mu$ of probability Radon measures on $Y$, the sequence of measures $\mu_n \circ f^{-1}$ on $X$ is weakly convergent to $\mu \circ f^{-1}$.

We define a topological space $X$ to be *almost metrizable* if there exists a bijective continuous proper mapping $f\colon M \to X$ from a metrizable space $M$. If $M$ is discrete, then $X$ is called *almost discrete*.

One can easily show by examples that an almost metrizable space may not be metrizable (such examples are given below). On the other hand, each almost metrizable $k$-space is metrizable. We recall that a topological space $X$ is a $k$-*space* if a subset $U \subset X$ is open in $X$ precisely when $U \cap K$ is open in $K$ for every compact subset $K \subset X$ (see [11]).

One can readily show that almost metrizable spaces and almost discrete spaces have the following properties.

**Proposition 2.3.** (i) *Any subspace of an almost metrizable space is almost metrizable.*

(ii) *A topological space is metrizable if and only if it is an almost metrizable $k$-space.*

(iii) *A topological space $X$ is almost metrizable if and only if the strongest topology inducing the original topology on each compact subset of $X$ is metrizable.*

(iv) *A topological space is almost discrete if and only if it contains no infinite compact subspaces.*

(v) *A countable product of almost metrizable spaces is almost metrizable.*

(vi*) The classes of almost metrizable and almost discrete spaces are stable under formation of arbitrary topological sums.*

(vii) *The images of almost metrizable and almost discrete spaces under continuous bijective proper mappings belong to the respective classes.*

The following theorem characterizes almost metrizable spaces possessing the strong Skorokhod property for Radon measures.

**Theorem 2.4.** *Any almost metrizable space has the uniformly tight strong Skorokhod property for Radon measures. Moreover, an almost metrizable space has the strong Skorokhod property for Radon measures if and only if it is sequentially Prokhorov.*

*Proof.* If a space $X$ is almost metrizable and sequentially Prokhorov, then it has the strong Skorokhod property by Theorem 2.1. The same reasoning proves also the first claim. Suppose now that $X$ is almost metrizable and has the strong Skorokhod property. Let Radon probability measures $\mu_n$ converge weakly to a Radon measure $\mu$ on $X$. We take Borel mappings $\xi_{\mu_n}\colon [0,1] \to X$ which converge almost everywhere to a Borel mapping $\xi_\mu\colon [0,1] \to X$ such that they transform Lebesgue measure $\lambda$ on $[0,1]$ to the measures $\mu_n$ and $\mu$, respectively. Then we fix a metric space $M$ that admits a proper bijective continuous mapping

$F$ onto $X$. As noted in the proof of Theorem 2.1, there exist unique Radon probability measures $\widehat{\mu_n}$ and $\widehat{\mu}$ whose images under $F$ are $\mu_n$ and $\mu$, respectively. Since the preimages under $F$ of all compact sets in $X$ are compact in $M$, it is readily seen that $G_n(t) := F^{-1}\xi_{\mu_n}(t) \to G(t) := F^{-1}\xi_\mu(t)$ in $M$ for every point $t$ at which $\xi_{\mu_n}(t) \to \xi_\mu(t)$. We observe that $\widehat{\mu_n} = \lambda \circ G_n^{-1}$ and $\widehat{\mu} = \lambda \circ G^{-1}$, since $(\lambda \circ G_n^{-1}) \circ F^{-1} = \mu_n$, $(\lambda \circ G^{-1}) \circ F^{-1} = \mu$ and the measures $\mu_n$ and $\mu$ have unique preimages under $F$. This shows that $\widehat{\mu_n} \Rightarrow \widehat{\mu}$. Taking into account that all the measures in question are Radon and $M$ is metrizable, we obtain by the Le Cam theorem (see, e.g., [2]) that the sequence $\widehat{\mu_n}$ is uniformly tight, hence $\{\mu_n\}$ is also. □

Since the countable product of sequentially Prokhorov spaces is also sequentially Prokhorov (see, e.g., [4, §8.3]), we arrive at the following statement.

**Corollary 2.5.** *The countable product of almost metrizable spaces with the strong Skorokhod property has the strong Skorokhod property.*

For almost discrete spaces we have even stronger results. We recall that a topological space $X$ is called *sequentially compact* if each sequence in $X$ contains a convergent subsequence, see [11, §3.10].

**Theorem 2.6.** *For a topological space $X$ the following conditions are equivalent:*

(i) *$X$ is an almost discrete space.*

(ii) *Each compact subset of $X$ is sequentially compact and each uniformly tight weakly convergent sequence $\mu_n \Rightarrow \mu$ of Radon probability measures on $X$ converges in the variation norm (equivalently, one has the convergence $\mu_n(x) \to \mu(x)$ for each $x \in X$).*

*Proof.* The implications (i) ⇒ (ii) follows from Remark 2.2, since every weakly convergent sequence of Radon measures on a discrete space converges in variation. To prove the reverse implication, assume that the space $X$ satisfies condition (ii). It suffices to prove that each compact subset $K$ of $X$ is finite. Suppose not. By the sequential compactness of $K$, find a nontrivial convergent sequence $x_n \to x_0$ in $K$. Then the sequence of Dirac's measures $\delta_{x_n}$ at the points $x_n$ is uniformly tight and converges weakly to Dirac's measure $\delta_{x_0}$. By our hypothesis, $\delta_{x_n}(x_0) \to \delta_{x_0}(x_0) = 1$. Then $x_n = x_0$ for all but finitely many numbers $n$. □

**Corollary 2.7.** *Let $X$ be an almost discrete sequentially Prokhorov space, let $E$ be a completely regular space, and let a sequence of Radon probability measures $\mu_n$ on $X \times E$ converge weakly to a Radon probability measure $\mu$. Then, for each $x \in X$, the restrictions of the measures $\mu_n$ to the set $x \times E$ converge weakly to the restriction of $\mu$, i.e., one has $\mu_n|_{x\times E} \Rightarrow \mu|_{x\times E}$.*

*Proof.* The projections $\eta_n$ of the measures $\mu_n$ to $X$ converge weakly to the projection $\eta$ of $\mu$. By Theorem 2.6, $\eta_n(x) \to \eta(x)$ and thus $\mu_n(x\times E) \to \mu(x\times E)$ for each $x \in X$. Since every set $x \times E$ is closed in $X \times E$, one has by the weak convergence on $X \times E$ that $\limsup\limits_{n\to\infty} \mu_n(Z) \le \mu(Z)$ for every closed subset of the space $x \times E$. Hence we obtain the desired weak convergence. □

Almost metrizable spaces need not be metrizable (we shall encounter below even countable almost metrizable nonmetrizable spaces). A somewhat unexpected example is the Banach space $l^1$ endowed with the weak topology. The space $l^1$ is known to have the Shur property. We recall that a Banach space $X$ has the *Shur property* if each weakly convergent sequence in $X$ is norm convergent.

**Theorem 2.8.** *Let $X$ be a Banach space with the Shur property and let $\tau$ be an intermediate topology between the norm and weak topologies on $X$. Then the space $(X, \tau)$ is almost metrizable and has the uniformly tight strong Skorokhod for Radon measures.*

*Proof.* According to Theorem 2.4 it suffices to prove that the identity map $X \to (X, \tau)$ is proper. To this end, let us fix a compact subset $K \subset (X, \tau)$. Then $K$ is weakly compact and by the Eberlein–Šmulyan theorem $K$ is sequentially compact in the weak topology. Assume that $K$ is not norm compact. Then $K$ contains a sequence $\{x_n\}_{n\in\mathbb{N}}$ without norm convergent subsequences. Since $K$ is sequentially compact in the weak topology, the sequence contains a weakly convergent subsequence, which contradicts the Shur property. □

It should be noted that, as shown in [14, p. 127], the space $l^1$ with the weak topology (as well as any infinite dimensional Banach space with the weak topology) is not sequentially Prokhorov.

We shall now construct an almost discrete space without the strong Skorokhod property.

**Example 2.9.** Let $X_n$, $n \in \mathbb{N}$, be pairwise disjoint finite sets in $\mathbb{N}$ with $Card\,(X_n) < Card\,(X_{n+1})$ for each $n$. Fix any point $\infty \notin \bigcup_{n\in\mathbb{N}} X_n$ and define a topology on the union $X = \{\infty\} \cup \bigcup_{n=1}^{\infty} X_n$ as follows. All points except for $\infty$ are isolated and the neighborhood base of a unique nonisolated point $\infty$ is formed by the sets $X \setminus F$, where $F \subset \bigcup_{n\in\mathbb{N}} X_n$ is a subset for which there is $m \in \mathbb{N}$ such that $Card\,(F \cap X_n) \le m$ for every $n$. It can be shown that the space $X$ is almost discrete and fails to have the strong Skorokhod property. To this end, it suffices to note that $X$ has no nontrivial convergent sequences. On the other hand, the sequence of measures $\mu_n$, where each $\mu_n$ is concentrated on $X_n$ and assigns equal values $[Card\,(X_n)]^{-1}$ to the points of $X_n$, converges weakly to Dirac's measure at $\infty$. Existence of a Skorokhod parameterization of a subsequence of $\mu_n$ would give a nontrivial convergent sequence. For the same reason, no subsequence in $\{\mu_n\}$ is uniformly tight (otherwise such a subsequence would be Skorokhod parameterizable by Remark 2.2).

Finally, we shall show that nonmetrizable almost metrizable spaces with the strong Skorokhod property do exist. Such spaces will be constructed as subspaces of extremally disconnected spaces. We recall that a topological space $X$

is said to be *extremally disconnected* if the closure $\overline{U}$ of any open subset $U$ of $X$ is open, see, e.g. [11]. A standard example of an extremally disconnected space is $\beta\mathbb{N}$, the Stone–Čech compactification of $\mathbb{N}$. More generally, the Stone–Čech compactification $\beta X$ of a Tychonoff space $X$ is extremally disconnected if and only if the space $X$ is extremally disconnected [11, §6.2].

**Theorem 2.10.** *Any countable subspace $X$ of an extremally disconnected Tychonoff space $K$ is almost discrete and has the strong Skorokhod property for Radon measures.*

*Proof.* Without loss of generality we may assume that $K$ is compact (otherwise we replace $K$ by its Stone–Čech compactification). Since extremally disconnected spaces contain no nontrivial convergent sequences, any countable subspace $X \subset K$ is almost discrete. According to Theorem 2.4, to show the strong Skorokhod property of the space $X$, it suffices to verify that $X$ is sequentially Prokhorov. So, suppose that a sequence of Radon probability measures $\mu_n$ on $X$ converges weakly to a Radon measure $\mu$. We shall show in fact that the weak convergence of countable sequences of probability measures on $X$ is equivalent to the convergence at every point of $X = \{x_n\}$, hence by the Scheffé theorem, to the convergence in the variation norm. So that instead of using Theorem 2.4 we could refer to the fact that $\mathbb{N}$ has the strong Skorokhod property. The measures $\mu_n$ regarded as measures on $K$ converge weakly to the measure $\mu$ on $K$. By a well known result of Grothendieck [15, Théorème 9], we have the convergence of $\{\mu_n\}$ to $\mu$ in the weak topology of the Banach space $\mathcal{M}(K)$ of all Radon measures on $K$ (where the norm is the variation norm). Therefore, the integrals of every bounded Borel function $f$ on $K$ against the measures $\mu_n$ converge to the integral of $f$ against $\mu$. This means that for every bounded real sequence $\{y_j\}$ one has $\lim_{n\to\infty} \sum_{j=1}^{\infty} y_j \mu_n(x_j) = \sum_{j=1}^{\infty} y_j \mu(x_j)$. It is well known (see, e.g., [8, p. 85]) that the sequence of vectors $v_n := \big(\mu_n(x_j)\big)_{j=1}^{\infty}$ is norm convergent to the vector $v = \big(\mu(x_j)\big)_{j=1}^{\infty}$ in the space $l^1$, which completes the proof. □

**Corollary 2.11.** *For every $p \in \beta\mathbb{N} \setminus \mathbb{N}$, the space $X = \{p\} \cup \mathbb{N}$ with the induced topology is a nonmetrizable almost discrete space with the strong Skorokhod property for Radon measures.*

A closer look at the proof of Theorem 2.10 reveals that it holds true for a wider class of spaces. We say that a Tychonoff space $X$ is a *Grothendieck space* if the space $C_b(X)$ with the sup-norm is a Grothendieck Banach space. We recall that a Banach space $E$ is said to be a Grothendieck Banach space if the $*$-weak convergence of countable sequences in $E^*$ is equivalent to the weak convergence (i.e., the convergence in the topology $\sigma(E^*, E^{**})$). According to the above cited Grothendieck theorem, each extremally disconnected Tychonoff space is Grothendieck (see also [9], [25], and [29] for a discussion and further generalizations of the Grothendieck theorem).

**Theorem 2.12.** *Any countable subspace $X$ of a Grothendieck space $K$ is almost discrete and has the strong Skorokhod property.*

*Proof.* Since Grothendieck spaces contain no nontrivial convergent sequences, any countable subspace $X \subset K$ is almost discrete. According to Theorem 2.4, in order to prove the strong Skorokhod property of the space $X$, it suffices to verify that $X$ is sequentially Prokhorov. Let a sequence of probability measures $\mu_n$ converge weakly to a measure $\mu$ on $X$. The measures $\mu_n$ can be considered as elements of the dual space $C_b^*(K)$ to the Banach space $C_b(K)$ and the convergence of the sequence $\{\mu_n\}$ corresponds to the $*$-weak convergence in $C_b^*(K)$. Since $C_b(K)$ is a Grothendieck Banach space, the sequence $\{\mu_n\}$ converges in the weak topology of $C_b^*(K)$. Let $L$ be the closed subspace of $C_b^*(K)$ generated by Dirac's measures $\delta_x$ for $x \in X$. It is readily verified that the space $L$ is (isometrically) isomorphic to the Banach space $l^1$. Clearly, one has $\mu_n \in L$ for all $n \in \mathbb{N}$. By the above mentioned property of the weak convergence in $l^1$, the sequence $\{\mu_n\}$ converges in norm, which implies that it is uniformly tight. □

**Corollary 2.13.** *A subspace $X$ of a Grothendieck space $K$ is almost discrete and has the strong Skorokhod property if and only if all compact subsets of $X$ are metrizable* (*equivalently, finite*). *In particular, this is true if $K$ is an extremally disconnected Tychonoff space.*

The next result follows by Theorem 2.4 and Corollary 2.7.

**Corollary 2.14.** *Let $X$ be the same as in Theorem* 2.12, *let $E$ be a completely regular space, and let a sequence of Radon probability measures $\mu_n$ on $X \times E$ converge weakly to a Radon measure $\mu$. Then, for each $x \in X$, the restrictions of the measures $\mu_n$ to the set $x \times E$ converge weakly to the restriction of $\mu$, i.e., $\mu_n|_{x\times E} \Rightarrow \mu|_{x\times E}$.*

The space $\{p\} \cup \mathbb{N}$, $p \in \beta\mathbb{N}\backslash\mathbb{N}$, is probably the simplest example of a nonmetrizable space with the strong Skorokhod property. The fact that it is not metrizable is seen from the property that $p$ belongs to the closure of $\mathbb{N}$, but there are no infinite convergent sequences with elements from $\mathbb{N}$ (if such a sequence $\{n_i\}$ converges, then the function $f(n_{2i}) = 0$, $f(n_{2i+1}) = 1$ has no continuous extensions to $\beta\mathbb{N}$).

It should be noted that although the weak convergence of countable sequences of probability measures on the space $X$ in Corollary 2.11 is the same one that corresponds to the discrete metric on $X$, the two weak topologies on the space of probability measures are different (otherwise $X$ would be metrizable in the topology from $\beta\mathbb{N}$).

Thus, in the class of countable spaces with a unique nonisolated point, there are almost metrizable nonmetrizable spaces which have (or have not) the strong Skorokhod property.

On the other hand, all countable spaces with a unique non-isolated point have the strong Skorokhod property for uniformly tight families of Radon measures.

**Proposition 2.15.** *Any uniformly tight collection of probability measures on a countable space with a unique nonisolated point has the strong Skorokhod property.*

*Proof.* Given a uniformly tight family $\mathcal{M}$ of probability measures on a countable space $X$ with a unique nonisolated point $x_\infty$, find for every $n \in \mathbb{N}$ a compact subset $K_n \subset X$ such that $\mu(K_n) > 1-2^{-n}$, $\forall\, \mu \in \mathcal{M}$. Without loss of generality we may assume that $x_\infty \in K_n \subset K_{n+1}$ for each $n$. The compact sets $K_n$, being countable, are metrizable. Consequently, the topological sum $Y = \oplus_{n\in\mathbb{N}} K_n$ is metrizable as well. Next, consider the projection $Y \to \bigcup_{n=1}^\infty K_n \subset X$. Our statement will be proved as soon as we show that the induced map $\mathcal{P}_r(Y) \to \mathcal{P}_r(X)$ between the spaces of measures has a continuous section $\mathcal{M} \to \mathcal{P}_r(Y)$. To this end, we shall decompose each measure $\mu \in \mathcal{M}$ into a series $\sum_{n=1}^\infty \mu_n$, where $\mu_n$ is a measure on $K_n$ such that $\mu_n(K_n) = 2^{-n}$ and the correspondence $\mu \mapsto \mu_n$ is continuous in $\mu \in \mathcal{M}$.

We proceed by induction. Write $K_1 \setminus \{x_\infty\} = \{x_i\colon 1 \le i < Card\,(K_1)\}$. Since $X$ has a unique nonisolated point, any compact set is either finite or has a unique nonisolated point $x_\infty$. Given a measure $\mu \in \mathcal{M}$, let $N(\mu) = \sup\Bigl\{m\colon \sum_{i<m} \mu(x_i) < \frac{1}{2}\Bigr\}$ and $\mu_1 = \Bigl(\frac{1}{2} - \sum_{i<N(\mu)} \mu(x_i)\Bigr)\delta_{x_{N(\mu)}} + \sum_{i<N(\mu)} \mu(x_i)\delta_{x_i}$. It is readily verified that $\mu_1 \le \mu$, $\mu_1(K_1) = 1/2$ and the so defined measure $\mu_1$ depends continuously on $\mu \in \mathcal{M}$. In order to show the continuity, let us consider a sequence of measures $\mu^m \in \mathcal{M}$ weakly convergent to $\mu \in \mathcal{M}$. Let us fix a continuous function $f$ on $X$ with $|f| \le 1$ and $\varepsilon > 0$. We can assume that $K$ is infinite and then the sequence $\{x_n\}$ converges to $x_\infty$. Hence there is $N$ such that $|f(x_n) - f(x_\infty)| < \varepsilon$ for all $n > N$. Then we have

$$\Bigl|\int f\, d\mu_1 - \int f\, d\mu_1^m\Bigr| \le \Bigl|\int_{\{x_n\colon\, n\le N\}} f\,[d\mu_1 - d\mu_1^m]\Bigr|$$

$$+\Bigl|\int_{\{x_n\colon\, n>N\}\cup\{x_\infty\}} [f - f(x_\infty)]\,[d\mu_1 - d\mu_1^m]\Bigr| + \Bigl|\int_{\{x_n\colon\, n>N\}\cup\{x_\infty\}} f(x_\infty)\,[d\mu_1 - d\mu_1^m]\Bigr|$$

$$\le \sum_{n\le N} |\mu_1(x_n) - \mu_1^m(x_n)| + \varepsilon + \Bigl|\frac{1}{2} - \sum_{n\le N} \mu_1(x_n) - \Bigl(\frac{1}{2} - \sum_{n\le N} \mu_1^m(x_n)\Bigr)\Bigr|$$

$$\le \varepsilon + 2\sum_{n\le N} |\mu_1(x_n) - \mu_1^m(x_n)|.$$

It remains to note that the right-hand side tends to zero as $m \to \infty$, since $\mu^m(x_n) \to \mu(x_n)$ for every fixed $n < \infty$. Applying this procedure to the measure $\mu - \mu_1$, we find a measure $\mu_2 \le \mu - \mu_1$ on $K_2$ such that $\mu_2(K_2) = \frac{1}{4}$. Proceeding in this way we obtain the desired decomposition $\mu = \sum_{n=1}^\infty \mu_n$. □

Note that the nonmetrizable spaces with the strong Skorokhod property which have been constructed so far are not topologically homogeneous. Let us show that there exist also nonmetrizable topologically homogeneous spaces with the strong Skorokhod property for Radon measures. A topological space $X$ is called *topologically homogeneous* if for every pair of points $x, y \in X$, there is a homeomorphism $h\colon X \to X$ such that $h(x) = y$. As a rule, pathological examples of topologically homogeneous spaces are constructed by using the technique of left topological groups. We recall that a *left topological group* is a group $(G, *)$ endowed with a left invariant topology, i.e., a topology $\tau$ such that for each $g \in G$, the left shift $l_g\colon x \mapsto g * x$ is continuous on $(G, \tau)$. A rich theory of left topological groups has been developed by I.V. Protasov, see [22], [23]. According to [23, Theorem 4.1], each infinite group $G$ admits a nondiscrete regular extremally disconnected left invariant topology. The theorem cited together with Theorem 2.12 implies the following assertion.

**Corollary 2.16.** *Each countable group $G$ admits a left invariant topology $\tau$ such that $(G, \tau)$ is a nonmetrizable countable almost discrete topologically homogeneous extremally disconnected space with the strong Skorokhod property for Radon measures.*

A countable nondiscrete extremally disconnected Boolean topological group was constructed by S. Sirota [27] (see also [21]) under the continuum hypothesis CH (we recall that a group $G$ is *Boolean* if $x^2 = 0$ for every $x \in G$). This fact yields the following result.

**Corollary 2.17.** *Under CH, there exists a countable nonmetrizable almost discrete extremally disconnected topological Boolean group with the strong Skorokhod property for Radon measures.*

The following questions remain open.

**Question 1.** Is there an infinite extremally disconnected compact space with the strong Skorokhod property for Radon measures? In particular, does $\beta\mathbb{N}$ possess the strong Skorokhod property for Radon measures?

Since compact subsets of almost metrizable spaces are metrizable, we conclude that each Radon measure $\mu$ on an almost metrizable space $X$ is concentrated on a $\sigma$-compact space $C \subset X$ with a countable network in the sense that $\mu(C) = 1$. We recall that a space $X$ has a *countable network* if there is a countable family $\mathcal{N}$ of subsets of $X$ such that, for every point $x \in X$ and every neighborhood $U \subset X$ of $x$, there is an element $N \in \mathcal{N}$ with $x \in N \subset U$.

**Question 2.** Is it true that every Radon probability measure $\mu$ on a space with the strong Skorokhod property for Radon measures is concentrated on a subspace $C \subset X$ with a countable network?

A topological space $X$ is called *sequential* if for every nonclosed subset $F \subset X$, there is a sequence $\{x_n\} \subset F$ converging to a point $x_0 \notin F$. It is clear that each metrizable space is sequential and each almost metrizable sequential space is metrizable. A topological space $X$ is called a *Fréchet–Urysohn space* if, for every subset $A \subset X$ and every point $x \in \overline{A} \setminus A$, there is a sequence $\{x_n\} \subset A$

convergent to the point $x_0$. It is clear that each Fréchet–Urysohn space is sequential.

A standard example of a nonmetrizable Urysohn space is the Fréchet–Urysohn fan $V$. The Fréchet–Urysohn fan $V$ is defined as follows (cf. [11, 1.6.18]):

$$V := \{k + (n+1)^{-1}\colon\ k, n \in \mathbb{N}\} \cup \{0\}$$

is endowed with the following topology: every point $k + (n+1)^{-1}$ has its usual neighborhoods from the space $V\backslash\{0\}$ and the point 0 has an open base formed by all sets

$$U_{n_1,\dots,n_j,\dots} := \{k + (n+1)^{-1}\colon\ k \in \mathbb{N}, n \geq n_k\} \cup \{0\},$$

where $\{n_j\}$ is a sequence of natural numbers.

There are also sequential spaces which are not Fréchet–Urysohn. The simplest example is *the Arens fan* $A_2$, i.e., the space $A_2 = \{(0,0), (1/i, 0), (1/i, 1/j)\colon\ 1 \leq i \leq j < \infty\}$ endowed with the strongest topology inducing the original topology on each compact $K_n = \{(0,0), (1/k, 0), (1/i, 1/j)\colon\ k \in \mathbb{N},\ 1 \leq i \leq n,\ i \leq j < \infty\}$.

A topological space $X$ is called *scattered* if each subspace $E$ of $X$ has an isolated point. It is well known that each Radon measure on a scattered space is discrete, see [17, Lemma 294].

**Question 3.** Is it true that any scattered sequential (Fréchet–Urysohn) countable space with the strong Skorokhod property is metrizable?

**Question 4.** Do the Fréchet–Urysohn and Arens fans have the strong Skorokhod property?

Let us note that according to Proposition 2.15 the Fréchet–Urysohn fan has the strong Skorokhod property for uniformly tight families of probability measures.

## 3. The strong Skorokhod Property of Spaces Whose Topology is Generated by a Linear Order

Any linear order $\leq$ on a set $X$ generates two natural topologies on $X$. The usual *interval topology* is generated by the pre-basis consisting of the rays $(\leftarrow, a) = \{x \in X\colon\ x < a\}$ and $(a, \rightarrow) = \{x \in X\colon\ x > a\}$, where $a \in X$. The *Sorgenfrey topology* on $X$ is generated by the pre-basis consisting of the rays $(a, \rightarrow)$ and $(\leftarrow, a] = \{x \in X\colon\ x \leq a\}$, where $a \in X$. The space $X$ endowed with the interval topology will be denoted by $\bigl(X, (\leq)\bigr)$. The space $X$ endowed with the Sorgenfrey topology will be denoted by $\bigl(X, (\leq]\bigr)$. According to [11, 2.7.5] the space $\bigl(X, (\leq)\bigr)$ is hereditarily normal, while the space $\bigl(X, (\leq]\bigr)$ is Tychonoff and zero-dimensional.

**Theorem 3.1.** *If $\leq$ is a linear order on a set $X$, then the spaces $\bigl(X, (\leq)\bigr)$ and $\bigl(X, (\leq]\bigr)$ have the strong Skorokhod property for discrete probability measures.*

*Proof.* Fix any point $x_0 \in X$. Given a discrete probability measure $\mu$ on $X$, consider the countable set $S(\mu) := \{x \in X\colon \mu(\{x\}) > 0\}$ and to each point $x \in S(\mu)$ assign the open interval $I_x = \big(\mu(\leftarrow, x), \mu(\leftarrow, x]\big) \subset [0,1]$. It is clear that for distinct $x, y \in S(\mu)$, the intervals $I_x$ and $I_y$ are disjoint and Lebesgue measure of the union $I(\mu) = \bigcup_{x\in S(\mu)} I_x$ is 1. Let $\xi_\mu\colon [0,1] \to X$ be the Borel function defined by

$$\xi_\mu(t) = \begin{cases} x & \text{if } t \in I_x \text{ for some } x \in S(\mu), \\ x_0 & \text{otherwise.} \end{cases}$$

Since $\lambda(I_x) = \mu(\{x\})$, we get $\mu = \lambda \circ \xi_\mu^{-1}$ where $\lambda$ stands for the standard Lebesgue measure on $[0,1]$. Our crucial observation is that for each $t \in I(\mu)$ we have

$$\mu\big(\leftarrow, \xi_\mu(t)\big) = \mu(\leftarrow, x) < t < \mu(\leftarrow, x] = \mu\big(\leftarrow, \xi_\mu(t)\big],$$

where $x \in S(\mu)$ is such that $t \in I_x$.

Now let us show that the family $\{\xi_\mu\}$ turns the spaces $\big(X, (\le)\big)$ and $\big(X, (\le]\big)$ into spaces with the strong Skorokhod property for discrete probability measures. Assume that $\mu_n \Rightarrow \mu_0$ is a weakly convergent sequence of discrete probability measures on $\big(X, (\le)\big)$ such that $\mu_0$ is discrete. We shall show that for each $t \in \bigcap_{n=0}^\infty I(\mu_n)$ the sequence $\{\xi_{\mu_n}(t)\}$ converges to $\xi_{\mu_0}(t)$. It suffices to verify that for each pre-basic neighborhood $W$ of $\xi_{\mu_0}(t)$, all but finitely many points $\xi_{\mu_n}(t)$ lie in $W$.

If $W = (a, \to) \ni \xi_{\mu_0}(t)$ for some $a \in X$, then $\mu_0(\leftarrow, a] \le \mu_0\big(\leftarrow, \xi_{\mu_0}(t)\big) < t$. Since the ray $(\leftarrow, a]$ is closed in the space $\big(X, (\le)\big)$, from the weak convergence of $\{\mu_n\}$ to $\mu_0$ we obtain that $\mu_n(\leftarrow, a] < t$ for almost all $n$. Then $\mu_n(\leftarrow, a] < t < \mu_n\big(\leftarrow, \xi_{\mu_n}(t)\big)$ and $a < \xi_{\mu_n}(t)$ for all but finitely many $n$. Hence $\xi_{\mu_n}(t) \in W$ for all but finitely many $n$.

Next, assume that $W = (\leftarrow, a)$ for some $a \in X$. Then $\xi_{\mu_0}(t) < a$ and $t < \mu_0(\leftarrow, \xi_{\mu_0}(t)] \le \mu_0(\leftarrow, a)$. Since the ray $(\leftarrow, a)$ is open in the space $\big(X, (\le)\big)$, we get $\mu_n(\leftarrow, a) > t$ for all but finitely many $n$. Then $\mu_n(\leftarrow, a) > t > \mu_n(\leftarrow, \xi_{\mu_n}(t))$ and hence $a > \xi_{\mu_n}(t)$ for all but finitely many $n$.

Now assume that the sequence $\{\mu_n\}$ converges weakly to $\mu_0$ in the space $\big(X, (\le]\big)$, where all the measures in question are discrete. In order to show that the sequence $\{\xi_{\mu_n}(t)\}$ converges to $\xi_{\mu_0}(t)$ for each $t \in \bigcap_{n\ge 0} I(\mu_n)$, fix any pre-basic neighborhood $W$ of the point $\xi_{\mu_0}(t)$ in $\big(X, (\le]\big)$. If $W = (a, \to)$ for some $a \in X$, then repeating the above argument we prove that $\xi_{\mu_n}(t) \in W$ for all but finitely many $n$. So assume that $W = (\leftarrow, a]$ for some $a \in X$. Then $\xi_{\mu_0}(t) \le a$ and $t < \mu_0(\leftarrow, \xi_{\mu_0}(t)] \le \mu_0(\leftarrow, a]$. Since the sets $(\leftarrow, a]$ are open in $\big(X, (\le]\big)$ and $\mu_n \Rightarrow \mu_0$, we obtain $\mu_n(\leftarrow, a] > t$ for all but finitely many $n$. Since $\mu_n\big(\leftarrow, \xi_{\mu_n}(t)\big) < t < \mu_n(\leftarrow, a]$, we conclude that $\xi_{\mu_n}(t) \le a$ and $\xi_{\mu_n}(t) \in W$ for all but finitely many $n$. $\square$

A topological space $X$ is called *linearly ordered* if it carries the interval topology generated by some liner order on $X$. Standard examples of linearly ordered spaces are the real line and segments of ordinals.

**Corollary 3.2.** *If each Radon probability measure on a linearly ordered topological space $X$ is discrete, then the space $X$ has the strong Skorokhod property for Radon measures.*

**Corollary 3.3.** *Each scattered linearly ordered space has the strong Skorokhod property.*

**Corollary 3.4.** *For every ordinal $\alpha$, the segment $[0, \alpha]$ endowed with the usual order topology has the strong Skorokhod property for Radon measures.*

**Corollary 3.5.** *For any linear order $\le$ on a set $X$, the space $\big(X, (\le]\big)$ has the strong Skorokhod property for Radon measures.*

*Proof.* According to Theorem 3.1, it suffices to show that each Radon probability measure on $\big(X, (\le]\big)$ is discrete. It suffices to verify that each compact subspace of $\big(X, (\le]\big)$ is scattered. If it were not the case, we could find a compact subspace $K$ of $\big(X, (\le]\big)$ without isolated points. Clearly, the set $K$ has a minimal element $\min K$ (otherwise the family $\{(a, \rightarrow) \colon a \in K\}$ would be an open cover of $K$ without a finite subcover). Then the set $\{\min K\} = K \cap (\leftarrow, \min K]$ is open in $K$ and thus $\min K$ is an isolated point of $K$, which is a contradiction. □

**Corollary 3.6.** *The Sorgenfrey line $\big(\mathbb{R}, (\le]\big)$ has the strong Skorokhod property for Radon measures.*

**Question 5.** Does the product $[0, 1] \times [0, \omega_1]$ have the strong Skorokhod property for Radon measures or for discrete probability measures?

**Question 6.** Does every linearly ordered compact space have the strong Skorokhod property for Radon measures? In particular, does the Souslin line have the strong Skorokhod property for Radon measures?

We recall that a *Souslin line* is a linearly ordered nonseparable compact space with countable cellularity, see [11, 2.7.9]. It is known that the existence of a Souslin line is independent of the ZFC axioms.

It is worth noting that the Sorgenfrey line can be topologically embedded into the product $A = [0, 1] \times \{0, 1\}$ endowed with the interval topology generated by the lexicographic order $\le$: $(x, t) \le (y, \tau)$ if and only if $x < y$ or $x = y$ and $t \le \tau$. The space $A$ (called *two arrows of Alexandroff*) is well known as an example of a nonmetrizable separable first countable compact space.

Theorem 3.1 implies the following assertion.

**Corollary 3.7.** *The Alexandroff two arrows space $A$ has the strong Skorokhod property for discrete probability measures.*

Since the space $A$ admits a surjective continuous map onto the interval $[0, 1]$, it admits a nondiscrete probability measure $\mu$, namely, a measure whose projection is the standard Lebesgue measure on $[0, 1]$.

**Question 7.** Is it true that the Alexandroff two arrows space $A$ possess the strong Skorokhod property for Radon measures?

Finally, let us consider yet another interesting space whose topology is generated by a partial order. Let $T$ be the standard binary tree, i.e., the set $T = \bigcup_{0 \le n \le \omega} \{0, 1\}^n$ of all binary sequences (finite and infinite) with the natural partial order $\le$: $(x_i)_{i \le n} \le (y_i)_{i \le m}$ if and only if $n \le m$ and $x_i = y_i$ for all $i \le n$. The topology of the space $T$ is generated by the half-intervals $(a, b] = \{x \in T : a < x \le b\}$ where $a < b$ are points of $T$. The space $T$ endowed with this topology is scattered, separable, locally metrizable, and locally compact but not metrizable (since it contains the discrete subspace $\{0, 1\}^\omega$). In fact, the space $T$ is known as an example of a nonmetrizable Moore space. We recall (see [11], [16]) that a topological space $X$ is a *Moore space* if it admits a sequence $\{\mathcal{U}_n\}$ of open covers such that for every point $x \in X$ the family $\left\{ \bigcup_{x \in U_n \in \mathcal{U}} U \right\}_{n \in \mathbb{N}}$ forms a neighborhood base at $x$. By the Bing metrization criterion [11, 5.4.1], a Moore space is metrizable if and only if it is collectively normal.

**Question 8.** Is every Moore space with the strong Skorokhod property metrizable? In particular, does the Moore space $T$ have the strong Skorokhod property for Radon measures?

## 4. The strong Skorokhod Property of Compact Spaces

In this section, we study the strong Skorokhod property in nonmetrizable compact topological spaces. Probably, the simplest example of such a space is *the Alexandroff supersequence*, which is the one-point compactification $\alpha\aleph_1$ of a discrete space of the smallest uncountable size. We shall show that the Alexandroff supersequence does not have the strong Skorokhod property for Radon measures. To this end, we need one combinatorial lemma. Let $\infty$ denote the unique nonisolated point of $\alpha\aleph_1$ and let $\Delta$ be the diagonal of the square $(\alpha\aleph_1)^2$ of $\alpha\aleph_1$.

**Lemma 4.1.** *There is no continuous mapping $f\colon (\alpha\aleph_1)^2 \setminus \Delta \to M$ into a metric space $(M, d)$ such that $d\big(f(\infty, a), f(a, \infty)\big) \ge 1$ for every $a \in \aleph_1$.*

*Proof.* Assume that such a mapping $f$ exists. By the continuity of $f$ outside $\Delta$, for every $a \in \aleph_1$, we find a finite set $F(a) \subset \aleph_1$ such that $a \in F(a)$ and

$$\max\Big\{ d\Big(f(a, \infty), f(a, b)\Big),\ d\Big(f(\infty, a), f(b, a)\Big) \Big\} < 1/6$$

for every $b \in \aleph_1 \setminus F(a)$. By the $\Delta$-System Lemma in [19, 16.1], there exist an uncountable subset $A \subset \aleph_1$ and a finite set $F \subset \aleph_1$ such that $F(a) \cap F(a') = F$ for any distinct $a, a' \in A$. By transfinite induction, we may construct an uncountable subset $B \subset A \setminus F$ such that $b \notin F(a)$ for any distinct $a, b \in B$. Then, for any distinct points $a, b \in B$, we have $d\Big(f(a,\infty), f(a,b)\Big) < 1/6$ and $d\Big(f(\infty,b), f(a,b)\Big) < 1/6$. Now fix any three different points $a, b, c \in B$ and observe that the following estimates are true:

$$
\begin{aligned}
& d\Big(f(a,\infty), f(\infty,a)\Big) \le d\Big(f(a,\infty), f(b,\infty)\Big) + d\Big(f(b,\infty), f(\infty,a)\Big) \\
& \le d\Big(f(a,\infty), f(a,c)\Big) + d\Big(f(a,c), f(\infty,c)\Big) + d\Big(f(\infty,c), f(b,c)\Big) \\
& + d\Big(f(b,c), f(b,\infty)\Big) + d\Big(f(b,\infty), f(b,a)\Big) + d\Big(f(b,a), f(\infty,a)\Big) < 6 \cdot \frac{1}{6} = 1,
\end{aligned}
$$

which is a contradiction. □

**Theorem 4.2.** *The Alexandroff supersequence $\alpha\aleph_1$ fails to have the strong Skorokhod property even for probability measures with two-point supports.*

*Proof.* Let $\mathcal{M}$ denote the space of all measurable subsets $A \subset [0,1]$ with Lebesgue measure $\lambda(A) = 1/2$, endowed with the standard metric $d(A,B) = \lambda(A\Delta B)$ (to be more precise, we deal with the corresponding equivalence classes). Observe that $d(A,B) = 1$ for any sets $A, B \in \mathcal{M}$ such that $\lambda(A \cap B) = 0$. Assume that the Alexandroff supersequence $\alpha\aleph_1$ has the strong Skorokhod property for probability measures with two-point supports and let $\xi_\mu \colon [0,1] \to X$ be a Skorokhod parameterization of probability measures on $X$ with two-point supports. Next, define a function $f \colon (\alpha\aleph_1)^2 \setminus \Delta \to \mathcal{M}$ letting $f(a,b) = \xi^{-1}_{\frac{1}{2}\delta_a + \frac{1}{2}\delta_b}(a)$ for any distinct $a, b \in \alpha\aleph_1$. It can be shown that the map $f$ is continuous and $d\Big(f(a,\infty), f(\infty,a)\Big) = 1$ for every $a \in \alpha\aleph_1$, which contradicts Lemma 4.1. □

**Corollary 4.3.** *A topological space containing a copy of the Alexandroff supersequence $\alpha\aleph_1$ fails to have the strong Skorokhod property for probability measures with two-point supports.*

Since the Alexandroff supersequence is a scattered compact, Theorem 4.2 shows that there exist scattered compacta without the strong Skorokhod property for probability measures with two-point supports. On the other hand, by Corollary 3.4, the segment $[0, \omega_1]$ is an example of a nonmetrizable scattered compact with the strong Skorokhod property. We shall show that any nonmetrizable scattered compact with the strong Skorokhod property in some sense resembles the space $[0, \omega_1]$. Namely, it has infinite Cantor–Bendixson rank which is defined as follows.

Given a topological space $X$ let $X^{(1)}$ denote the set of all nonisolated points of $X$. By transfinite induction, for every ordinal $\alpha$ define the $\alpha$-th derived set $X^{(\alpha)}$ of $X$ letting $X^{(0)} = X$ and $X^{(\alpha)} = \bigcap_{\beta<\alpha}(X^{(\beta)})^{(1)}$. Thus, we get a decreasing transfinite sequence $(X^{(\alpha)})_\alpha$ of subsets of $X$. The smallest ordinal $\alpha_0$ such that $X^{(\alpha_0)} = X^{(\alpha)}$ for any $\alpha \ge \alpha_0$ is called the *Cantor–Bendixson rank* of $X$. It is clear that a space $X$ is scattered if and only if $X^{(\alpha)} = \varnothing$ for some $\alpha$.

It is well known that the Cantor–Bendixson rank of $[0, \omega_1]$ is equal to $\omega_1$, while the Cantor–Bendixson rank of the Alexandroff supersequence $\alpha\aleph_1$ is equal to 2.

**Theorem 4.4.** *Each nonmetrizable scattered compact space with the strong Skorokhod property for probability measures with two-point supports has infinite Cantor–Bendixson rank.*

*Proof.* It can be proved by induction that each nonmetrizable scattered compact space with finite Cantor–Bendixson rank contains a copy of the Alexandroff supersequence $\alpha\aleph_1$. □

**Question 9.** Is it true that any scattered compact space with the strong Skorokhod property for probability measures with two-point supports has uncountable Cantor–Bendixson rank?

We shall now show that in the class of dyadic compacta only metrizable ones enjoy the strong Skorokhod property.

**Theorem 4.5.** *Each dyadic compact with the strong Skorokhod property for probability measures with two-point supports is metrizable. In particular, if a dyadic compact has the strong Skorokhod property for probability measures with two-point supports, then it has the strong Skorokhod property for Radon measures.*

*Proof.* Our claim follows from Corollary 4.3 and the known fact that any nonmetrizable dyadic compact contains a copy of the Alexandroff supersequence, see [11, 3.12.12]. □

Besides the class of dyadic compact spaces, there are many interesting classes of compact spaces for which their relation to the class of spaces with the strong Skorokhod property has not yet been clarified. In this respect, it would be interesting to investigate the classes of Eberlein, Corson, and Rosenthal compacta (see, e.g., [1], [12]). We recall that a compact space $K$ is defined to be

- an *Eberlein compact* if $K$ is homeomorphic to a compact subset of the $\Sigma^*$-product $\Sigma^*(\tau) = \Big\{(x_i)_{i\in\tau} \in \mathbb{R}^\tau : \forall\varepsilon > 0 \text{ the set } \{i \in \tau : |x_i| > \varepsilon\} \text{ is finite}\Big\} \subset \mathbb{R}^\tau$ for some cardinal $\tau$;
- a *Corson compact* if $K$ is homeomorphic to a compact subset of the $\Sigma$-product $\Sigma(\tau) = \Big\{(x_i)_{i\in\tau} \in \mathbb{R}^\tau : \text{the set } \{i \in \tau : x_i \neq 0\} \text{ is countable}\Big\} \subset \mathbb{R}^\tau$ for some cardinal $\tau$;
- a *Rosenthal compact* if $K$ is homeomorphic to a compact subset of the space $B_1(P) \subset \mathbb{R}^P$ of all functions of the first Baire class on a Polish space $P$.

It is clear that any Eberlein compact is Corson. It is known that each separable Corson compact as well as each Eberlein compact with countable cellularity is metrizable. The Alexandroff two arrows space is a standard example of a Rosenthal compact which is not a Corson compact. The Alexandroff supersequence is both an Eberlein and Rosenthal compact. Thus, there exist Eberlein and Rosenthal compacta without the strong Skorokhod property. The space $[0, \omega_1]$ is neither Corson nor Rosenthal, see [7, pp. 256, 259].

Other examples of nonmetrizable compacta which are both Eberlein and Rosenthal are supplied by the construction of the Alexandroff doubling. Given a topological space $X$, let $f\colon X \to Y$ be a bijective map onto a set $Y$ disjoint with $X$. On the union $D(X) = X \cup Y$ introduce the topology consisting of the sets of the form $A \cup U \cup \Bigl(f(U) \setminus F\Bigr)$, where $U \subset X$ is open in $X$, $A$ is a subset of $Y$, and $F$ is a finite subset of $Y$. The obtained topological space $D(X)$ is called *the Alexandroff doubling* of the space $X$. It is known that for each metrizable compact space $K$, its Alexandroff doubling $D(K)$ is a first countable Eberlein Rosenthal compact (observe that $D(K)$ is homeomorphic to a subspace of the product $K \times \alpha|K|$, where $\alpha|K|$ is the one-point compactification of $K$ endowed with the discrete topology).

**Question 10.** Is every Eberlein (Corson, Rosenthal) compact with the strong Skorokhod property metrizable? In particular, does the Alexandroff doubling $D([0, 1])$ of the segment have the strong Skorokhod property (or the strong Skorokhod property for probability measures with two-point supports)?

It is worth noting that each countable family $\mathcal{M}$ of probability measures on an Eberlein compact $E$ does admit a Skorokhod representation. Indeed, each compact subset of $E$ is metrizable provided it has countable cellularity, which implies the existence of a metrizable compact subset $K \subset E$ with $\mu(K) = 1$ for all $\mu \in \mathcal{M}$.

The Alexandroff doubling of a metrizable compact as well as the Alexandroff two arrows space are examples of nonmetrizable spaces admitting a continuous finite-to-one map onto a metrizable compact.

**Question 11.** Suppose a compact $K$ has the strong Skorokhod property and admits a continuous finite-to-one map onto a metrizable compact. Is $K$ metrizable?

We recall that the strong Skorokhod property for Radon measures is inherited by arbitrary subspaces, which enables one to construct more examples of spaces with this property on the basis of the examples above.

Now several remarks on open problems are in order.

**Question 12.** Stability of the class of spaces with the strong Skorokhod property (or with the other related properties mentioned above) with respect to formation of finite and countable products.

**Question 13.** In particular, it would be interesting to investigate whether the product $X \times Y$, where $X$ has the strong property and $Y$ is separable metric (say, $Y = \mathbb{N}^{\mathbb{N}}$), retains the same Skorokhod property. We do not know the answer even for finite or countable $Y$ (in particular, we do not know whether the topological sum of two spaces with the strong Skorokhod property has that property).

**Question 14.** Preservation of the strong Skorokhod property (or the other related properties mentioned above) by continuous mappings (with certain additional properties).

Some positive results concerning continuous images have been proved above and in [5]. The example of the space $\mathbb{R}_0^\infty$ shows that some additional assumptions are needed in order to guarantee that the image space has the strong Skorokhod property. However, the assumptions imposed in [5] seem to be rather restrictive.

It might be also reasonable to look at weaker Skorokhod-type properties inherited by the continuous images of spaces with stronger properties (as is the case for the space $\mathbb{R}_0^\infty$).

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