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# NON-NOETHER SYMMETRIES IN SINGULAR DYNAMICAL SYSTEMS

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**Abstract**. The Hojman–Lutzky conservation law establishes a certain correspondence between non-Noether symmetries and conserved quantities. In the present paper the extension of the Hojman–Lutzky theorem to singular dynamical systems is carried out.

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## 1. Introduction

Noether's theorem associates conservation laws with particular continuous symmetries of the Lagrangian. According to the Hojman's theorem [1–3] there exist the definite correspondence between non-Noether symmetries and conserved quantities. In 1998 M. Lutzky showed that several integrals of motion might correspond to a single one-parameter group of non-Noether transformations [4]. In the present paper, the extension of the Hojman–Lutzky theorem to singular dynamical systems is considered.

First let us recall some basic knowledge of the description of regular dynamical systems (see, e.g., [5]). Since the trajectories are solutions of the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v^a} - \frac{\partial L}{\partial q^a} = 0,$$

the tangent vector of trajectory satisfies Hamilton's equation

$$i_{X_h}\omega_L + dh = 0,$$

where $\omega_L$ is the closed ($d\omega_L = 0$) and non-degenerate ($i_X\omega_L = 0 \Leftrightarrow X = 0$) 2-form, $h$ is the Hamiltonian and $i_X\omega_L = X\rfloor\omega_L$ denotes the contraction of $X$ to $\omega_L$. In the local coordinates $\omega_L = d\theta_L$, where $\theta_L = \frac{\partial L}{\partial v^a}dq^a$ and $h = \frac{\partial L}{\partial v^a}v^a - L$. Since $\omega_L$ is non-degenerate, this gives rise to an isomorphism between the vector fields and the 1-forms given by $i_X\omega_L + \alpha = 0$. The vector field is said to be the Hamiltonian if it corresponds to the exact form

$$i_{X_f}\omega_L + df = 0.$$

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The Poisson bracket is defined as follows:

$$\{f,g\} = X_f g = -X_g f = i_{X_f} i_{X_g}\omega.$$

By introducing a bivector field $W$ satisfying

$$i_X i_Y \omega = W \rfloor i_X\omega \wedge i_Y\omega,$$

the Poisson bracket can be rewritten as

$$\{f,g\} = W \rfloor df \wedge dg.$$

It is easy to show that

$$i_X i_Y L_Z \omega = [Z,W] \rfloor i_X\omega \wedge i_Y\omega, \tag{1}$$

where the bracket [ , ] is actually a supercommutator (for an arbitrary bivector field $W = \sum_i V^i \wedge U^i$ we have $[X,W] = \sum_i [X,V^i]\wedge U^i + \sum_i V^i \wedge [X,U^i]$). Equation (1) is based on the following useful property of the Lie derivative:

$$L_X i_W \omega = i_{[X,W]}\omega + i_W L_X \omega.$$

Indeed, for an arbitrary bivector field $W = \sum_i V^i \wedge U^i$ we have

$$L_X i_W \omega = L_X i_{\sum_i V^i\wedge U^i}\omega = L_X \sum_i i_{U^i} i_{V^i}\omega = \sum_i i_{[X,U^i]} i_{V^i}\omega$$
$$+ \sum_i i_{U^i} i_{[X,V^i]}\omega + \sum_i i_{U^i} i_{V^i} L_X\omega = i_{[X,W]}\omega + i_W L_X \omega,$$

where $L_Z$ denotes the Lie derivative along the vector field $Z$. According to Liouville's theorem the Hamiltonian vector field preserves $\omega_L$

$$L_{X_f}\omega = 0;$$

therefore it commutes with $W$:

$$[X_f, W] = 0.$$

In the local coordinates $\xi_i$, where $\omega = \omega^{ij} d\xi_i \wedge \xi_j$, the bivector field $W$ has the form $W = W^{ij}\frac{\partial}{\partial\xi_i}\wedge\frac{\partial}{\partial\xi_j}$, where $W^{ij}$ is the matrix inverse to $\omega^{ij}$.

## 2. The Hojman–Lutzky Theorem for Regular Lagrangian Systems

We can say that a group of transformations $g(a) = e^{aL_E}$ generated by the vector field $E$ maps the space of solutions of the equation onto itself if

$$i_{X_h} g_*(\omega_L) + g_*(dh) = 0. \tag{2}$$

For $X_h$ satisfying

$$i_{X_h}\omega_L + dh = 0$$

Hamilton's equation. It is easy to show that the vector field $E$ should satisfy $[E, X_h] = 0$ (Indeed, $i_{X_h} L_E \omega_L + dL_E h = L_E(i_{X_h}\omega_L + dh) = 0$ since $[E, X_h] = 0$).

When $E$ is not Hamiltonian, the group of transformations $g(a) = e^{aL_E}$ is non-Noether symmetry (in a sense that it maps solutions onto solutions but does not preserve action).

**Theorem (Lutzky, 1998).** *If the non-Hamiltonian vector field $E$ generates non-Noether symmetry, then the following functions are constant along solutions:*

$$I^{(k)} = W^k \rfloor \omega_E^k, \quad k = 1, \dots, n,$$

*where $W^k$ and $\omega_E^k$ are the outer powers of $W$ and $L_E \omega$.*

*Proof.* We have to prove that $I^{(k)}$ is constant along the flow generated by the Hamiltonian. In other words, we should find that $L_{X_h} I^{(k)} = 0$ is fulfilled. Let us consider

$$L_{X_h} I^{(1)} = L_{X_h}(W \rfloor \omega_E) = [X_h, W] \rfloor \omega_E + W \rfloor L_{X_h} \omega_E,$$

where, according to Liouville's theorem, both terms ($[X_h, W] = 0$ and $W \rfloor L_{X_h} L_E \omega = W \rfloor L_E L_{X_h} \omega = 0$ since $[E, X_h] = 0$ and $L_{X_h}\omega = 0$) vanish. In the same manner one can verify that $L_{X_h} I^{(k)} = 0$. □

*Note* 1. The theorem is valid for a larger class of generators $E$. Namely, if $[E, X_h] = X_f$ where $X_f$ is an arbitrary Hamiltonian vector field, then $I^{(k)}$ is still conserved. Such a symmetries map the solutions of the equation $i_{X_h}\omega_L + dh = 0$ on the solutions of the equation $i_{X_h} g_*(\omega) + d(g_* h + f) = 0$.

*Note* 2. Discrete non-Noether symmetries give rise to the conservation of $I^{(k)} = W^k \rfloor g_*(\omega)^k$ where $g_*(\omega)$ is transformed $\omega$.

*Note* 3. If $I^{(k)}$ is a set of conserved quantities associated with $E$ and $f$ is any conserved quantity, then the set of functions $\{I^{(k)}, f\}$ (which due to the Poisson theorem are integrals of motion) is associated with $[X_h, E]$. Namely, it is easy to show by taking the Lie derivative of (2) along vector field $E$ that $\{I^{(k)}, f\} = i_{[W,X_h]^k} \omega_E^k$ is fulfilled. As a result, the conserved quantities associated with Non-Noether symmetries form a Lie algebra under the Poisson bracket.

## 3. The Case of Irregular Lagrangian Systems

The singular Lagrangian (a Lagrangian with the vanishing Hessian $\det \frac{\partial L}{\partial v^i \partial v^j} = 0$) leads to the degenerate 2-form $\omega_L$ and we no longer have an isomorphism between the vector fields and the 1-forms. Since there exists a set of "null vectors" $u^k$ such that $i_{u^k}\omega = 0$, $k = 1, 2, \dots, n - \mathrm{rank}(\omega)$, every Hamiltonian vector field is defined up to a linear combination of vectors $u^k$. By identifying $X_f$ with $X_f + C_k u^k$, we can introduce an equivalence class $\tilde{X}_f$ (then all $u^k$ belong to $\tilde{0}$ ). The bivector field $W$ is also far from being unique, but if $W_1$ and $W_2$ both satisfy

$$i_X i_Y \omega = W_{1,2} \rfloor i_X \omega \wedge i_Y \omega,$$

then

$$(W_1 - W_2) \rfloor i_X \omega \wedge i_Y \omega = 0 \quad \forall X, Y$$

is fulfilled. This is possible only when

$$W_1 - W_2 = v_k \wedge u^k,$$

where $v_k$ are some vector fields and $i_{u^k}\omega = 0$ (in other words, when $W_1 - W_2$ belongs to the class $\tilde{0}$.

**Theorem.** *If the non-Hamiltonian vector field $E$ satisfies the commutation relation $[E, \tilde{X}_h] = \tilde{0}$ (generates non-Noether symmetry), then the functions*

$$I^{(k)} = W^k \rfloor \omega_E^k, \quad k = 1, \ldots, \operatorname{rank}(\omega),$$

*(where $\omega_E = L_E\omega$) are constant along the trajectories.*

*Proof.* Let us consider $I^{(1)}$:

$$L_{\tilde{X}_h} I^{(1)} = L_{\tilde{X}_h}(W \rfloor \omega_E) = [\tilde{X}_h, W]\rfloor \omega_E + W \rfloor L_{\tilde{X}_h}\omega_E = 0$$

The second term vanishes since $[E, \tilde{X}_h] = \tilde{0}$ and $L_{\tilde{X}_h}\omega = 0$. The first one is zero as far as $[\tilde{X}_h, \tilde{W}] = \tilde{0}$ and $[E, \tilde{0}] = \tilde{0}$ are satisfied. So $I^{(1)}$ is conserved. Similarly, one can show that $L_{X_h} I^{(k)} = 0$ is fulfilled. $\square$

*Note* 1. $W$ is not unique, but $I^{(k)}$ does not depend on choosing a representative from the class $\tilde{W}$.

*Note* 2. Theorem is also valid for the generators $E$ satisfying $[E, \tilde{X}_h] = \tilde{X}_f$.

*Note* 3. Theorem can be applied to irregular Hamiltonian systems (Hamiltonian systems with degenerate $\omega$).

**Example.** Hamiltonian description of the relativistic particle leads to the action

$$\int \sqrt{p^2 + m^2}\,dx_0 + p_k dx^k$$

with the vanishing canonical Hamiltonian and degenerate 2-form

$$\frac{1}{\sqrt{p^2 + m^2}}(p^k dp_k \wedge dx_0 + \sqrt{p^2 + m^2}\,dp_k \wedge dx^k).$$

$\omega$ possesses the "null vector field" $i_u\omega = 0$,

$$u = \sqrt{p^2 + m^2}\frac{\partial}{\partial x_0} + p_k \frac{\partial}{\partial x^k}.$$

One can check that the non-Hamiltonian vector field

$$E = \sqrt{p^2 + m^2}\,x_0 \frac{\partial}{\partial x_0} + p_1 x^1 \frac{\partial}{\partial x^1} + \cdots + p_n x^n \frac{\partial}{\partial x^n}$$

generates non-Noether symmetry. Indeed, $E$ satisfies $[E, \tilde{X}_h] = \tilde{0}$ because of $\tilde{X}_h = \tilde{0}$ and $[E, u] = u$. The corresponding integrals of motion are combinations of momenta:

$$I^{(1)} = \sqrt{p^2 + m^2} + p_1 + \cdots + p_n = \sum_\mu p_\mu; \quad I^{(2)} = \sum_{\mu\nu} p_\mu p_\nu; \quad \ldots \quad I^{(n)} = \prod_\mu p_\mu.$$

This example shows that the set of conserved quantities can be obtained from a single one-parameter group of non-Noether transformations.

## 4. The Case of Dynamical Systems on Poisson Manifold

The previous two sections dealt with dynamical systems on symplectic and presymplectic manifolds. Now let us consider the case of dynamical systems on the Poisson manifold. In general, the Poisson manifold is an even-dimensional manifold equipped with the Poisson bracket which can be defined by means of the bivector field $W$ satisfying $[W, W] = 0$ as follows:

$$\{f, g\} = i_W df \wedge dg$$

Due to skewsymmetry of the Poisson bracket $W$ is also skewsymmetric and, in general, it is degenerate. The commutation relation $[W, W] = 0$ (where $[\ ,\ ]$ denotes the supercommutator of vector fields) ensures that the Poisson bracket satisfies the Jacobi identity. Like in the case of symplectic (presymplectic) manifold, we have the correspondence between the vector fields and the 1-forms

$$\beta(X) + \alpha \wedge \beta(W) = 0 \quad \forall \beta \in \Omega^1(M).$$

The vector field is called the Hamiltonian if it corresponds to exact the 1-form

$$\beta(X_h) + dh \wedge \beta(W) = 0 \quad \forall \beta \in \Omega^1(M).$$

According to Liouville's theorem such a vector field preserves bivector field $W$ satisfying $[X_h, W] = 0$. Now let us consider the one-parameter group of transformations $g(a) = e^{aL_E}$ generated by the vector field $E$. When $W$ is nondegenerate, it is easy to show that every vector field $E$ satisfying $[E, X_h] = 0$ generates the symmetry of Hamilton's equation (maps the space of solutions onto itself). In the case of degenerate $W$ there exists a subspace of 1-forms $A$ such that

$$\alpha \wedge \beta(W) = 0 \quad \forall \alpha \in A \quad \forall \beta \in \Omega^1(M).$$

This set of 1-forms gives rise to the set of constraints

$$\alpha \wedge dh(W) = 0 \quad \forall \alpha \in A.$$

These constraints will be preserved if $A$ is an invariant subspace under the action of $g$, in other words, if $L_E\alpha \in A\ \forall \alpha \in A$. So if the vector field $E$ satisfies

$$[E, X_h] = 0 \quad \text{and} \quad L_E\alpha \in A \quad \forall \alpha \in A,$$

then it generates the symmetry of Hamilton's equation. Now let us consider the correspondence between such symmetries and the conservation laws.

**Theorem.** *If the non-Hamiltonian vector field $E$ generates the symmetry of Hamilton's equation, then the set of functions $I^{(k)} = i_{W^k}\omega_E^k$ is conserved.*

Here $\omega$ is the 2-form defined by $W^k(\omega) = W^{k-1}$ and $\omega_E = L_E\omega$; such a form always exists, but in the case of degenerate $W$ it is far from being unique.

*Proof.* Let us consider $I^{(1)}$. We have to prove that $L_{X_h} I^{(1)} = 0$ . Indeed,

$$L_{X_h} I^{(1)} = L_{X_h} i_W \omega_E = i_{[X_h,W]} \omega_E + i_W L_{X_h} \omega_E.$$

In this expression the first term vanishes since according to Liouville's theorem $[X_h, W] = 0$. Using $[E, X_h] = 0$, the second term can be rewritten as $i_W L_{X_h} L_E \omega = i_W L_E L_{X_h} \omega$. Now from the definition of $\omega$ one can show that $L_{X_h} \omega(W^k) = 0$. This means that $L_{X_h} \omega$ can be expressed in the following form:

$$L_{X_h} \omega = \sum_i \alpha_i \wedge \beta_i, \quad \alpha_i \in A, \quad \beta_i \in \Omega^{(1)}(M).$$

As far as $A$ is invariant under the action of $E$ we have

$$i_W L_E L_{X_h} \omega = i_W L_E \sum_i \alpha_i \wedge \beta_i = \sum_i i_W L_E(\alpha_i \wedge \beta_i) = \sum_i i_W (L_E(\alpha_i) \wedge \beta_i)$$
$$- \sum_i i_W (\alpha_i \wedge L_E(\beta_i)) = \sum_i i_W \tilde{\alpha}_i \wedge \beta_i - \sum_i i_W \alpha_i \wedge \tilde{\beta}_i = 0$$

because of $\alpha, \tilde{\alpha} \in A$. The proof of $L_{X_h} I^{(k)} = 0$ is similar. □

*Note* 1. The 2-form $\omega$ is far from being unique, but if $\omega_1$ and $\omega_2$ both satisfy $W^k(\omega) = W^{k-1}$, then $\omega_1 - \omega_2$ can be expressed as $L_{X_h} \omega = \sum_i \alpha_i \wedge \beta_i$, $\alpha_i \in A$, $\beta_i \in \Omega^{(1)}(M)$, and therefore it does not contribute in $I^{(k)}$.

*Note* 2. The theorem is valid for a larger class of generators $E$ satisfying

$$[E, X_h] = 0 \quad \text{and} \quad L_E \alpha \in A \quad \forall \alpha \in A.$$

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