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OPTIMAL MEAN-VARIANCE ROBUST HEDGING UNDER ASSET PRICE MODEL MISSPECIFICATION

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Abstract. The problem of constructing robust optimal in the mean-variance sense trading strategies is considered. The approach based on the notion of sensitivity of a risk functional of the problem w.r.t. small perturbation of asset price model parameters is suggested. The optimal mean-variance robust trading strategies are constructed for one-dimensional diffusion models with misspecified volatility.

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1. Introduction and statement of the problem. Let $(\Omega, \mathcal{F}, F, P)$ be a filtered probability space with a fitration $F = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions, where $T \in (0,\infty]$ is a fixed time horizon.

Let for each $\varepsilon > 0$

$$\Lambda_\varepsilon := \Big\{\lambda :\ \lambda = \lambda^0 + \varepsilon h,\ h \in \mathcal{H}\Big\}, \tag{1}$$

where λ^0 is a fixed F-predictable vector or matrix valued process satisfying some additional integrability conditions (see, e.g., (7), (9), (18) below),

$$\mathcal{H} := \Big\{h :\ h \text{ bounded},\ F\text{-predictable, vector or matrix valued process},\\ h \in \mathrm{Ball}_L(0,R),\ 0 < R < +\infty\Big\}. \tag{2}$$

Here $\mathrm{Ball}_L(0,R)$ denotes the closed R-radius ball of processes h in an appropriate metric space L with center at the origin.

The class $\mathcal{H}$ is called the class of alternatives.

Let X^λ, $\lambda \in \Lambda_\varepsilon$, be a continuous R^d-valued semimartingale describing the misspecified discounted price of a risky asset (stock) in a frictionless financial market. A contingent claim is an $\mathcal{F}_T$-measurable square-integrable random variable (r.v.) H, and a trading strategy θ is a F-predictable R^d-valued process such that the stochastic integral $G(\lambda,\theta) := \int \theta\, dX^\lambda$, $\lambda \in \Lambda_\varepsilon$, is a well-defined real-valued square-integrable semimartingale.

Intuitively, H models the payoff from a financial product one is interested in and for each λ, $G(\lambda,\theta)$ describes the trading gains induced by the self-financing

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portfolio strategy associated with θ when the asset price process follows the semimartingale X^λ.

For each $\lambda \in \Lambda_\varepsilon$, the total loss of a hedger, who srarts with the initial capital x, uses the strategy θ, believes that the stock price process follows X^λ and has to pay a random amount H at the date T, is thus $H - x - G_T(\lambda, \theta)$.

The robust mean-variance hedging means solving the optimization problem

$$\text{minimize} \quad \sup_{\lambda \in \Lambda_\varepsilon} E\Big(H - x - G_T(\lambda, \theta)\Big)^2 \quad \text{over all strategies} \quad \theta. \tag{3}$$

Denote by $J(\lambda, \theta)$ the risk functional of problem (3),

$$J(\lambda, \theta) := E\Big(H - x - G_T(\lambda, \theta)\Big)^2,$$

and consider the following approximation (which is common in the robust statistics theory, see, e.g., [1], [2]):

$$\begin{aligned}
\sup_{\lambda \in \Lambda_\varepsilon} J(\lambda, \theta) &= \exp\Big\{ \sup_{h \in \mathcal{H}} \ln J(\lambda^0 + \varepsilon h; \theta) \Big\} \\
&\simeq \exp\left\{ \sup_{h \in \mathcal{H}} \left[\ln J(\lambda^0, \theta) + \varepsilon \, \frac{DJ(\lambda^0, h; \theta)}{J(\lambda^0, \theta)} \right] \right\} \\
&= J(\lambda^0, \theta) \exp\left\{ \varepsilon \, \frac{\sup_{h \in \mathcal{H}} DJ(\lambda^0, h; \theta)}{J(\lambda^0, \theta)} \right\},
\end{aligned}$$

where

$$DJ(\lambda^0, h; \theta) := \frac{d}{d\varepsilon} \, J(\lambda^0 + \varepsilon h; \theta) \Big|_{\varepsilon = 0} = \lim_{\varepsilon \to 0} \frac{J(\lambda^0 + \varepsilon h; \theta) - J(\lambda^0, \theta)}{\varepsilon} \,, \tag{4}$$

is the Gateaux differential of the functional J at the point λ^0 in the direction h.

Our approach consists in approximating (in the leading order ε) the optimization problem (3) by the problem

$$\text{minimize} \quad J(\lambda^0, \theta) \exp\left\{ \varepsilon \, \frac{\sup_{h \in \mathcal{H}} DJ(\lambda^0, h; \theta)}{J(\lambda^0, \theta)} \right\} \quad \text{over all strategies} \quad \theta, \tag{5}$$

and note that every solution θ^* of problem (5) minimizes $J(\lambda^0, \theta)$ under the constraint

$$\frac{\sup_{h \in \mathcal{H}} DJ(\lambda^0, h; \theta)}{J(\lambda^0, \theta)} \leq k := \frac{\sup_{h \in \mathcal{H}} DJ(\lambda^0, h; \theta^*)}{J(\lambda^0, \theta^*)} \,.$$

This gives a characterization of an optimal strategy θ^* of problem (5), and thus leads to

Definition 1. The trading strategy θ^* is called optimal mean-variance robust against the class of alternatives $\mathcal{H}$ if it is a solution of the optimization problem

$$\text{minimize } \; J(\lambda^0,\theta) \; \text{ over all strategies } \; \theta, \; \text{ subject to constraint}$$
$$\frac{\sup\limits_{h\in\mathcal{H}} DJ(\lambda^0,h;\theta)}{J(\lambda^0,\theta)} \le c \tag{6}$$

(c is some general constant).

In the present paper we consider first a simple diffusion model with zero drift and show (see the Proposition in Section 2) that the solutions of problems (3) and (6) coincide. Then we pass to a more complicated diffusion model with nonzero drift and a deterministic mean-variance tradeoff process and solve the optimization problem (6) which will be at the same time an approximation (in leading order ε) solution of problem (3) (see the Theorem in Section 3).

The consideration of misspecified asset price models was initiated by Avellaneda et al. [3], Avelaneda and Paras [4], and Avellaneda and Lewicki [5]. They obtained pricing and hedging bounds in markets with bounds on uncertain volatility. El Karoui et al. [6] investigate the robustness of the Black–Scholes formula, C. Gallus [7] give an estimate of the variance of additional costs at maturity if the hedger uses the classical Black–Scholes strategy, but the volatility is uncertain. H. Ahn et al. [8] consider the Black–Scholes model with misspecified volatility of the form $\widetilde{\sigma}^2=\sigma^2+\delta S(t,x)$, $|S(t,x)|\le 1$. The trading strategies are also the Black–Scholes ones, and the risk functional is an expected exponencial utility. Based on Feynman–Kac formula, they write the partial differential equation for the corresponding optimization problem whose solution cannot be obtained in explicit form. Instead, they find an approximate solution in the functional form.

2. Diffusion model with zero drift. Let a standard Wiener process $w=(w_t)_{0\le t\le T}$ be given on the complete probability space $(\Omega,\mathcal{F},P)$. Denote by $F^w=(\mathcal{F}_t^w,\,0\le t\le T)$ the P-augmentation of the natural filtration $\mathcal{F}_t^w=\sigma(w_s,\,0\le s\le t)$, $0\le t\le T$, generated by w.

Let the stock price process be modeled by the equation

$$dX_t^\lambda = X_t^\lambda\cdot\lambda_t\,dw_t,\quad X_0^\lambda>0,\quad 0\le t\le T, \tag{7}$$

where $\lambda\in\Lambda_\varepsilon$, see (1), (2), with

$$\int_0^T(\lambda_t^0)^2\,dt<\infty,$$

P-a.s., and $h\in\mathrm{Ball}_{L_\infty(dt\times dP)}(0,R)$, $0<R<\infty$. All considered processes are real-valued.

Denote by R^λ the yield process, i.e.,

$$dR_t^\lambda=\lambda_t\,dw_t,\quad R_0=0,\quad 0\le t\le T, \tag{8}$$

and let $\theta = (\theta_t)_{0\le t\le T}$ be the dollar amount invested in the stock X^λ.

Define the class of admissible strategies $\Theta = \Theta(\Lambda_\varepsilon) = \Theta(\lambda^0, \mathcal{H})$.

Definition 2. A class of admissible strategies $\Theta = \Theta(\lambda^0, \mathcal{H})$ is a class of F^w-predictable real-valued processes $\theta = (\theta_t)_{0\le t\le T}$ such that

$$E\int_0^T \theta_t^2 (\lambda_t^0)^2\,dt < \infty, \quad E\int_0^T \theta_t^2 h_t^2\,dt < \infty, \;\; \forall h \in \mathcal{H},$$

or, equivalently,

$$E\int_0^T \theta_t^2 (\lambda_t^0)^2\,dt < \infty, \quad E\int_0^T \theta_t^2\,dt < \infty. \tag{9}$$

The corresponding gain process has the form

$$G_t(\lambda,\theta) = \int_0^t \theta_s\,dR_s^\lambda, \;\; 0 \le t \le T, \tag{10}$$

(recall that θ is the dollar amount invested in the risky asset rather than the number of shares). Evidently, $G_T(\lambda,\theta) \in L^2(P)$ for each $\lambda \in \Lambda_\varepsilon$. The wealth at maturity T, with the initial endowment x, is equal to

$$V_T^{x,\theta}(\lambda) = x + \int_0^T \theta_t\,dR_t^\lambda.$$

Let, further, the contingent claim H be $\mathcal{F}_T^w$-measurable P-square-integrable r.v.

For simplicity, we suppose that a risk-free interest rate $r \equiv 0$; hence the corresponding bond price $B_t \equiv 1$, $0 \le t \le T$.

Consider the optimization problem (3). It is easy to see that if $\lambda \in \Lambda_\varepsilon$; then

$$\lambda_t^0 - \varepsilon R \le \lambda_t \le \lambda_t^0 + \varepsilon R, \;\; 0 \le t \le T, \;\; P\text{-a.s.}$$

By the martingale representation theorem

$$H = EH + \int_0^T \varphi_t^H\,dw_t, \;\; P\text{-a.s.}, \tag{11}$$

where φ^H is the F^w-predictable process with

$$E\int_0^T (\varphi_t^H)^2\,dt < \infty. \tag{12}$$

Hence

$$E\Big(H - V_T^{x,\theta}(\lambda)\Big)^2 = (EH - x)^2 + E\int_0^T (\varphi_t^H - \lambda_t\theta_t)^2\,dt.$$

From this it directly follows that the process

$$\lambda_t^*(\theta) = (\lambda_t^0 - \varepsilon R) I_{\{\frac{\varphi_t^H}{\theta_t} \geq \lambda_t^0\}} I_{\{\theta_t \neq 0\}} + (\lambda_t^0 + \varepsilon R) I_{\{\frac{\varphi_t^H}{\theta_t} < \lambda_t^0\}} I_{\{\theta_t \neq 0\}}, \quad 0 \leq t \leq T, \tag{13}$$

is a solution of the optimization problem

$$\text{maximize } E\Big(H - V_T^{x,\theta}(\lambda)\Big)^2 \text{ over all } \lambda \in \Lambda_\varepsilon, \text{ with a given } \theta \in \Theta.$$

It remains to minimize (w.r.t. θ) the expression

$$E\int_0^T \Big(\varphi_t^H - \lambda_t^*(\theta)\theta_t\Big)^2 dt.$$

From (13) it easily follows that the equation (w.r.t. θ)

$$\varphi_t^H - \lambda_t^*(\theta)\theta_t = 0,$$

has no solution, but

$$\theta_t^* = \frac{\varphi_t^H}{\lambda_t^0} I_{\{\lambda_t^0 \neq 0\}}, \quad 0 \leq t \leq T, \tag{14}$$

solves problem (3). We assume that $0/0 := 0$.

Consider now the optimization problem (6).

For each fixed h

$$\begin{aligned} J(\lambda,\theta) &= E\Big(H - x - \int_0^T \theta_t \, dR_t^\lambda\Big)^2 \\ &= E\Big(H - x - \int_0^T \theta_t \lambda_t^0 \, dw_t - \varepsilon \int_0^T \theta_t h_t \, dw_t\Big)^2 \\ &= J(\lambda^0,\theta) - 2\varepsilon E\Big[\Big(EH - x + \int_0^T \Big(\varphi_t^H - \theta_t\lambda_t^0\Big) dw_t\Big) \int_0^T \theta_t h_t \, dw_t\Big] \\ &+ \varepsilon^2 E\int_0^T \theta_t^2 h_t^2 \, dt, \end{aligned}$$

and hence

$$DJ(\lambda^0, h; \theta) = 2E\int_0^T \Big(\theta_t\lambda_t^0 - \varphi_t^H\Big)\theta_t h_t \, dt, \tag{15}$$

as follows from (9), (12), the definition of the class $\mathcal{H}$ and the estimation

$$\Big(E\int_0^T \big(\theta_t\lambda_t^0-\varphi_t^H\big)\theta_t h_t\,dt\Big)^2 \le E\int_0^T \big(\theta_t\lambda_t^0-\theta_t^H\big)^2\,dt\;E\int_0^T \theta_t^2 h_t^2\,dt$$

$$\le \text{const}\cdot R^2\Big(E\int_0^T \theta_t^2(\lambda_t^0)^2\,dt+E\int_0^T(\varphi_t^H)^2\,dt\Big)E\int_0^T\theta_t^2\,dt<\infty. \tag{16}$$

Since, further, $DJ(\lambda^0,h;\theta)=0$ for $h\equiv 0$, using (16) we get

$$0\le \sup_{h\in\mathcal{H}} DJ(\lambda^0,h;\theta)<\infty.$$

Hence we can take $0\le c<\infty$ in problem (6). Now if we substitute θ^* from (14) into (15), we get $DJ(\lambda^0,h;\theta^*)=0$ for each h, and thus

$$\frac{\sup\limits_{h\in\mathcal{H}} DJ(\lambda^0,h;\theta^*)}{J(\lambda^0,\theta^*)}=0.$$

If we recall that $\theta^*=\arg\min\limits_{\theta\in\Theta_{\Lambda_\varepsilon}} J(\lambda^0,\theta)$, we get that θ^* defined by (14) is a solution of the optimization problem (6) as well.

Thus we prove the following

Proposition. *In scheme* (7), (8) *under assumptions* (9)*:*

(a) *the optimal mean-variance robust trading strategy* $\theta^*=(\theta_t^*)_{0\le t\le T}$ *for the optimization problem* (6) *is given by the formula*

$$\theta_t^*=\frac{\varphi_t^H}{\lambda_t^0}\,I_{\{\lambda_t^0\ne 0\}};$$

(b) *this strategy is an approximation* (*in leading order* ε) *strategy for the optimization problem* (3) *and coincides with the exact optimal strategy of this problem.*

3. Diffusion model with nonzero drift. Let us consider the filtered probability space $(\Omega,\mathcal{F},F^w=(\mathcal{F}_t^w)_{0\le t\le T},P)$ with a given standard Wiener process $w=(w_t,\mathcal{F}_t^w)$, $0\le t\le T$, and a given P-square-integrable $\mathcal{F}_T^w$-measurable r.v. H. Let the stock price process be defined by the equation

$$dX_t^\lambda=X_t^\lambda\big(\mu_t\,dt+\lambda_t dw_t\big),\quad X_0^\lambda>0, \tag{17}$$

where

$$\mu_t=k_t\lambda_t,\quad 0\le t\le T, \tag{18}$$

and $k=(k_t)_{0\le t\le T}$ is a bounded deterministic function, $\lambda=(\lambda_t)_{0\le t\le T}\in\Lambda_\varepsilon$, i.e.,

$$\lambda_t=\lambda_t^0+\varepsilon h_t,$$

where λ^0 is an F^w-predictable process with $\int\limits_0^T (\lambda_t^0)^2\,dt < \infty$, P-a.s. $h \in \mathcal{H}$, with $L = L_\infty(dt \times dP)$, i.e., h is a bounded F-pedictable process, $h \in \mathrm{Ball}_{L_\infty(dt\times dP)}(0,R)$, $0 < R < +\infty$. All processes in (17), (18) are real-valued.

Consider the optimization problem (6). Denote, as in the previous section, by R^λ the yield process defined by the equation

$$dR_t^\lambda = \lambda_t(k_t dt + dw_t), \quad R_0^\lambda = 0, \quad 0 \le t \le T, \tag{19}$$

and for each $\theta = (\theta_t)_{0 \le t \le T} \in \Theta(\lambda^0, \mathcal{H})$ (see Definition 2) introduce the risk functional of the problem

$$J(\lambda, \theta) = E\Big(H - x - \int\limits_0^T \theta_t\,dR_t^\lambda\Big)^2.$$

Note that since the mean-variance tradeoff process $(\int\limits_0^t k_s^2\,ds)_{0 \le t \le T}$ is continuous and bounded, the space $\{G_T(\lambda, \theta) :\ \theta \in \Theta\}$ is closed in $L^2(P)$ for each $\lambda \in \Lambda_\varepsilon$, see, e.g., Corollary 4 of [9]. Further, there exists a unique equivalent martingale measure (which does not depend on the parameter λ) given by the relation

$$d\widetilde{P} = \widetilde{z}_T\,dP,$$

where $\widetilde{z}_T = \mathcal{E}_T(-k.w)$, $\widetilde{z}_T > 0$, $E\widetilde{z}_T = 1$, $\mathcal{E}_t(-k.w)$ is the Dolean exponential of the martingale $-k.w_t = -\int\limits_0^t k_s\,dw_s$, $0 \le t \le T$. Moreover, the process $G(\lambda, \theta)$ belongs to S^2, the set of square integrable semimartingales, for each $\lambda \in \Lambda_\varepsilon$.

Now, following [10] and [11], introduce the objects: $\widetilde{z}_t = E^{\widetilde{P}}(\widetilde{z}_T/\mathcal{F}_t^w)$, $0 \le t \le T$, $d\widetilde{Q} = \frac{\widetilde{z}_T}{\widetilde{z}_0}\,d\widetilde{P}$ (and thus $d\widetilde{Q} = \frac{\widetilde{z}_T^2}{\widetilde{z}_0}\,dP$). Since $\widetilde{z} > 0$ is a strictly positive $\widetilde{P}$-martingale, $\widetilde{Q}$ is a probability measure with $\widetilde{Q} \approx P$.

Introduce, further, the process

$$w_t^0 = w_t + \int\limits_0^t k_s\,ds, \quad 0 \le t \le T.$$

Then $\mathcal{F}_t^{w^0} = \mathcal{F}_t^w$, $0 \le t \le T$, because k is deterministic, and hence the process $w^0 = (w_t^0)_{0 \le t \le T}$ is a standard $(\widetilde{P}, F^w)$-Wiener process. Consider now the new filtered probability space $(\Omega, \mathcal{F}, F^w, \widetilde{P})$, rewrite the process R^λ in the form

$$dR_t^\lambda = \lambda_t\,dw_t^0, \quad R_0^\lambda = 0, \quad 0 \le t \le T, \tag{20}$$

and decompose the r.v. $\widetilde{z}_T$ w.r.t. w^0:

$$\widetilde{z}_T = \widetilde{z}_0 + \int\limits_0^T \zeta_t\,dw_t^0. \tag{21}$$

In this notation, based on Proposition 5.1 of [10], we can write

$$\begin{aligned} J(\lambda,\theta) &= E\,\frac{\widetilde{z}_T^2}{\widetilde{z}_0^2}\,\frac{\widetilde{z}_0^2}{\widetilde{z}_T^2}\Big(H-x-\int_0^T \theta_t\,dR_t^\lambda\Big)^2 \\ &= \widetilde{z}_0^{-1}E^{\widetilde{Q}}\frac{\widetilde{z}_0^2}{\widetilde{z}_T^2}\Big(H-x-\int_0^T \theta_t\lambda_t\,dw_t^0\Big)^2 \\ &= \widetilde{z}_0^{-1}E^{\widetilde{Q}}\bigg(\frac{H}{\widetilde{z}_T}\,\widetilde{z}_0 - x - \int_0^T \psi_t^0(\lambda)\,d\frac{\widetilde{z}_0}{\widetilde{z}_t} - \int_0^T \psi_t^1(\lambda)\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\bigg)^2 \\ &:= J(\lambda,\psi^0,\psi^1). \end{aligned} \tag{22}$$

Here

$$\psi_t^1(\lambda)=\theta_t\lambda_t, \quad \psi_t^0(\lambda)=x+\int_0^t \theta_s\lambda_s\,dw_s^0-\theta_t\lambda_t w_t^0, \;\; 0\le t\le T.$$

Thus

$$\psi_t^1(\lambda)=\psi_t^1(\lambda^0)+\varepsilon\psi_t^1(h), \quad \psi_t^0(\lambda)=\psi_t^0(\lambda^0)+\varepsilon\overline{\psi}_t^{\,0}(h), \tag{23}$$

where $\overline{\psi}_t^{\,0}(h)=\psi_t^0(h)-x$.

If now

$$\frac{H}{\widetilde{z}_T}\,\widetilde{z}_0 = E^{\widetilde{Q}}\Big(\frac{H}{\widetilde{z}_T}\,\widetilde{z}_0\Big)+\int_0^T \psi_t^{0,H}\,d\frac{\widetilde{z}_0}{\widetilde{z}_t}+\int_0^T \psi_t^{1,H}\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0 \tag{24}$$

is the Galtchouk–Kunita–Watanabe decomposition of the r.v. $\frac{H}{\widetilde{z}_T}\,\widetilde{z}_0$ w.r.t. $(\widetilde{Q},F^w)$-local martingales $\frac{\widetilde{z}_0}{\widetilde{z}_t}$ and $\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0$, then, using (22), (23) and (24), we get for each fixed h

$$\begin{aligned} J(\lambda,\psi^0,\psi^1) &= J\Big(\lambda^0,\psi^0(\lambda^0),\psi^1(\lambda^0)\Big)+\varepsilon 2\widetilde{z}_0^{-1}E^{\widetilde{Q}}\bigg\{\bigg[\Big(x-E^{\widetilde{Q}}\Big(\frac{H}{\widetilde{z}_T}\,\widetilde{z}_0\Big)\Big) \\ &\quad + \int_0^T \Big(\psi_t^0(\lambda^0)-\psi_t^{0,H}\Big)\,d\frac{\widetilde{z}_0}{\widetilde{z}_t}+\int_0^T \Big(\psi_t^1(\lambda^0)-\psi_t^{1,H}(\lambda^0)\Big)\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\bigg] \\ &\quad \times\Big(\int_0^T \overline{\psi}_t^{\,0}(h)\,d\frac{\widetilde{z}_0}{\widetilde{z}_t}+\int_0^T \psi_t^1(h)\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\Big)\bigg\} \\ &\quad + \varepsilon^2\widetilde{z}_0^{-1}E^{\widetilde{Q}}\bigg(\int_0^T \overline{\psi}_t^{\,0}(h)\,d\frac{\widetilde{z}_0}{\widetilde{z}_t}+\int_0^T \psi_t^1(h)\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\bigg)^2. \end{aligned} \tag{25}$$

Consequently,

$$DJ(\lambda^0,h;\psi^0,\psi^1)=2\widetilde{z}_0^{\,-1}\bigg\{E^{\widetilde{Q}}\int_0^T\Big(\psi_t^0(\lambda^0)-\psi_t^{0,H}\Big)\overline{\psi}_t^{\,0}(h)\,d\Big\langle\frac{\widetilde{z}_0}{\widetilde{z}_t}\Big\rangle$$

$$+E^{\widetilde{Q}}\int_0^T\Big[\Big(\psi_t^1(\lambda^0)-\psi_t^{1,H}\Big)\overline{\psi}_t^{\,0}(h)+\Big(\psi_t^0(\lambda^0)-\psi_t^{0,H}\Big)\psi_t^1(h)\Big]\,d\Big\langle\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0,\frac{\widetilde{z}_0}{\widetilde{z}_t}\Big\rangle$$

$$+E^{\widetilde{Q}}\int_0^t\Big(\psi_t^1(\lambda^0)-\psi_t^{1,H}\Big)\psi_t^1(h)\,d\Big\langle\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\Big\rangle\bigg\}. \tag{26}$$

From the definition of $\psi^1(h)$ and $\overline{\psi}^{\,0}(h)$ (see (23)) it follows that for $h\equiv 0$, $\psi^1(h)=0$ and $\overline{\psi}^{\,0}(h)=0$. Hence

$$DJ(\lambda^0,0;\psi^0,\psi^1)=0,$$

and thus

$$\sup_{h\in\mathcal{H}}DJ(\lambda^0,h;\psi^0,\psi^1)\geq 0. \tag{27}$$

We show now that

$$\sup_{h\in\mathcal{H}}DJ(\lambda^0,h;\psi^0,\psi^1)<\infty. \tag{28}$$

For this it is sufficient to estimate the expression (as it easily follows from (25))

$$I=E^{\widetilde{Q}}\bigg(\int_0^T\overline{\psi}_t^{\,0}(h)\,d\frac{\widetilde{z}_0}{\widetilde{z}_t}+\int_0^T\psi^1(h)\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}^0\bigg)^2.$$

But using Proposition 8 of [11], we have for each h

$$\frac{\widetilde{z}_0}{\widetilde{z}_t}\,G_T\Big(h,\Theta(0,\mathcal{H})\Big)$$

$$=\bigg\{\int_0^T\overline{\psi}_t^{\,0}(h)\,d\frac{\widetilde{z}_0}{\widetilde{z}_t}+\int_0^T\psi^1(h)\,d\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\,\bigg|\ \ \overline{\psi}^{\,0}(h),\ \psi^1(h)\in L^2\Big(\frac{\widetilde{z}_0}{\widetilde{z}},\frac{w^0}{\widetilde{z}}\,\widetilde{z}_0,\widetilde{Q}\Big)\bigg\},$$

where $L^2(\frac{\widetilde{z}_0}{\widetilde{z}},\frac{w^0}{\widetilde{z}}\,\widetilde{z}_0,\widetilde{Q})$ is the space of F^w-predictable processes (ψ^0,ψ^1) such that $\int\psi^0\,d\frac{\widetilde{z}_0}{\widetilde{z}}+\int\psi^1\,d\frac{w^0}{\widetilde{z}}\,\widetilde{z}_0$ is in the space $\mathcal{M}^2(\widetilde{Q},F^w)$ of martingales.

Hence, using notation (10), we have

$$I=E^{\widetilde{Q}}\frac{\widetilde{z}_0^2}{\widetilde{z}_T^2}\,G_T^2(h,\theta)=\widetilde{z}_0EG_T^2(h,\theta)=\widetilde{z}_0E\bigg(\int_0^T\theta_t\,dR_t^h\bigg)^2$$

$$=\widetilde{z}_0E\bigg(\int_0^T\theta_th_t(k_t\,dt+dw_t)\bigg)^2$$

$$\le \widetilde{z}_0 \operatorname{const}\cdot\biggl(E\int_0^T \theta_t^2 h_t^2 k_t^2\,dt + E\int_0^T \theta_t^2 h_t^2\,dt\biggr)$$
$$\le \widetilde{z}_0 \operatorname{const}\cdot R^2\biggl(E\int_0^T \theta_t^2\,dt\biggr) < \infty,$$

by the definitions of the classes Θ, $\mathcal{H}$, and the boundedness of the function $k = (k_t)_{0\le t\le T}$.

From (27) and (28), as in the previous section, it follows that we can take $0 \le c < \infty$ in problem (6).

Now if we substitute $\psi^{1,*}(\lambda^0) := \psi^{1,H}$ and $\psi^{0,*}(\lambda^0) := \psi^{0,H}$ into $J(\lambda^0,\psi^0,\psi^1)$ and $DJ(\lambda^0,h,\psi^0,\psi^1)$, we get

$$J\Bigl(\lambda^0,\psi^{0,*},\psi^{1,*}\Bigr) = \min_{\psi^0,\psi^1} J(\lambda^0,\psi^0,\psi^1)$$

(see Lemma 5.1 of [10]), and

$$\sup_{h\in\mathcal{H}} \frac{DJ(\lambda^0,h;\psi^{0,*},\psi^{1,*})}{J(\lambda^0,\psi^{0,*},\psi^{1,*})} = 0$$

(hence the constraint of problem (6) is satisfied).

Consequently, using Proposition 8 of [11], we arrive at the following

Theorem. *In model* (17)–(19) *the optimal mean-variance robust trading strategy* (*in the sense of Definition* 1) *is given by the formula*

$$\theta_t^* = \biggl[\frac{\psi_t^{1,H}}{\lambda_t^0} + \frac{\zeta_t}{\lambda_t^0}\Bigl(V_t^* - \psi_t^{0,H}\,\frac{\widetilde{z}_0}{\widetilde{z}_t} - \psi_t^{1,H}\,\frac{w_t^0}{\widetilde{z}_t}\,\widetilde{z}_0\Bigr)\biggr] I_{\{\lambda_t^0\ne 0\}},\quad 0\le t\le T,$$

where

$$V_t^* = \frac{\widetilde{z}_0}{\widetilde{z}_t}\biggl(x + \int_0^t \psi_s^{0,H}\,d\frac{\widetilde{z}_0}{\widetilde{z}_s} + \int_0^t \psi_s^{1,H}\,d\frac{w_s^0}{\widetilde{z}_s}\,\widetilde{z}_0\biggr),$$

$\psi^{0,H}$ *and* $\psi^{1,H}$ *are given by relation* (24), ζ_t *is defined in* (21).

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