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HILBERT SPACES FORMED BY STRONGLY HARMONIZABLE STABLE PROCESSES

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Abstract. A strongly harmonizable continuous time symmetric α-stable process is considered. By using covariations, a Hilbert space is formed from the process elements and used for a purpose of moving average representation and prediction.

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1. Introduction

Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a strongly harmonizable symmetric α-stable process, $1 < \alpha \leq 2$, SH(SαS)P. Then $X(t)$ is the Fourier transform of a SαS random measure with independent increments Φ,

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\Phi(d\lambda). \tag{1.1}$$

Then $f(\lambda) = \frac{\|\Phi(d\lambda)\|_\alpha^\alpha}{d\lambda}$, where $\|\cdot\|_\alpha$ is the Schilder's norm, defines the spectral density of the process [2]. The closed linear span of $X(t)$, $t \in \mathbf{R}$, under $\|\cdot\|_\alpha$, denoted by $(\mathcal{A}, \|\cdot\|_\alpha)$, forms the time domain of the process which is a Banach space of jointly SαS random variables. The spectral domain is $L^\alpha(f)$. Since 1984 this space has been used intensively to explore Banach space techniques for time series analysis of the process, [3], [12], [9], [6] among others. In most of the situations the methods are different from those for the Gaussian processes, which rely on the geometry of a Hilbert space and the properties of inner products. To make some of the Gaussian techniques accessible for stable processes, it is natural to raise a question if it is possible to construct a Hilbert space by the elements of the SH(SαS)P. In this paper we provide an affirmative answer to this question.

2. Hilbert Space

Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a strongly harmonizable SαS process given by (1.1) with spectral density f. Also let $\mathcal{S}_0$ be the linear span of the elements of

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the set $\{X(t),\ t \in \mathbf{R}\}$. For $Y_1 = \sum_{l=1}^{n} d_l X(t_l)$ and $Y_2 = \sum_{j=1}^{m} b_j X(s_j)$ in $\mathcal{S}_0$ define

$$\begin{aligned}\langle Y_1, Y_2\rangle &= \sum_{l,j} d_l b_j^* [X(t_l), X(s_j)]_\alpha \\ &= 2\pi \sum_{l,j} d_l b_j^* f^\vee(s_j - t_l),\end{aligned}$$

where $f^\vee(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-itu}du$ and $*$ stands for the complex conjugate, also $\widehat{f}(u) = \int_{-\infty}^{\infty} f(t)e^{itu}dt$. Then $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{S}_0$. The space $(\mathcal{S}_0, \langle\cdot,\cdot\rangle)$ is not complete, but theoretically it has a completion in the form of Theorem 7.4.9 from [14]. Theorem 2.1 given below specifies the completion of $(\mathcal{S}_0, \langle\cdot,\cdot\rangle)$ for regular SH(SαS)P. A stable process $X = \{X(t),\ t \in \mathbf{R}\}$ is called regular if

$$\bigcap_{s\le 0} \overline{\mathrm{sp}}\{X(t),\ t \le s\} = 0,$$

where $\overline{\mathrm{sp}}$, the span closure, is taken in $(\mathcal{A}, \|\cdot\|_\alpha)$. The SH(S$\alpha$S) regular processes were studied in [3] and [8]. Under the regularity assumption the spectral density f exists, stays away from zero so that $\log f \in L^1(\mathbf{R})$, and therefore f can be written as $f = |h_\alpha|^\alpha$, where h_α is an outer function in the Hardy space H^α. Now let $h = h_\alpha^{\alpha/2}$; then h is an outer function of the class H^2 and $f = |h|^2$. Define

$$M(A) = \int_A \frac{1}{h^*} d\Phi, \quad \text{for Borel sets } A. \tag{2.1}$$

Then M possesses the properties of an independently scattered SαS random measure for which for every $g \in L^2$, $\int g dM \in \mathcal{A}$, and there is a universal constant C such that

$$\| \int g dM \|_\alpha \le C\|g\|_{L^2}, \quad g \in L^2. \tag{2.2}$$

The random measure M enables us to specify the completion of $(\mathcal{S}_0, \langle\cdot,\cdot\rangle)$.

Theorem 2.1. *Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a regular SH(SαS)P given by* (1.1), *then the completion of $(\mathcal{S}_0, \langle\cdot,\cdot\rangle)$ denoted by $(\mathcal{S}\langle\cdot,\cdot\rangle_{\mathcal{S}})$, is a Hilbert space of jointly symmetric stable random variables for which:*

(i) *$\mathcal{S} \subset \mathcal{A}$, as a point inclusion;*

(ii) *for every $Y \in \mathcal{S}$, $\|Y\|_\alpha \le C\|Y\|_{\mathcal{S}}$, where C is a constant independent of Y.*

(iii) *$\|Y\|_\alpha$ and $\|Y\|_{\mathcal{S}}$ can be evaluated for $Y \in \mathcal{S}$.*

Proof. [9]. Let $\mathcal{S} = \{\int g dM, g \in L^2\}$, and for $g, k \in L^2$ define

$$\left\langle \int g dM, \int k dM \right\rangle_{\mathcal{S}} = \langle g, k\rangle_{L^2}.$$

Then $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ is the completion of $(\mathcal{S}_0,\ \langle\cdot,\cdot\rangle)$. Indeed,

$$\begin{aligned}\Big\langle \sum_{l=1}^{n} d_l X(t_l), \sum_{j=1}^{m} b_j X(s_j)\Big\rangle_{\mathcal{S}} &= \Big\langle \int \sum_{l=1}^{n} d_l e^{it_l\lambda} h^*(\lambda) dM, \int (\sum_{j=1}^{m} b_j e^{is_j\lambda} h^*(\lambda) dM\Big\rangle_{\mathcal{S}} \\ &= \Big\langle \sum_{l=1}^{n} d_l e^{it_l\lambda} h^*(\lambda), \sum_{j=1}^{m} b_j e^{is_j\lambda} h^*(\lambda)\Big\rangle_{L^2} \\ &= 2\pi \sum_{l,j} d_l b_j^* f^\vee(s_j - t_l) \\ &= \Big\langle \sum_{l=1}^{n} d_l X(t_l), \sum_{j=1}^{m} b_j X(s_j)\Big\rangle.\end{aligned}$$

Clearly, $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ is complete since L^2 is complete. Now suppose, for $Y \in \mathcal{S}$, $\langle Y, X(t)\rangle_{\mathcal{S}} = 0$ for every $t \in \mathbf{R}$. Then $Y = \int g dM$ for $g \in L^2$ and $\int g h e^{-i\lambda t} d\lambda = 0,\ t \in \mathbf{R}$. Therefore $gh = 0$ for a.e. λ. But since h is outer $h \neq 0$ for a.e. λ. Consequently $g = 0$ for a.e. λ and thus $Y = 0$. Hence $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ is the completion of $(\mathcal{S}_0, \langle\cdot,\cdot\rangle)$. The properties (i) and (ii) follow from (2.2). For (iii) note that if $Y \in \mathcal{S}$, then $Y = \int g dM = \int \frac{g}{h^*} d\Phi$; thus $\|Y\|_{\mathcal{S}} = \|g\|_{L^2}$ and $\|Y\|_\alpha = \int |g|^\alpha |h|^{2-\alpha} d\lambda$. $\square$

It is customary to consider $(\mathcal{A},\ \|\cdot\|_\alpha)$ as the time domain of a given a stable process. In the case of stable harmonizable process it is more convenient to work with $(\mathcal{S}, \langle,\rangle_{\mathcal{S}})$ as the later is a Hilbert space, and as will be shown, for prediction the classical L^2-approximation theory can be applied.

It is well known that the classical moving average representation does not exist for X in $\mathcal{A}$ [9], [3]. But a moving average representation against a stable random measure Z, whose Fourier transform has independent increments, exists in $\mathcal{A}$ [9]. Moreover, a moving average representation exists in $\mathcal{S}$, and Z has orthogonal increments in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$.

Theorem 2.2. *Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a regular process, then*

$$X(t) = \int_{-\infty}^{t} h^\vee(t-s) dZ(s), \quad t \in \mathbf{R}, \tag{2.3}$$

in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ and, consequently, in $(\mathcal{A}, \|\cdot\|_\alpha)$, where for bounded Borel sets $A \subset \mathbf{R}$, $Z(A) = \int \widehat{I}_A(\lambda) dM(\lambda)$, where M is the random measure given by (2.1). Furthermore, $\langle Z(A), Z(B)\rangle_{\mathcal{S}} = 0$, $A \cap B = \phi$, and $\mathcal{S}_t(X) = \mathcal{S}_t(\Delta Z)$, $t \in \mathbf{R}$, where $\mathcal{S}_t(X) = \overline{\mathrm{sp}}\{X(s),\ s \le t\}$ in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ and $\mathcal{S}_t(\Delta Z) = \overline{\mathrm{sp}}\{Z(A):\ A \subset (-\infty, t]$ and A is bounded$\}$.

Proof. (2.3) is given in [9]. For the properties of Z note that $\langle Z(A), Z(B)\rangle_{\mathcal{S}} = \langle \widehat{I}_A, \widehat{I}_B\rangle_{L^2} = 2\pi\langle I_A, I_B\rangle_{L^2} = 0$. The isomorphisms $X(t) \leftrightarrow h^\vee(t-\cdot) \leftrightarrow e^{it\lambda}h^*(\lambda)$ together with the classical Beurling's theorem imply that $\mathcal{S}_t(X) = \mathcal{S}_t(\Delta Z)$ for every $t \in \mathbf{R}$. $\square$

Prediction. The orthogonality in $\mathcal{A}$ is in the sense of the James orthogonality which was clarified in [2]. The linear approximation problems in $\mathcal{A}$ or, equivalently, in $L^\alpha(\nu)$ were established by Rajput and Sundberg [12]. The best linear predictor of $X(t+T)$ based on $\mathcal{A}_t(X) = \overline{\text{sp}}\{X(s),\ s \le t\}$ in $(\mathcal{A}, \|\cdot\|_\alpha)$ is given by

$$\widetilde{X}(t,T) = \int_{-\infty}^{\infty} e^{i(t+T)u} \left\{1 - \frac{(\mathcal{P}_T(h_\alpha^{\alpha/2}))^{2/\alpha}(u)}{h_\alpha(u)}\right\}^* d\Phi(u),$$

where

$$\mathcal{P}_T(g)(z) = \int_0^T e^{iuz} g^\vee(u)du, \quad z \in \mathbf{C}, \quad g \in H^2,$$

and the error is

$$E(T) = \left\{2\pi \int_0^T |h_\alpha^{\alpha/2^\vee}(u)|^2 du\right\}^{\frac{1}{\alpha}} = \left\{2\pi \int_0^T |h^\vee(u)|^2 du\right\}^{\frac{1}{\alpha}}. \tag{2.4}$$

The classical L^2 theory can be applied to obtain $\widehat{X}(t,T)$, the best linear predictor of $X(t+T)$, based on $\mathcal{S}_t(X)$ in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$. Indeed,

$$\widehat{X}(t,T) = \mathbf{P}_{\mathcal{S}_t(X)} X(t+T) = \int_{-\infty}^{\infty} g_{T,t}(\lambda) M(d\lambda),$$

where

$$\begin{aligned}
g_{T,t}(\lambda) &= \mathbf{P}_{\bar{sp}\{e^{is\lambda},\ s\le t\}} e^{i(t+T)\lambda} h^*(\lambda),\\
g_{T,0}(\lambda) &= (\mathbf{P}_{H^2} e^{-iT\lambda} h(\lambda))^* = \left[\int_T^\infty h^\vee(u) e^{-iu\lambda} du\right] e^{iT\lambda},\\
g_{T,t}(\lambda) &= e^{it\lambda} g_{T,0}(\lambda),
\end{aligned}$$

see [5].

Theorem 2.3. *Let $\{X(t),\ t \in \mathbf{R}\}$ be a regular SH(SαS)P given by* (1.1) *with spectral density $f = |h|^2$. Then, in $\mathcal{S}$, the best linear predictor of $X(t+T)$ based on $\{X_s,\ s \le t\}$ is given in the spectral form by*

$$\widehat{X}(t,T) = \int_{-\infty}^{\infty} \frac{1}{h^*(\lambda)} e^{i(t+T)\lambda} \int_T^\infty h^\vee(u) e^{-iu\lambda} du\, d\Phi(\lambda), \tag{2.5}$$

and in the moving average form by

$$\widehat{X}(t,T) = \int_{-\infty}^{t} h^\vee(t+T-u) dZ(u), \tag{2.6}$$

where $\{Z(A)\}$ *is the random measure given in Theorem* 2.2. *The error term is given by*

$$e(T) = \left\{ 2\pi \int_0^T |h^\vee(u)|^2 du \right\}^{1/2}. \tag{2.7}$$

We remark that (2.5) *and* (2.6) *are valid both in* $\mathcal{S}$ *and in* $\mathcal{A}$.

The proof is omitted as it is similar to the Gaussian case presented in [5]. It follows from (2.3) and (2.6) that

$$e^2(T) = E^\alpha(T).$$

The maximum relative deviation of $E(T)$ from $e(T)$ can be specified as

$$e(T) = \left(\int_{-\infty}^{+\infty} f(x)dx \right)^{(\alpha-2)/(2\alpha)} E(T), \quad T \to +\infty.$$

Examples. Let us present the following four spectral densities:

$$\mathbf{A}: \quad f(x) = \left(\frac{1}{x^2 + a^2} \right)^{\alpha/2}, \quad a > 0,$$

$$\mathbf{B}: \quad f(x) = \left(\frac{1}{\sqrt{x^2 + a^2}(x^2 + b^2)} \right)^{\alpha}, \quad a > b > 0,$$

$$\mathbf{C}: \quad f(x) = \frac{1}{(x^2 + a^2)(x^2 + b^2)^{\alpha/2}}, \quad a, b > 0, \quad \beta > \frac{-\alpha + 1}{2},$$

$$\mathbf{D}: \quad f(x) = \frac{1}{(a^2 + x^2)^{\alpha/2}} e^{-2b/(a^2+x^2)}, \quad a, b > 0.$$

The corresponding outer factors together with the Fourier transforms are given in Table 1, where

$$\gamma(w, r) = \int_0^r u^{w-1} e^{-u} du,$$

$${}_1F_1(w; r; y) = \frac{\Gamma(r)}{\Gamma(w)} \sum_{k=0}^{\infty} \frac{\Gamma(w + k)}{\Gamma(r + k)} \frac{y^k}{k!},$$

$$J_w(y) = \sum_{k=1}^{\infty} \frac{(-1)^k \, (\frac{y}{2})^{w+2k}}{k! \, \Gamma(w + k + 1)}$$

are the truncated gamma function, the confluent hypergeometric function and the Bessel function, respectively.

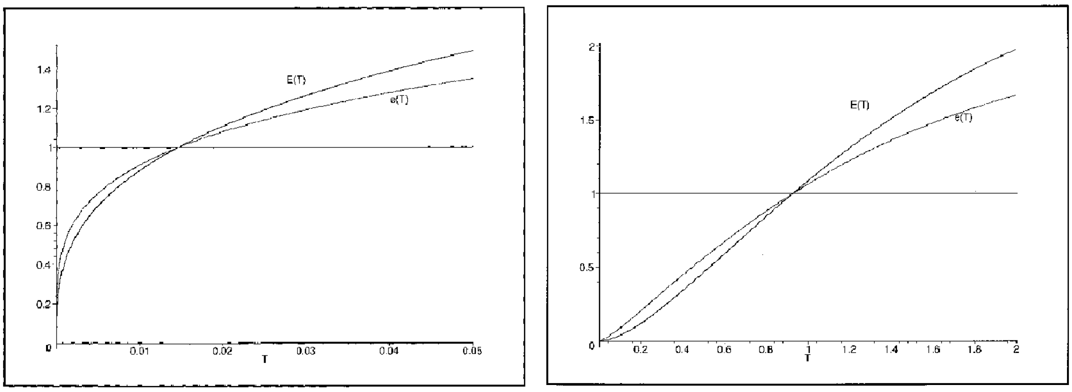

Figure 1: Left: $E(T)$ and $e(T)$ for density **A**, $a = 1$.
Right: $E(T)$ and $e(T)$ for density **B**, $a = 1,\ b = 0.5$.

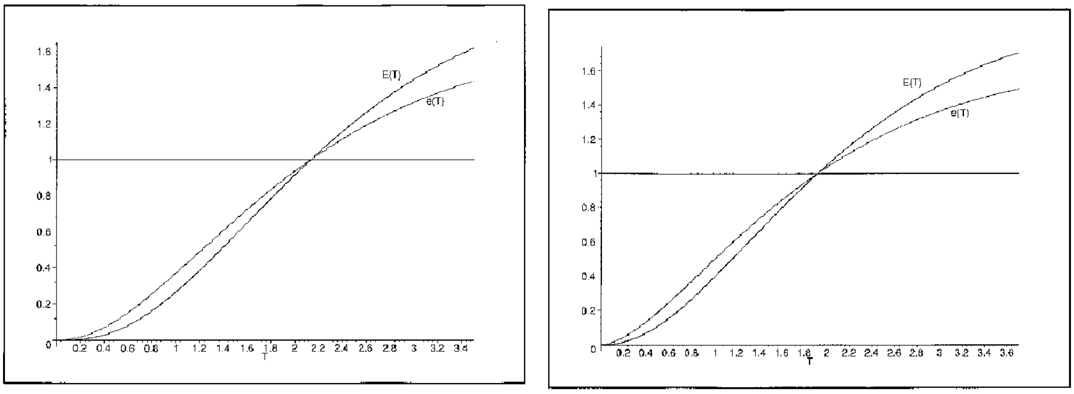

Figure 2: Left: $E(T)$ and $e(T)$ for density **C**, $a = 1,\ b = 0.5,\ \beta = 1$;
Right: $E(T)$ and $e(T)$ for density **D**, $a = 1,\ b = 1$.

$f(x)$	$h_\alpha(x)$	$h(x)$	$h^\vee(u)$
A	$\frac{i}{ai+x}$	$\left(\frac{i}{ai+x}\right)^{2/\alpha}$	$\frac{1}{\Gamma(\alpha/2)}u^{\alpha/2-1}e^{-au\alpha/2}I_{[0,\infty)}(u)$
B	$\frac{i}{ai+x}\left(\frac{i}{bi+x}\right)^{2/\alpha}$	$\left(\frac{i}{ai+x}\right)^{2/\alpha}\frac{i}{bi+x}$	$\frac{e^{-bu}}{\Gamma(\alpha/2)(a-b)^{\alpha/2}}\gamma(\alpha/2, au-bu)I_{[0,\infty)}(u)$
C	$\left(\frac{i}{ai+x}\right)^{\frac{2\beta}{\alpha}}\left(\frac{i}{bi+x}\right)$	$\left(\frac{i}{ai+x}\right)^{\beta}\left(\frac{i}{bi+x}\right)^{\alpha/2}$	$\frac{e^{-au}u^{\beta+\frac{\alpha}{2}-1}}{\Gamma(\beta+\frac{\alpha}{2})}\times$ ${}_1F_1(\alpha/2;\alpha/2+\beta;au-bu)I_{[0,\infty)}(u)$
D	$\frac{i}{ai+x}e^{\frac{-2bi}{\alpha(ai+x)}}$	$\left(\frac{i}{ai+x}\right)^{2/\alpha}e^{\frac{-bi}{ai+x}}$	$e^{-au}\left(\frac{u}{b}\right)^{\frac{\alpha-2}{4}}J_{\frac{\alpha}{2}-1}[2(bu)^{1/2}]I_{[0,\infty)}(u)$

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