Georgian Mathematical Journal
Volume 8 (2001), Number 1, 165-179

INEQUALITIES OF CALDERON–ZYGMUND TYPE FOR TRIGONOMETRIC POLYNOMIALS

K. RUNOVSKI AND H.-J. SCHMEISSER

Abstract. We give a unified approach to inequalities of Calderon–Zygmund type for trigonometric polynomials of several variables based on the Fourier analytic methods. Sharp results are achieved for the full range of admissible parameters $p,\ 0 < p \leq +\infty$. The results obtained are applied to the problem of the image of the Fourier transform in the scale of Besov spaces.

2000 Mathematics Subject Classification: 42B15, 42B05, 42A05.
Key words and phrases: Inequalities for trigonometric polynomials in L_p, $0 < p \leq +\infty$, necessary and sufficient conditions, Fourier multipliers, Besov spaces.

1. Introduction

In the present paper we treat the problem of finding all p for which the inequality

$$\|P_m(\mathcal{D})t\|_p \leq c(d;p;P_m;\beta) \cdot n^{m-2\beta} \|\Delta^\beta t\|_p, \quad t \in \mathcal{T}_n, \ \ n \geq 1, \tag{1.1}$$

is valid for all real or complex trigonometric polynomials $t(x)$ in the space $\mathcal{T}_n$ spanned by harmonics e^{ikx} $(kx = k_1x_1 + \cdots + k_dx_d)$ with $|k| = (k_1^2 + \cdots + k_d^2)^{1/2} \leq n$ and for all $n \in \mathbb{N}$ with some constant independent of t and n. In (1.1) $\|\cdot\|_p$ is the usual $L_p -$ norm (the quasi-norm if $0 < p < 1$) on the d -dimensional torus $\mathbb{T}^d = [0, 2\pi)^d$, $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, $\beta \geq 0$,

$$P_m(\mathcal{D}) = i^{-m} \cdot \sum_{k \in \mathbb{Z}^d,\ |k|_1 = m} \alpha_k \mathcal{D}^k \ ; \ (|k|_1 = k_1 + \cdots + k_d);$$

$$\mathcal{D}^k = \frac{\partial^{|k|_1}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}, \ k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \ ,$$

and Δ^β is the power of the Laplacian given on the space $\mathcal{T}$ of all trigonometric polynomials by

$$\Delta^\beta t(x) = (-1)^\beta \sum_{k \in \mathbb{Z}^d} |k|^{2\beta} c_k e^{ikx}, \ ((-1)^\beta = e^{i\pi\beta}) \ \left(t(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ikx} \in \mathcal{T} \right).$$

ISSN 1072-947X / $8.00 /

Some famous inequalities are special cases of (1.1). The Calderon–Zygmund inequality [9, p. 59]

$$\left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_p \leq c(d;p) \cdot \|\Delta f\|_p \tag{1.2}$$

for C^2-smooth 2π-periodic functions or for functions having a compact support is of type (1.1) with $P_m(\xi) = -\xi_j \xi_k$ and $\beta = 1$. As is well-known [9, p. 59], (1.2) is valid for $1 < p < +\infty$. The Bernstein inequality (for references see, for instance, [2, Chapter 4, §§1, 3])

$$\|t^{(m)}\|_p \leq cn^m \|t\|_p, \ t \in \mathcal{T}_n \ , \ n \in \mathbb{N} \ , \tag{1.3}$$

corresponds to $d = 1$, $P_m(\xi) = (i\xi)^m$, $\beta = 0$. It holds for all $0 < p \leq +\infty$. We notice that the best possible constant in (1.3) is equal to 1 ([1]).

Inequalities (1.2) and (1.3) have been studied separately by using specific methods and approaches. In the present paper we propose a unified approach to inequalities of type (1.1). We shall show that this problem has a complete solution. More precisely, if $P_m(\xi) \cdot |\xi|^{-2\beta}$ does not coincide on $\mathbb{R}^d \setminus \{0\}$ with a polynomial, then inequality (1.1) holds if and only if $\frac{d}{d+(m-2\beta)} < p \leq +\infty$, for $m > 2\beta$, and if and only if $1 < p < +\infty$, for $m = 2\beta$. Clearly, if $P_m(\xi) \cdot |\xi|^{-2\beta}$ is a polynomial, that is, β is a non-negative integer and $P_m(\xi)$ is divisible by $|\xi|^{2\beta}$, then (1.1) is an immediate consequence of the Bernstein inequality and it is valid for all $0 < p \leq +\infty$. Considering the harmonics e^{ikx} it will be proved below that (1.1) fails for all $0 < p \leq +\infty$ if $m < 2\beta$.

To prove inequality (1.1) for $\frac{d}{d+(m-2\beta)} < p \leq +\infty$ if $m > 2\beta$, and for $1 < p < +\infty$ if $m = 2\beta$, we use standard methods of harmonic analysis like Fourier multipliers theorems in the scale of Bessel potential spaces [8, p. 150] and the Marcinkiewicz theorem for periodic multipliers [5, p. 57]. The proof of the inverse result is more difficult, in particular, we apply some facts of the theory of homogeneous distributions [3, §§3.2, 7.1].

In Section 2 we formulate the main result of the paper and give some of its consequences. In Section 3 we describe the asymptotic behavior of the Fourier transform of the function $P_m(\xi) \cdot |\xi|^{-2\beta} \psi(\xi)$, where ψ belongs to the Schwartz space $\mathcal{S}$ of test functions and satisfies some additional conditions. Section 4 is devoted to the proof of the main result. In Section 5 we prove the sharpness of the smoothness order in the Szasz theorem on mapping properties of the Fourier transform.

2. Main Results

Henceforth we say that inequality (1.1) is valid for p if it holds in the p-norm (quasi-norm for $0 < p < 1$) for all $t \in \mathcal{T}_n$ and for all $n \in \mathbb{N}$ with some positive constant that does not depend on t and n. We call the set of all p, for which (1.1) is valid, its range of validity.

For $m \in \mathbb{N}$ we denote by

$$\Pi_m = \left\{ \sum_{k\in\mathbb{Z}^d,\ |k|_1=m} \alpha_k \xi^k \ :\ \alpha_k \in \mathbb{C} \right\} \ (\xi^k = \xi_1^{k_1} \cdots \xi_d^{k_d}) \tag{2.1}$$

the class of homogeneous polynomials of order m.

The main result of the paper is given by the following

Theorem 2.1. *Suppose $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, $m \geq 2\beta > 0$ and $P_m(\xi) \in \Pi_m$. If $P_m(\xi)\cdot|\xi|^{-2\beta}$ is not identical on $\mathbb{R}^d \setminus \{0\}$ with a polynomial, then the inequality*

$$\|P_m(\mathcal{D})t\|_p \leq c(d;p;P_m;\beta)\cdot n^{m-2\beta}\|\Delta^\beta t\|_p,\ t\in\mathcal{T}_n,\ n\geq 1,$$

is valid if and only if $\frac{d}{d+m-2\beta} < p \leq +\infty$ for $m > 2\beta$ and if and only if $1 < p < +\infty$ for $m = 2\beta$. If $\beta = 0$, the inequality is valid for all $0 < p \leq +\infty$.

For the sake of simplifying our notations we shall often write

$$A_n = n^{-m}\cdot P_m(\mathcal{D})\ ;\ B_n = (-1)^\beta\cdot n^{-2\beta}\Delta^\beta\ ,$$

so that inequality (1.1) can be rewritten in the form

$$\|A_n t\|_p \leq C(d;p;P_m;\beta)\cdot\|B_n t\|_p,\ t\in\mathcal{T}_n,\ n\in\mathbb{N}.$$

Remark 2.1. If $P_m(\xi)\cdot|\xi|^{-2\beta} = Q(\xi)$ for $\xi \in \mathbb{R}^d \setminus \{0\}$, where $Q(\xi) \in \Pi_s$, $s \in \mathbb{N}_0$, then $P_m(\xi) = Q(\xi)\cdot|\xi|^{2\beta}$ for each $\xi \in \mathbb{R}^d$ and (1.1) follows for any $0 < p \leq +\infty$ from the inequality

$$\|Q(\mathcal{D})t\|_p \leq C\cdot n^s\|t\|_p,\ t\in\mathcal{T}_n,\ n\in\mathbb{N},$$

which is an immediate consequence of (1.3).

Remark 2.2. If $m < 2\beta$, the function $P_m(\xi)\cdot|\xi|^{-2\beta}$ is unbounded at 0; therefore there exist sequences $\{k_s\}_{s=1}^{+\infty} \subset \mathbb{Z}^d$, $\{n_s\}_{s=1}^{+\infty} \subset \mathbb{N}$ satisfying

$$\lim_{s\to+\infty}\frac{k_s}{n_s} = 0,\ \lim_{s\to+\infty}\left|P_m\left(\frac{k_s}{n_s}\right)\right|\cdot\left|\frac{k_s}{n_s}\right|^{-2\beta} = +\infty.$$

Then

$$\lim_{s\to+\infty}\frac{\|A_{n_s}(e^{ik_s\cdot})\|_p}{\|B_{n_s}(e^{ik_s\cdot})\|_p} = \lim_{s\to+\infty}\left|P_m\left(\frac{k_s}{n_s}\right)\right|\cdot\left|\frac{k_s}{n_s}\right|^{-2\beta} = +\infty$$

and (1.1) fails for all $0 < p \leq +\infty$.

Remarks 2.1 and 2.2 show that the conditions of Theorem 2.1 are natural.

Remark 2.3. The condition "$P_m(\xi)\cdot|\xi|^{-2\beta}$ does not coincide with a polynomial on $\mathbb{R}^d \setminus \{0\}$" means that β is not an integer or otherwise $P_m(\xi)$ is not divisible by $|\xi|^{2\beta}$. In the second case the range of the validity of (1.1) depends essentially on algebraic properties of the polynomial $P_m(\xi)$. We give one example. It is easy to check that the polynomial $\sum_{j=1}^{d} \xi_j^6$ is divisible by $\sum_{j=1}^{d} \xi_j^2$ if and only if $d = 2$; therefore the range of the validity of the inequality

$$\left\| \sum_{j=1}^{d} \frac{\partial^6 t}{\partial x_j^6} \right\|_p \le c(d;p) \cdot n^4 \|\Delta t\|_p, \ t \in \mathcal{T}_n, \ n \in \mathbb{N},$$

is $\left(\frac{d}{d+4}, +\infty\right]$ for $d > 2$, but it is $(0, +\infty]$ for $d = 2$.

3. Fourier Transform of Some Functions

In this section we deal with the Fourier transform of the function $f_{m;\beta} \cdot \psi$, where

$$f_{m;\beta} = \begin{cases} P_m(\xi) \cdot |\xi|^{-2\beta}, & \xi \in \mathbb{R}^d \setminus \{0\} \\ 0, & \xi = 0 \end{cases} \tag{3.1}$$

and ψ belongs to the Schwartz space $\mathcal{S}$ and satisfies some additional conditions. We use some facts of the theory of homogeneous distributions that can be found in [3].

Clearly, $f(\xi)$ is homogeneous of order $a = m - 2\beta \ge 0$, that is, in spherical coordinates

$$f(\xi) = r^a \Phi(u), \ r = |\xi| > 0, \ u \in \mathcal{S}^{d-1} \ ,$$

where $\mathcal{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. Clearly, $\Phi(u)$ is bounded on $\mathcal{S}^{d-1}$ and f has at most polynomial growth at infinity. Therefore it is a regular element of the space $\mathcal{S}'$ of distributions on $\mathcal{S}$, that is

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(\xi) \varphi(\xi) d\xi, \ \varphi \in \mathcal{S} \ . \tag{3.2}$$

We recall that for $g \in \mathcal{S}'$ and $k \in \mathbb{N}_0^d$ the derivative $\mathcal{D}^k g$ is defined by

$$\langle \mathcal{D}^k g, \varphi \rangle = (-1)^{|k|_1} \langle g, \mathcal{D}^k \varphi \rangle, \ \varphi \in \mathcal{S} \ . \tag{3.3}$$

The Fourier transform of $g \in \mathcal{S}'$ is given by

$$\langle \widehat{g}, \varphi \rangle = \langle g, \widehat{\varphi} \rangle, \ \varphi \in \mathcal{S} \ ,$$

where

$$\widehat{\varphi}(x) = (2\pi)^{-\frac{d}{2}} \cdot \int_{\mathbb{R}^d} \varphi(\xi) e^{-ix\xi} d\xi \ .$$

We notice that if $g \in \mathcal{S}' \cap C^\infty(\mathbb{R}^d \setminus \{0\})$, the restriction of $\mathcal{D}^k g$ defined by (3.3) to $\mathcal{S}_0 = \{\varphi \in \mathcal{S} \ : \ \text{supp}\varphi \subset \mathbb{R}^d \setminus \{0\}\}$ coincides as an element of the dual space $\mathcal{S}_0'$ with the pointwise derivative of g.

A preliminary estimate of the asymptotic behaviour of the Fourier transform of $f\psi$, where $\psi \in \mathcal{S}$ is given by the following

Lemma 3.1. *Let* f *be defined by* (3.1) *and* $a = m - 2\beta$. *Then for any* $\psi \in \mathcal{S}$

$$|\widehat{f\psi}(x)| \le c \cdot (1 + |x|)^{-[a]-d+1}\ ,\ x \in \mathbb{R}^d\ , \tag{3.4}$$

where c *does not depend on* x.

Proof. We put $l = [a] + d - 1$. Let $\nu \in \mathbb{N}_0^d$ and $|\nu|_1 \le l$. Then $\mathcal{D}^\nu f$ is homogeneous of order $a - |\nu|_1$ as an element of $\mathcal{S}'$ [3, p. 95-96]. With the help of the remarks given above, $\mathcal{D}^\nu f$ is a regular element of $\mathcal{S}_0'$ and

$$\mathcal{D}^\nu f(\xi) = r^{a-|\nu|_1} \cdot \Phi_\nu(u),\ r = |\xi| > 0,\ u \in S^{d-1}\ .$$

Since $\Phi_\nu(u)$ is bounded on S^{d-1},

$$\begin{aligned}\|\mathcal{D}^\nu f\|_{L_1(\overline{D}_1)} &= \int\limits_{\mathcal{S}^{d-1}} \int\limits_0^1 r^{a-|\nu|_1+d-1} \cdot \Phi_\nu(u) dr d\mathcal{S}(u) \\ &\le c(\nu) \int\limits_0^1 r^{a-|\nu|_1+d-1} dr < +\infty\ ,\end{aligned} \tag{3.5}$$

where $d\mathcal{S}(u)$ is the surface element of $\mathcal{S}^{d-1}$, $D_r = \{x \in \mathbb{R}^d :\ |x| < r\}$, $\overline{D}_r = \{x \in \mathbb{R}^d :\ |x| \le r\}$. Applying the Leibnitz formula for the derivative of the product we deduce from (3.5) that $\mathcal{D}^\nu(f\psi) \in L_1(\mathbb{R}^d)$ for $|\nu|_1 \le l$.

Since $f\psi \in L_1(\mathbb{R}^d)$, the inequality (3.4) is valid for $|x| \le 1$. Let now $|x| > 1$. Using

$$x^\nu \widehat{f\psi}(x) = (-i)^{|\nu|_1} \widehat{\mathcal{D}^\nu(f\psi)}(x),\ x \in \mathbb{R}^d\ ,$$

we obtain for $|\nu|_1 \le l,\ x \in \mathbb{R}^d$

$$|x^\nu| \cdot |\widehat{f\psi}(x)| \le \|\mathcal{D}^\nu(f\psi)\|_{L_1(\mathbb{R}^d)}$$

and

$$\begin{aligned}|\widehat{f\psi}(x)| &\le \left(\sum_{|\nu|_1=l} |x^\nu|\right)^{-1} \cdot \sum_{|\nu|_1=l} \|\mathcal{D}^\nu(f\psi)\|_{L_1(\mathbb{R}^d)} \\ &\le c' \cdot \left(\sum_{|\nu|_1=l} |x^\nu|\right)^{-1} \le c'' \cdot |x|^{-l}\ .\end{aligned}$$

This completes the proof. □

By $\mathcal{X}^d$ we denote the space of radial real-valued functions $\psi \in \mathcal{S}$ with $\psi(0) = 1$.

Theorem 3.1. *Let* f *be defined by* (3.1) *and* $a = m - 2\beta$. *If* $f(\xi)$ *is not a polynomial and* $\psi(\xi) \in \mathcal{X}^d$, *then* $\widehat{f\psi}(x) \in L_p(\mathbb{R}^d)$ *if and only if* $\frac{d}{d+a} < p \le +\infty$.

Proof. We prove that

$$|\widehat{f\psi}(x)| \le c_1(1+|x|)^{-(d+a)}\ ,\ x \in \mathbb{R}^d, \tag{3.6}$$

and

$$|\widehat{f\psi}(x)| \ge c_2 \cdot |x|^{-(d+a)}\ ,\ x \in \Omega, \tag{3.7}$$

where

$$\Omega \equiv \Omega(\rho,\theta,u_0) = \{x = ru:\ r \ge \rho,\ u \in \mathcal{S}^{d-1}\ , \cos\theta \le (u,u_0) \le 1\}$$

for some $\rho > 0,\ 0 < \theta < \frac{\pi}{2}$ and $u_0 \in \mathcal{S}^{d-1}$.

Clearly, Theorem 3.1 follows from (3.6) and (3.7). Indeed, as $a \ge 0$, f is bounded on $\overline{D}_1$ and $\widehat{f\psi} \in L_\infty(\mathbb{R}^d)$. If $\frac{d}{d+a} < p < +\infty$, then $\sigma \equiv d-1-p(d+a) < -1$ and we obtain from (3.6)

$$\begin{aligned}\|\widehat{f\psi}\|^p_{L_p(\mathbb{R}^d)} &\le c\cdot\left\{1+\int\limits_{\mathcal{S}^{d-1}}\int\limits_1^{+\infty} r^{-p(d+a)}\cdot r^{d-1}\,dr dS(u)\right\}\\ &\le c'\cdot\left\{1+\int\limits_1^{+\infty} r^\sigma dr\right\} < +\infty\ .\end{aligned}$$

Let now $0 < p \le \frac{d}{d+a}$. Then $\sigma \ge -1$ and we get from (3.7)

$$\begin{aligned}\|\widehat{f\psi}\|^p_{L_p(\mathbb{R}^d)} \ge \|\widehat{f\psi}\|^p_{L_p(\Omega)} &\ge c\cdot\int\limits_{\cos\theta\le(u,u_0)\le 1}\ \int\limits_\rho^{+\infty} r^{-p(d+1)}\cdot r^{d-1}dr dS(u)\\ &= c'\cdot\int\limits_\rho^{+\infty} r^\sigma dr = +\infty\ .\end{aligned}$$

First we prove (3.6) and (3.7) for functions ψ in

$$\mathcal{X}_0^d = \left\{\psi \in \mathcal{X}^d:\ \widehat{\psi} \ge 0,\ \mathrm{supp}\widehat{\psi} \subset D_{3/4}\right\}\ . \tag{3.8}$$

Since f is bounded on $\overline{D}_1$, $f\psi$ belongs to $L_1(\mathbb{R}^d)$ that implies (3.6) for $|x| \le 1$. Let now $|x| > 1$. By the properties of the Fourier transform of homogeneous distributions [3, p. 203-205, Theorems 7.1.16, 7.1.18], $\widehat{f}$ is homogeneous of order $-(d+a)$ as an element of $\mathcal{S}'$ and it belongs to $\mathbb{C}^\infty(\mathbb{R}^d\backslash\{0\})$, in particular, it is a regular element of $\mathcal{S}_0'$ and

$$\widehat{f}(x) = r^{-(d+a)}\cdot\Psi(u),\ r = |x| > 0,\ u \in S^{d-1}\ . \tag{3.9}$$

Noticing that $\widehat{\psi}(x-\cdot)$ belongs to $\mathcal{S}_0$ for $|x|>1$ and applying the properties of convolution [3, p. 202, Theorem 7.1.15] as well as (3.8) and (3.9) we obtain

$$
\begin{aligned}
|\widehat{f\psi}(x)| &= c\cdot|<\widehat{f},\widehat{\psi}(x-\cdot)>| = c\cdot\left|\int\limits_{\mathbb{R}^d}\widehat{f}(y)\widehat{\psi}(x-y)dy\right| \\
&= c\cdot\left|\int\limits_{|x-y|\le 3/4}\widehat{f}(y)\widehat{\psi}(x-y)dy\right| \le c\cdot\max_{|x-y|\le 3/4}|\widehat{f}(y)|\cdot\int\limits_{\mathbb{R}^d}\widehat{\psi}(y)dy \\
&\le c'\cdot\max_{|x-y|\le 3/4}|y|^{-(d+a)} \le c''\cdot|x|^{-(d+a)}\ ,
\end{aligned}
$$

that proves (3.6).

To show the lower estimate, we observe first that since f is not a polynomial, $\widehat{f}$ cannot be concentrated at 0 and, therefore, there is $u_0\in\mathcal{S}^{d-1}$ such that $\Psi(u_0)\ne 0$. Without loss of generality we may assume that $Re\Psi(u_0)>0$. We choose $0<\theta<\frac{\pi}{2}$ from the condition

$$
Re\Psi(u)\ge\frac{1}{2}Re\Psi(u_0),\ u\in S^{d-1},\ \cos 2\theta\le(u,u_0)\le 1.
$$

Let $\rho>1$ be so large that the conditions $x\in\Omega(\rho,\theta,u_0)$, $|y-x|\le\frac{3}{4}$ imply $y\in\Omega(1,2\theta,u_0)$. Then for $x\in\Omega(\rho,\theta,u_0)$ we obtain

$$
\begin{aligned}
|\widehat{f\psi}(x)| &= c\cdot\left|\int\limits_{|x-y|\le 3/4}\widehat{f}(y)\widehat{\psi}(x-y)dy\right| \\
&\ge c\cdot\int\limits_{|x-y|\le 3/4}|y|^{-(d+a)}\cdot Re\Psi\left(\frac{y}{|y|}\right)\widehat{\psi}(x-y)dy \\
&\ge\frac{c}{2}Re\Psi(u_0)\cdot\int\limits_{|x-y|\le 3/4}|y|^{-(d+a)}\widehat{\psi}(x-y)dy \\
&\ge c2^{-(d+a)-1}\cdot Re\Psi(u_0)\cdot|x|^{-(d+a)}\int\limits_{\mathbb{R}^d}\widehat{\psi}(y)dy \\
&= c'\cdot|x|^{-(d+a)}\ ,
\end{aligned}
$$

where $c'=c2^{-(d+a)-1}Re\Psi(u_0)(2\pi)^{d/2}\psi(0)>0$. Inequality (3.7) is proved.

Let now ψ be an arbitrary function in $\mathcal{X}^d$. We set

$$
f\psi=f\varphi+f(\psi-\varphi),\ \varphi\in\mathcal{X}_0^d\ .
$$

Clearly,

$$
\psi(\xi)-\varphi(\xi)=\alpha|\xi|^2\cdot\psi_1(\xi)\ ,
$$

where $\psi_1\in\mathcal{X}^d$ and

$$
\alpha=\lim_{\xi\to 0}\frac{\psi(\xi)-\varphi(\xi)}{|\xi|^2}\ .
$$

Therefore,

$$f(\xi)\psi(\xi) = f(\xi)\varphi(\xi) + \alpha f(\xi) \cdot |\xi|^2 \cdot \psi_1(\xi) \ . \tag{3.10}$$

Noticing that $g(\xi) = f(\xi) \cdot |\xi|^2$ is homogeneous of order $a+2$, we obtain by Lemma 3.1 that

$$|\widehat{g\psi_1}(x)| \le c(1+|x|)^{-(d+[a]+1)} \ , \ x \in \mathbb{R}^d \ . \tag{3.11}$$

From (3.6) for functions in $\mathcal{X}_0^d$, (3.10) and (3.11) we get for $x \in \mathbb{R}^d$

$$\begin{aligned} |\widehat{f\psi}(x)| &\le |\widehat{f\varphi}(x)| + |\alpha| \cdot |\widehat{g\psi_1}(x)| \\ &\le c' \left(1 + |x|^{-(d+a)} + |x|^{-(d+[a]+1)}\right) \le c'' \left(1 + |x|^{-(d+a)}\right) \ . \end{aligned}$$

Hence estimate (3.6) is proved for $\psi \in \mathcal{X}^d$.

Let $\Omega(\rho, \theta, u_0)$ be the domain, where (3.7) is valid for φ. We put

$$\tilde\rho = \max\left\{\rho, \ \left(\frac{2|\alpha|c}{c_2}\right)^{\frac{1}{1-\{a\}}}\right\} \ .$$

Then for $|x| \ge \tilde\rho$

$$c_2 - c|\alpha| \cdot |x|^{\{a\}-1} \ge \frac{c_2}{2} \ . \tag{3.12}$$

From (3.7) for $\varphi \in \mathcal{X}_0^d$, (3.11) and (3.12) we have for $x \in \Omega(\tilde\rho, \theta, u_0)$

$$\begin{aligned} |\widehat{f\psi}(x)| &\ge |\widehat{f\varphi}(x)| - |\alpha| \cdot |\widehat{g\psi_1}(x)| \\ &\ge c_2 |x|^{-(d+a)} - c|\alpha| \cdot |x|^{-(d+[a]+1)} \\ &\ge |x|^{-(d+a)} \cdot \left(c_2 - c|\alpha| \cdot |x|^{\{a\}-1}\right) \ \ge \frac{c_2}{2} |x|^{-(d+a)} \ . \end{aligned}$$

The proof of Theorem 3.1 is complete. □

4. Proof of the Main Result

To simplify our notations we omit the index n in A_n and B_n defined in Section 2. We notice that the operator B^{-1} is well-defined on

$$\mathcal{T}^\circ = \left\{ t(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ikx} \in \mathcal{T} : c_0 = 0 \right\}$$

by the formula

$$B^{-1} t(x) = \sum_{k \in \mathbb{Z}^d, \ k \ne 0} \left| \frac{k}{n} \right|^{-2\beta} \cdot c_k e^{ikx} \ (t(x) \in \mathcal{T}^\circ \ ). \tag{4.1}$$

By P we denote the projection operator

$$P \left(\sum_{k \in \mathbb{Z}^d} c_k e^{ikx} \right) = \sum_{k \in Z^d, \ k \ne 0} \ c_k e^{ikx}$$

that maps $\mathcal{T}$ into $\mathcal{T}^\circ$.

We split the proof of Theorem 2.1 into 4 steps.

Step 1. *If* $m > 2\beta$, *then* (1.1) *is valid for* $\frac{d}{d+(m-2\beta)} < p \le +\infty$.

Indeed, as it follows from Theorem 3.1, $\widehat{f\psi}(x) \in L_q(\mathbb{R}^d)$ $(\psi \in \mathcal{X}^d)$ for all $q \in \left(\frac{d}{d+(m-2\beta)}, +\infty\right]$; therefore $\widehat{f\psi}(x) \in L_{\tilde p}(\mathbb{R}^d)$, where $\tilde p = \min(1,p)$. It was proved in [8, p. 150–151] that

$$\left\| \sum_{|k|\le n} f\left(\frac{k}{n}\right) c_k e^{ikx} \right\|_p \le c \cdot \|\widehat{f\psi}\|_{L_{\tilde p}(\mathbb{R}^d)} \cdot \|t\|_p \ , \ t \in \mathcal{T}_n \ ,$$

where $\psi(\xi) = 1$ on $\overline{D}_1$ and $\psi(\xi) = 0$ outside D_2. Hence the inequality

$$\|AB^{-1}Pt\|_p \le c\|t\|_p, \ t \in \mathcal{T}_n, \ n \in \mathbb{N}, \tag{4.2}$$

is valid for $\frac{d}{d+(m-2\beta)} < p \le +\infty$. For $t(x) = \sum\limits_{k\in Z^d} c_k e^{ikx} \in \mathcal{T}_n$ we put $\tau(x) = Bt(x)$. Then we get from (4.2)

$$\|AB^{-1}P\tau\|_p \le c\|\tau\|_p \ . \tag{4.3}$$

Since $B^{-1}P\tau(x) = t(x) - c_0$ and $P_m(0) = 0$, $AB^{-1}P\tau(x) = At(x) - Ac_0 = At(x)$ and (4.3) can be rewritten in the form

$$\|At\|_p \le c \cdot \|Bt\|_p \ ,$$

that is, (1.1) is valid.

Step 2. *If* $m = 2\beta$, (1.1) *is valid for* $1 < p < +\infty$.

Since $m = 2\beta$, f is homogeneous of order 0 and $\mathcal{D}^{(m)} f \equiv \frac{\partial^m f}{\partial \xi_1 \cdots \partial \xi_m}$ is homogeneous of order $-m$ for each $1 \le m \le d$ [3, p. 95-96], that is,

$$\mathcal{D}^{(m)} f(\xi) = r^{-m}\Phi_m(u), \ r = |\xi| > 0, \ u \in \mathcal{S}^{d-1}. \tag{4.4}$$

For each dyadic rectangle

$$I_l = \prod_{s=1}^{m} \left[2^{|l_s|-1}, \ 2^{|l_s|}\right] \subset \{\eta \in \mathbb{R}^m : \ r_l \le |\eta| \le 2r_l\}, \ l = (l_1, \dots, l_m) \in \mathbb{Z}^m,$$

where $r_l = \left(\sum\limits_{s=1}^{m} 2^{2(|l_s|-1)}\right)^{\frac{1}{2}}$, for $k \in \mathbb{N}^d$ we have from (4.4)

$$\begin{aligned} \sup_{k_{m+1},\dots,k_d} & \sum_{k_1=2^{|l_1|-1}}^{2^{|l_1|-1}} \cdots \sum_{k_m=2^{|l_m|-1}}^{2^{|l_m|-1}} |\Delta_1 \cdots \Delta_m f(k)| \\ & \le \sup_{k_{m+1},\dots,k_d} \int_{I_l} \left|\mathcal{D}^{(m)} f(\xi_1, \dots, \xi_m, k_{m+1}, \dots, k_d)\right| \, d\xi_1 \cdots d\xi_m \\ & \le \max_{m\in\mathcal{S}^{d-1}} |\Phi_m(u)| \int_{r_l}^{2r_l} \frac{d\xi_1 \cdots d\xi_m}{(\xi_1^2 + \cdots + \xi_m^2)^{\frac{m}{2}}} \le C_m \ , \end{aligned}$$

where $\Delta_j f = f(\dots, k_j + 1, \dots) - f(\dots, k_j, \dots)$ is the first difference of the sequence $f(k)$ with respect to the variable j and C_m does not depend on l. If l_s is equal to 0, the corresponding sum is extended only to $k_s = 0$. The same estimate is obviously valid for any other set of variables $1 \le j_1 < \dots < j_m \le d$.

We have checked the conditions of the Marcinkiewicz theorem on periodic multipliers [5, p. 57]. In view of this theorem we conclude that the inequality

$$\|AB^{-1}Pt\|_p \le c\|t\|_p, \ t \in \mathcal{T},$$

is valid for all $1 < p < +\infty$. As it was shown in Step 1, this implies (1.1).

Step 3. *If* (1.1) *is valid for some* $0 < p \le +\infty$, *then* $\frac{d}{d+(m-2\beta)} < p$.

Obviously, it is enough to consider the case $0 < p \le 1$. For a function $\psi \in \mathcal{X}^d$ with support in $\overline{D}_1$ we consider the polynomial

$$\Psi(x) \equiv \Psi_n(x) = \sum_{k \in \mathbb{Z}^d} \psi\left(\frac{k}{n}\right) e^{ikx} \ \in \ \mathcal{T}_n \ .$$

We choose $h \equiv h_n$ such that

$$\|\Delta_h AB^{-1}P\Psi\|_p = \max_{y \in \mathbb{R}^d} \|\Delta_y AB^{-1}P\Psi\|_p \ ,$$

where $\Delta_y g(\cdot) = g(\cdot + y) - g(\cdot)$. We put

$$t(x) = B^{-1}\Delta_h \Psi_n(x) \ .$$

Clearly, $t(x) \in \mathcal{T}_n \cap \mathcal{T}^\circ$. We notice that

$$\left\| \sum_{k \in \mathbb{Z}^d} \psi\left(\frac{k}{n}\right) e^{ikx} \right\|_p \le cn^{d(1-1/p)} \ , \ n \in \mathbb{N}, \tag{4.5}$$

where c does not depend on n. For $0 < p \le 1$ this was proved in [4] with the help of the Poisson summation formula. Using a Jackson type inequality (obviously, the proof given in [7] fits for complex-valued functions), (1.1) for $t(x)$ and (4.5) we get

$$\begin{aligned} E_0(AB^{-1}P\Psi)_p &\le c \cdot \|\Delta_h AB^{-1}P\Psi\|_p \\ &= c \cdot \|AB^{-1}P\Delta_h\Psi\|_p = c \cdot \|AB^{-1}\Delta_h\Psi\|_p \\ &= c \cdot \|At\|_p \le c'\|Bt\|_p \le c' \cdot \|\Delta_h\Psi\|_p \\ &\le 2^{1/p}c'\|\Psi\|_p \le c'' \cdot n^{d(1-\frac{1}{p})} \ , \end{aligned} \tag{4.6}$$

where $E_0(g)_p = \inf\limits_{z\in\mathbb{R}^d} \|g(x)-z\|_p$ is a best approximation to g by constants. Recalling that $f(\xi)$ is bounded on $\overline{D}_1$, we get with the help of Hölder's inequality

$$\begin{aligned}\|AB^{-1}P\Psi\|_p &\le (2\pi)^{d(\frac{1}{p}-\frac{1}{2})}\|AB^{-1}P\Psi\|_2 \\ &= (2\pi)^{\frac{d}{p}} \cdot \left\{\sum_{k\in\mathbb{Z}^d} \left| f\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)\right|^2\right\}^{1/2} \\ &\le c\cdot\{\mathrm{card}(\mathrm{supp}\psi\cap Z^d)\}^{1/2} \le c'\cdot n^{d/2} \ . \end{aligned} \tag{4.7}$$

We choose $\sigma\equiv\sigma_n$ such that

$$E_0(AB^{-1}P\Psi)_p \ge 2^{-\frac{1}{p}}\|AB^{-1}P\Psi-\sigma\|_p \ . \tag{4.8}$$

From (4.7) and (4.8) we obtain

$$\begin{aligned}(2\pi)^d|\sigma|^p &\le 2\cdot E_0(AB^{-1}P\Psi)_p^p + \|AB^{-1}P\Psi\|_p^p \\ &\le 3\|AB^{-1}P\Psi\|_p^p \le cn^{\frac{pd}{2}} \ ,\end{aligned}$$

in particular,

$$\lim_{n\to+\infty} n^{-d}\cdot\sigma_n = 0 \ . \tag{4.9}$$

We note that in different formulas the constans c can also be different, but all of them do not depend on n.

We consider the functions $\{F_n(x)\}_{n=1}^{+\infty}$ given by

$$F_n(x) = \begin{cases} n^{-dp}\cdot\left|\sum\limits_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)e^{i\frac{k}{n}x}-\sigma_n\right|^p, & x\in[-\pi n,\pi n]^d, \\ 0, & \text{otherwise} \ . \end{cases} \tag{4.10}$$

Clearly, the functions $F_n(x)$, $n\in\mathbb{N}$, are non-negative and measurable. Let $x_0\in\mathbb{R}^d$. Then there exists $n_0\in\mathbb{N}$, such that $x_0\in[-\pi n,\pi n]^d$ for $n\ge n_0$. The function $f(\xi)\psi(\xi)e^{i\xi x_0}$ of variable ξ is integrable in the Riemann sense on $[-1,1]^d$. By the definition of the Riemann integral we get

$$\begin{aligned}&\lim_{n\to+\infty} n^{-d}\cdot\sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)e^{i\frac{k}{n}x_0} \\ &= \int\limits_{[-1,1]^d} f(\xi)\psi(\xi)e^{i\xi x_0}d\xi = (2\pi)^{d/2}\cdot\widehat{f\psi}(-x_0) \ .\end{aligned}$$

Therefore by (4.9) we have

$$\lim_{n\to+\infty} F_n(x_0) = (2\pi)^{dp/2}\cdot|\widehat{f\psi}(-x_0)|^p \ . \tag{4.11}$$

From (4.6) and (4.8) we get

$$\int_{\mathbb{R}^d} F_n(x)dx = n^{-dp} \int_{[-\pi n,\pi n]^d} \left| \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\cdot\psi\left(\frac{k}{n}\right) e^{ik\frac{x}{n}} - \sigma_n\right|^p dx$$

$$= n^{d(1-p)} \left\| \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\cdot\psi\left(\frac{k}{n}\right) e^{iky} - \sigma_n\right\|_p^p$$

$$= n^{d(1-p)} \|A_n B_n^{-1} P\Psi_n - \sigma_n\|_p^p \le 2n^{d(1-p)} E_0(A_n B_n^{-1} P\Psi_n)_p^p \le c\ .$$

Thus we have proved that the sequence $\{F_n(x)\}_{n=1}^{+\infty}$ satisfies all conditions of Fatou's lemma. Hence the integral of its limit can be estimated by the same constant, that is, $\widehat{f\psi} \in L_p(\mathbb{R}^d)$. On the basis of Theorem 3.1 we obtain $\frac{d}{d+(m-2\beta)} < p$.

Step 4. *In the case* $m = 2\beta$, (1.1) *fails for* $p = +\infty$.

Let us assume that (1.1) is valid for $p = +\infty$. For $t(x) = \sum_{k\in Z^d} c_k e^{ikx} \in \mathcal{T}_n$ we put $\tau(x) = B^{-1}(t(x) - c_0)$. From (1.1) for $\tau(x)$ we get

$$\|AB^{-1}(t(x) - c_0)\|_\infty \le c\cdot\|t(x) - c_0\|_\infty\ . \tag{4.12}$$

Noticing that $t(x) - c_0 = Pt(x)$ and $\|t(x) - c_0\|_\infty \le 2\|t\|_\infty$, we obtain from (4.12) that

$$\|M_n(f)t\|_\infty \le c\|t\|_\infty\ ,\ t\in\mathcal{T}_n,\ n\in\mathbb{N}, \tag{4.13}$$

where $M_n(f) \equiv A_n B_n^{-1} P^{-1} \equiv AB^{-1}P$.

Next we shall use the principle of duality. For a function $\varphi(x) \in \mathcal{X}^d$ that is equal to 1 on D_1 and to 0 outside of D_2 we consider the polynomial

$$\Phi_n(x) = \sum_{k\in\mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) e^{ikx}\ \in\ \mathcal{T}_{2n}\ .$$

For $g \in L_\infty$ we have

$$M_n(\varphi)g \equiv \sum_{k\in\mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) g^\wedge(k) e^{ikx} = (2\pi)^{-d}\cdot\int_{\mathbb{T}^d} g(x+h)\Phi_n(h)dh\ ,$$

where

$$g^\wedge(k) = (2\pi)^{-d}\cdot\int_{\mathbb{T}^d} g(x)e^{-ikx}dx,\ k\in\mathbb{Z}^d\ ,$$

and by virtue of (4.5)

$$\|M_n(\varphi)g\|_\infty \le (2\pi)^{-d}\cdot\|g\|_\infty\cdot\|\Phi_n\|_1 \le c'\cdot\|g\|_\infty\ , \tag{4.14}$$

where c' does not depend on g and n.

Noticing that inequality (4.13) being valid for f is also valid for $\overline{f}$, we get by (4.14)

$$\begin{aligned}
\|M_n(f)\Psi_n\|_1 &= \sup_{\|g\|_\infty\le 1} |(M_n(f)\Psi_n, g)| \\
&= (2\pi)^d \sup_{\|g\|_\infty\le 1} \left| \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\psi\left(\frac{k}{n}\right)\overline{g^\wedge(k)} \right| \\
&= (2\pi)^d \sup_{\|g\|_\infty\le 1} \left| \sum_{k\in\mathbb{Z}^d} \psi\left(\frac{k}{n}\right)\overline{\overline{f}\left(\frac{k}{n}\right)\varphi\left(\frac{k}{n}\right)g^\wedge(k)} \right| \\
&= \sup_{\|g\|_\infty\le 1} |(\Psi_n,\ M_n(\overline{f})(M_n(\varphi)g))| \\
&\le \|\Psi_n\|_1 \cdot \sup_{\|g\|_\infty\le 1} \|M_n(\overline{f})(M_n(\varphi)g)\|_\infty \le c.
\end{aligned} \tag{4.15}$$

We put in (4.10) $\sigma_n = 0,\ p = 1$. Then we obtain from (4.15)

$$\begin{aligned}
\int_{\mathbb{R}^d} F_n(x)dx &= n^{-d} \int_{[-\pi n,\pi n]^d} \left| \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\cdot\psi\left(\frac{k}{n}\right) e^{ik\frac{x}{n}} \right| dx \\
&= \left\| \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{n}\right)\cdot\psi\left(\frac{k}{n}\right) e^{iky} \right\|_p^p \\
&= \|M_n(f)\Psi_n\|_1 \le c\ .
\end{aligned}$$

Thus, by virtue of Fatou lemma, $\widehat{f\psi} \in L_1(\mathbb{R}^d)$. Since f is homogeneous of order 0, we obtain a contradiction to Theorem 3.1.

The second statement of Theorem 2.1 concerning the case $\beta = 0$ is a direct consequence of the classical Bernstein inequality.

The proof is complete.

5. Some Estimates for the Fourier Transform

Mainly, the proof of Theorem 2.1 was based on Theorem 3.1. In this Section we give one of its further possible applications that concerns mapping properties of the Fourier transform. We consider the Besov spaces $B^s_{2;p},\ s\in\mathbb{R},\ 0<p\le +\infty$, that can be defined as

$$B^s_{2;p} = \left\{ g\in\mathcal{S}' \ :\ \left(\sum_{j=0}^{+\infty} 2^{spj}\cdot\|\widehat{g}\|^p_{L_2(K_j)}\right)^{\frac{1}{p}} \right\},\ p<+\infty;$$

$$B^s_{2;\infty} = \left\{ g\in\mathcal{S}' \ :\ \sup_{j=0,1,\ldots} 2^{sj}\cdot\|\widehat{g}\|_{L_2(K_j)} \right\}\ ,\ p=+\infty\ ,$$

where $K_0 = \{x\in\mathbb{R}^d :\ |x|\le 1\}$; $K_j = \{x\in\mathbb{R}^d :\ 2^{j-1}\le |x|\le 2^j\}$, $j\in\mathbb{N}$. The following estimate is known (see [6, pp. 9-11]; [8, p. 55]).

Theorem. *Let* $\alpha \geq 0,\ 0 < p \leq 1,\ \sigma \equiv \sigma(d;p;\alpha) = \alpha + d\left(\frac{1}{p} - \frac{1}{2}\right)$. *Then*

$$\|(1+|x|)^\alpha \cdot \widehat{g}\|_p \leq c \cdot \|g|B^\sigma_{2;p}\|\ ,\ g \in B^\alpha_{2;p},$$

where the positive constant c *does not depend on* g.

We will show that the order of smoothness σ is sharp. More precisely, the following theorem holds.

Theorem 5.1. *Let* $\alpha \geq 0,\ 0 < p \leq 1$. *For each* $s < \sigma$ *and for each* $0 < q \leq +\infty$ *there exists a function* $g \in B^s_{2;q}$, *such that the function* $(1+|x|)^\alpha \cdot \widehat{g}(x)$ *does not belong to* L_p.

Proof. We consider a function $f = P_m(\xi) \cdot |\xi|^{-2\beta}$ such that $a \equiv m - 2\beta = \alpha + d\left(\frac{1}{p} - 1\right)$. We put $g = f\psi$, where $\psi \in \mathcal{X}^d$. By (3.7) we obtain

$$\|(1+|x|)^\alpha \cdot \widehat{g}\|_p^p \geq c \cdot \int\limits_\rho^{+\infty} r^{\alpha p} \cdot r^{-(d+a)p} \cdot r^{d-1} dr = \int\limits_\rho^{+\infty} r^{-1} = +\infty.$$

Since $d + a = \alpha + \frac{d}{p}$, we get for $q < +\infty$

$$\|g|B^s_{2;q}\|^q = c \cdot \left(1 + \sum_{j=1}^{+\infty} 2^{sqj} \cdot \left(\int\limits_{2^{j-1}}^{2^j} r^{-2(d+a)} \cdot r^{d-1} dr\right)^{\frac{q}{2}}\right)$$

$$\leq c' \cdot \left(1 + \sum_{j=1}^{+\infty} 2^{qj(s+\frac{d}{2}-(d+a))}\right) \leq c'' \cdot \sum_{j=0}^{+\infty} 2^{qj(s-\sigma)} < +\infty.$$

The case $q = +\infty$ follows from the embedding $B^s_{2,q} \subset B^s_{2,\infty}$. □

Acknowledgement

We would like to express our deep gratitude to Professor Hans Triebel for his interest in our work.

References

1. V. V. Arestov, On integral inequalities for trigonometric polynomials and their derivatives. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **45**(1981), 3–22.
2. R. DeVore and G. G. Lorentz, Constructive approximation. *Springer-Verlag, Berlin Heidelberg,* 1993.
3. L. Hörmander, The analysis of linear partial differential operators, I. *Springer-Verlag, Berlin, Heidelberg,* 1983.
4. V. I. Ivanov and V. A. Yudin, On trigonometric system in L_p, $0 < p < 1$. (Russian) *Mat. Zametki* **28**(1980), 859–868.
5. S. M. Nikolski, Approximation of functions of several variables and embedding theorems. (Russian) *Nauka, Moscow,* 1977.

6. J. Peetre, Applications de la theorie des espaces d'interpolation dans l'analyse harmonique. *Ricerche Mat.* **15**(1966), 1–34.

7. K. Runovski, On approximation by families of linear polynomial operators in L_p-spaces, $0 < p < 1$. *Russian Acad. Sci. Sb. Math.* **82** (1995), 2, 441–459.

8. H.-J. Schmeisser and H. Triebel, Topics in Fourier analysis and function spaces. *Geest & Portig, Leipzig,* 1987 and *Wiley, Chichester,* 1987.

9. E. Stein, Singular integrals and Differentiability Properties of Functions. *Princeton University Press, Princeton, N.J.,* 1970.

(Received 4.12.2000)

Authors' addresses:

K. Runovski
Mathematisches Institut
Friedrich-Schiller-University-Jena
Ernst-Abbe-Platz, 1-4
07743 Jena
Germany
E-mail: runovski@minet.uni-jena.de

H.-J. Schmeisser
Mathematisches Institut
Friedrich-Schiller-University-Jena
Ernst-Abbe-Platz, 1-4
07743 Jena
Germany
E-mail: mhj@minet.uni-jena.de