

## POLYMERISATIONS OF A DISK WITH CRITICAL POINTS ON THE BOUNDARY

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ABSTRACT. The existence of an interior extension with critical points on the boundary is proved for a given closed normal curve and for nonnormal curves of specific kind.

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### INTRODUCTION

The problem of the existence of an interior extension for a given closed normal curve was solved by Ch. Titus [1] in 1961. In 1967 S. Blanc [2] gave another solution of this problem. Subsequently, many authors investigated different aspects of this theory [3]–[9], but in all these papers the existence of interior extension was proved under the assumption that the images of critical points of this mapping do not lie on the given curve. On the other hand, M. Morse [10] introduced the concept of a partial branch point of the inverse function and extended his formula connecting the angular order of a curve with the sum of multiplicities of critical points for interior mappings possessing critical points on the boundary.

In this paper we investigate a problem of the existence of an interior extension with critical points on the boundary for a given closed normal curve (Section 3). In Section 4 an interior extension is constructed for a closed curve which is not normal.

### 1. PRELIMINARIES

Two continuous mappings (paths)  $g : [a, b] \rightarrow \mathbb{R}^2$  and  $h : [c, d] \rightarrow \mathbb{R}^2$  are called equivalent if there exists a sense preserving homeomorphism  $\chi : [a, b] \rightarrow [c, d]$  such that  $g = h \circ \chi$ . The set of all equivalent paths is called a curve and each path belonging to the curve is called a representation of this curve. Let  $\gamma$  be a curve, and let  $\gamma(x)$  be some representation of  $\gamma$  defined on  $[a, b]$ . The point  $i(\gamma) = \gamma(a)$  is called the initial point of  $\gamma$  and  $t(\gamma) = \gamma(b)$  the terminal point of  $\gamma$ . By  $\langle \gamma \rangle$  is denoted the set of images of  $\gamma(x)$ . Let  $\gamma_1$  and  $\gamma_2$  be two curves such that  $i(\gamma_2) = t(\gamma_1)$ , and let  $\gamma_1(x)$ ,  $x \in [a, b]$  and  $\gamma_2(x)$ ,  $x \in [c, d]$

be the corresponding representations. Take some  $b' > b$  and a homeomorphism  $\chi : [b, b'] \rightarrow [c, d]$ ,  $\chi(b) = c$  and consider the mapping defined on  $[a, b']$

$$(\gamma_1 \cdot \gamma_2)(x) = \begin{cases} \gamma_1(x), & \text{when } x \in [a, b], \\ \gamma_2(\chi(x)), & \text{when } x \in [c, d]. \end{cases}$$

The curve containing the mapping  $(\gamma_1 \cdot \gamma_2)(x)$  will be denoted by  $\gamma_1 \cdot \gamma_2$ .

Let  $\gamma(x) \in \gamma$ ,  $x \in [a, b]$  and  $[c, d] \subseteq [a, b]$ . By  $\gamma[c, d]$  we denote the curve generated by the restriction of  $\gamma(x)$  to  $[c, d]$  and by  $\gamma[d, c]$  the curve generated by  $\gamma(\omega(x))$ , where  $\omega(x)$  is a sense reversing homeomorphism of  $[c, d]$  onto itself.

The point  $v \in \langle \gamma \rangle$  is called a vertex (or node) if for some path  $\psi \in \gamma$ ,  $\text{card } \psi^{-1}(v) > 1$  [4], [6]. It is clear that the definition of a vertex does not depend on the choice of representation. A vertex  $v$  is called a double point if  $\psi^{-1} = \{x_1, x_2\}$  and  $\langle \psi[x_1 - \varepsilon, x_1 + \varepsilon] \rangle$  separates  $\langle \psi[x_2 - \varepsilon, x_2 + \varepsilon] \rangle \setminus \psi(x_2)$  for some  $\varepsilon > 0$ . The curve is called normal if it has a finite number of vertices and each of them is a double point [9].

Let  $f$  be an interior mapping [11], [12] of a domain  $D$  into  $\mathbb{R}^2$ . For every  $z \in D$  there exists a neighborhood  $U(z)$  such that the restriction of  $f$  to  $U(z) \setminus \{z\}$  is a  $(\mu(f, z) + 1)$ -sheeted covering of  $f(U(z)) \setminus \{f(z)\}$ . If  $\mu(f, z) > 0$ , we say that the covering  $f|_{U(z)}$  is branched and the point  $z$  is called a critical point of the multiplicity  $\mu(f, z)$ . After G. Francis [6] we call an interior mapping having a finite number of critical points in  $D$  a polymersion.

A closed curve  $\gamma$  is called an interior boundary [1], [2] if there exists an interior mapping (polymersion) of the unit disk  $D^2$ , continuous in  $\overline{D}^2$  such that the restriction of  $f$  to the counterclockwise oriented unit circle  $S^1$  is a path belonging to  $\gamma$ . Another expression of this fact is:  $f$  is an interior extension of  $\gamma$ .

## 2. THE EXISTENCE OF AN INTERIOR EXTENSION FOR A NORMAL CLOSED CURVE

In this section a summary of the main result of the paper [5] is given for convenience.

Let  $w = \overline{\gamma(\zeta)}$ ,  $\zeta \in S^1$ , be a representation of a closed normal curve and let  $\alpha_j$ ,  $j = \overline{0, N}$ , be the collection (raying) of paths (rays) with the following properties:

1.  $i(\alpha_j) = a_j$  lies in some bounded component of  $\mathbb{R}^2 - \langle \gamma \rangle$ ,  $t(\alpha_j) = b_j$  lies in the unbounded component of  $\mathbb{R}^2 \setminus \langle \gamma \rangle$ . There is at least one  $a_j$  in each bounded component of  $\gamma$ .
2. Each  $\alpha_j$  is transverse to  $\gamma$ .
3.  $\alpha_j$ 's are pairwise disjoint and do not meet nodes of  $\gamma$ .

The crossing point of  $\gamma$  and  $\alpha_j$  is called positive if  $\gamma$  crosses  $\alpha_j$  from right to left, and negative otherwise. Index crossing points of  $\alpha_j$  and  $\gamma$  by  $(j, k, \pm 1)$  (called hereafter letters), where the sign of the third coordinate coincides with the sign of the crossing and second coordinates  $k$ , are arranged in an increasing

way along  $\alpha_j$ . In what follows the expression “a point  $(a, b, c)$ ” will mean the crossing point corresponding to the letter  $(a, b, c)$ .

Choose  $i(\gamma)$  and trace along  $\gamma$ . We make a sequence  $\alpha$  (called the word of  $\gamma$ ) of letters as they are encountered in the tracing of  $\gamma$ . Denote by  $\mathcal{G}(\gamma)$  (the grouping) and  $\mathcal{B}(\gamma)$  (the branching) collections of letters of  $\alpha$  satisfying following conditions

- (a) The sets of  $\mathcal{A}(\gamma) = \mathcal{G}(\gamma) \cup \mathcal{B}(\gamma)$  are mutually disjoint.
- (b) If  $A \in \mathcal{A}(\gamma)$  and  $A' \in \mathcal{A}(\gamma)$ , then  $\gamma^{-1}(A)$  is contained in one component of  $S^1 \setminus \gamma^{-1}(A')$ .
- (c) Every set in  $\mathcal{G}(\gamma)$  is of the form  $\{(j, k, -1), (j, k', 1)\}$  with  $k < k'$ .
- (d) Every negative letter (i.e., the letter with a negative third coordinate) occurs in some set of  $\mathcal{G}(\gamma)$ .
- (e) All letters belonging to some  $B \in \mathcal{B}(\gamma)$  have the same first coordinate. In other words, each  $B$  consists of letters corresponding to the crossing points of one path of the raying.

Such a set  $\mathcal{A}(\gamma)$  is called an assemblage for  $\gamma$ . The assemblage is called maximal if  $\sum_{B \in \mathcal{B}(\gamma)} (\text{card } B - 1) = \tau(\gamma)$ , where  $\tau(\gamma)$  is the tangent winding number of  $\gamma$ .

**Theorem ([5]).** *A normal closed curve has an interior extension iff it has a maximal assemblage. An assemblage determines multiplicities of critical points of corresponding interior extension  $f$ . Namely, for  $B \in \mathcal{B}(\gamma)$  there exists  $x \in D^2$  such that  $f(x) = a_j$  and  $\mu(f, x) = \text{card } B - 1$ .*

*Remark.* The investigation of the problem of the existence of an interior extension was carried out in [5] under the condition that  $\gamma$  is smooth. But it is easy to show that the above-stated result remains true for a piece-wise smooth curve as well.

### 3. THE EXISTENCE OF AN INTERIOR EXTENSION WITH PARTIAL CRITICAL POINTS

By  $U(z)$  we denote a neighborhood of  $z$  in  $\mathbb{R}^2$ . For a point  $\zeta \in S^1$  denote  $U_+(\zeta) = U(\zeta) \cap \overline{D}^2$ ,  $U_-(\zeta) = U(\zeta) \cap CD^2$ ,  $U^+(\zeta) = U(\zeta) \cap D^2$ ,  $U^-(\zeta) = U(\zeta) \cap \overline{D}^2$ . Let  $\gamma(\zeta)$  be a representation of a normal curve and let  $\gamma[\zeta_1, \zeta_2]$  be a simple arc of this curve. Let  $\zeta_0 \in \gamma[\zeta_1, \zeta_2]$  and let  $U(\gamma(\zeta_0))$  be a neighborhood of  $\gamma(\zeta_0)$  such that  $U(\gamma(\zeta_0)) \setminus \langle[\zeta_1, \zeta_2]\rangle$  consists of two components. Denote by  $U^+(\gamma(\zeta_0))$  the component of  $U(\gamma(\zeta_0)) \setminus \langle[\zeta_1, \zeta_2]\rangle$  lying to the left of  $\gamma[\zeta_1, \zeta_2]$  and by  $U^-(\gamma(\zeta_0))$  the other component. By  $U_+(\gamma(\zeta_0))$  and  $U_-(\gamma(\zeta_0))$  denote  $U^+(\gamma(\zeta_0)) \cup (\langle[\zeta_1, \zeta_2]\rangle \cap U(\gamma(\zeta_0)))$  and  $U^-(\gamma(\zeta_0)) \cup (\langle[\zeta_1, \zeta_2]\rangle \cap U(\gamma(\zeta_0)))$ , respectively.

Let  $f$  be an interior extension of  $\gamma$ . A point  $\tilde{\zeta} \in S^1$  is called a partial critical point of  $f$  (or a partial branch point of  $f^{-1}$ ) of multiplicity  $\mu(f, \tilde{\zeta}) > 0$  if there exists an extension  $F$  of  $f$  from  $U_+(\tilde{\zeta})$  to  $U(\tilde{\zeta})$  such that  $F : U(\tilde{\zeta}) \setminus \{\tilde{\zeta}\} \rightarrow$

$F(U(f(\tilde{\zeta})) \setminus \{f(\tilde{\zeta})\})$  is a  $\mu(f, \tilde{\zeta}) + 1$ -sheeted covering and  $F|_{U^-(\tilde{\zeta})} : U^-(\tilde{\zeta}) \rightarrow F(U^-(f(\tilde{\zeta})))$  is a homeomorphism.

The behavior of  $f(z)$  in  $U(\tilde{\zeta})$  can be described in the following way [12]:  $\tilde{\zeta} \in S^1$  is a partial critical point of  $f$  of multiplicity  $\mu(f, \tilde{\zeta})$  if there exist  $2\mu(f, \tilde{\zeta})$  simple arcs  $\ell_j$ ,  $j = \overline{1, 2\mu(f, \tilde{\zeta})}$ ,  $\bigcap_{j=1}^{2\mu(f, \tilde{\zeta})} \langle \ell_j \rangle = \{\tilde{\zeta}\}$  passing through some  $U^+(\tilde{\zeta})$

such that the restriction of  $f$  to each component of  $U^+(\tilde{\zeta}) \setminus \bigcap_{j=1}^{2\mu(f, \tilde{\zeta})} \langle \ell_j \rangle$  is schlicht and maps this component onto  $U^+(f(\tilde{\zeta}))$  or  $U^-(f(\tilde{\zeta}))$ . If we number these components in a consecutive way then the components with odd indices will be taken onto  $U^+(f(\tilde{\zeta}))$ , while the components with even indices onto  $U^-(f(\tilde{\zeta}))$ .

**Theorem 1.** *Let  $\gamma$  be a normal closed curve possessing a maximal assemblage  $\mathcal{A}(\gamma)$  and let  $B = \{(j, k_0, 1), (j, k_1, 1), \dots, (j, k_\mu, 1)\} \in \mathcal{B}(\gamma)$ ,  $k_i < k_{i+1}$ ,  $i = \overline{0, \mu-1}$ ,  $\mu > 0$ . Then there exists an interior extension  $\tilde{f}$  of  $\gamma$  such that  $\tilde{f}^{-1}$  has a partial branch point of multiplicity  $\mu$  at the point  $(j, k_0, 1)$ .*

Without loss of generality we can assume that the raying  $\alpha_0, \alpha_1, \dots, \alpha_N$  is numbered so that the set  $B$  consists of letters belonging to  $\alpha_0$ .

Let  $E$  be an open set such that it contains the ray  $\alpha_0$  and does not contain other rays of the raying and nodes of the curve  $\gamma$ . Moreover, we can choose  $E$  in such a way that every component of  $E \cap \langle \gamma \rangle$  contains at most one crossing point belonging to  $\alpha_0$ . Denote by  $\langle \gamma_{k_i} \rangle$  the component of  $E \cap \langle \gamma \rangle$  containing the point  $(0, k_i, 1) \in B$ ,  $i = \overline{0, \mu-1}$ .

Take a simple arc  $\gamma[\zeta', \zeta'']$  such that  $\langle \gamma[\zeta', \zeta''] \rangle \subset \langle \gamma_{k_0} \rangle$ . Let  $(0, k_0, 1) = \gamma(\tilde{\zeta}) = \gamma(e^{i\tilde{\theta}})$ ,  $\zeta' = e^{i\theta'}$ ,  $\zeta'' = e^{i\theta''}$ . Take  $2\mu + 2$  points  $\zeta_k = e^{i\theta_k}$ ,  $k = \overline{0, 2\mu+1}$ , such that  $\theta' < \theta_0 < \theta_2 < \dots < \theta_{2\mu} < \tilde{\theta} < \theta_{2\mu-1} < \theta_{2\mu-3} < \dots < \theta_1 < \theta_{2\mu+1} < \theta''$ . Let  $\tilde{U}(\gamma(\tilde{\zeta}))$  be a neighborhood such that  $\langle \gamma[\zeta', \zeta''] \rangle \subset \tilde{U}(\gamma(\tilde{\zeta})) \subset E$ .

Denote by  $g_1$  a simple arc such that  $i(g_1) = \gamma(\zeta_0)$ ,  $t(g_1) = \gamma(\zeta_1)$ ,  $\langle g_1 \rangle \setminus \{\gamma(\zeta_0), \gamma(\zeta_1)\} \subset \tilde{U}^+(\gamma(\tilde{\zeta}))$ , and by  $g_2$  a simple arc such that  $i(g_2) = \gamma(\zeta_1)$ ,  $t(g_2) = \gamma(\zeta_2)$ ,  $\langle g_2 \rangle \setminus \{\gamma(\zeta_1), \gamma(\zeta_2)\} \subset \tilde{U}^-(\gamma(\tilde{\zeta}))$  etc. Without loss of generality we can assume that all curves  $g_n$ ,  $n = \overline{1, 2\mu}$ , have no common points except end points of the adjacent curves. Evidently,  $g_{2\mu+1}$  is transversal to all curves  $g_n$  with odd indices. Denote  $g = g_1 \cdot g_2 \cdot \dots \cdot g_{2\mu+1}$  and parametrize this curve in some way by  $\theta$ ,  $\theta_0 \leq \theta \leq \theta_{2\mu+1}$ . Denote

$$\gamma^*(\theta) = \begin{cases} \gamma(e^{i\theta}), & 0 \leq \theta \leq \theta_0, \\ g(\theta), & \theta_0 \leq \theta \leq \theta_{2\mu+1}, \\ \gamma(e^{i\theta}), & \theta_{2\mu+1} \leq \theta \leq 2\pi. \end{cases}$$

Denote by  $D_{2k+1}$ ,  $k = \overline{0, \mu}$ , the domain bounded by the closed Jordan curve  $\gamma[\zeta_{2k}, \zeta_{2k+1}] \cdot g_{2k+1}^{-1}$ , and by  $D_{2k}$ ,  $k = \overline{1, \mu}$ , the domain bounded by  $g_{2k}^{-1} \cdot \gamma[\zeta_{2k}, \zeta_{2k-1}]$ . Take in each  $D_{2k}$ ,  $k = \overline{1, \mu}$ , a point  $a_{N+k}$  and draw simple disjoint paths  $\alpha_{N+k}$  through  $E$  on the left of  $\alpha_0$  such that  $i(\alpha_j) = a_j$  and

$t(\alpha_j)$  lie in the unbounded component of  $\mathbb{R}^2 \setminus \langle \gamma \rangle$ ,  $j = \overline{N+1, N+\mu}$ . Moreover, we can choose paths  $\alpha_j$ ,  $j = \overline{N+1, N+\mu}$ , in such a way that the crossing point of  $\alpha_j$  and  $g_{2\ell}$  precedes, on  $g$ , the crossing point of  $\alpha_s$  and  $g_{2\ell}$  when  $N+1 \leq j < s \leq \ell + N$ ,  $1 \leq \ell \leq \mu$ . Number the crossing points along each  $\alpha_j$ ,  $j = \overline{N+1, N+\mu}$ , consecutively so that the second coordinate of  $\langle \alpha_j \rangle \cap \langle g_{2\ell} \rangle$  is equal to  $\mu$ . It is evident that all crossings  $\langle \alpha_j \rangle \cap \langle g_{2\ell} \rangle$ ,  $j = \overline{N+1, N+\mu}$ ,  $1 \leq \ell \leq \mu$ , are negative. Choose two points  $p_1$  and  $p_2$  such that  $p_1 \in \langle \alpha_0 \rangle \cap \tilde{U}(\gamma(\zeta))$ ,  $p_2 \in \langle \alpha_0 \rangle \cap \tilde{U}(\gamma(\zeta))$ ,  $p_1$  precedes  $p_2$  on  $\alpha_0$  and  $(\langle \gamma^* \rangle \setminus \langle g \rangle) \cap \langle \alpha_0[p_1, p_2] \rangle = \emptyset$ . Denote by  $q$  an arc such that  $i(q) = p_1$ ,  $t(q) = p_2$ ,  $\langle q \rangle \cap (\langle \gamma \rangle \setminus \langle \gamma[\zeta', \zeta_0] \rangle) = \emptyset$  and  $\langle q \rangle \cap \langle \gamma[\zeta', \zeta_0] \rangle$  consists of a unique point. Denote  $\tilde{\alpha}_0 = \alpha_0[a_0, p_1] \cdot q \cdot \alpha_0[p_2, b_0]$ .

Consider the system of rays  $\tilde{\alpha}_0, \alpha_1, \dots, \alpha_N, \alpha_{N+1}, \dots, \alpha_{N+\mu}$ . As  $g_{2\mu+1}$  is transversal to all  $g_{2k+1}$ ,  $k = \overline{0, \mu-1}$ , we have at least one initial point of rays in each bounded component of  $\mathbb{R}^2 \setminus \langle \gamma^* \rangle$ . Thus we obtain the raying for  $\gamma^*$ .

Now we are going to construct an assemblage for  $\gamma^*$ . Since the existence of an assemblage does not depend on a choice of the initial point of the curve, we can assume that  $i(\gamma^*) = \gamma(\zeta_0)$ . Then the word for  $\gamma^*$  has a following form:

$$\begin{aligned}
 & (N+1, \mu, -1)(N+2, \mu, -1)(N+3, \mu, -1) \cdots (N+\mu, \mu, -1) \\
 & (N+2, \mu-1, -1)(N+3, \mu-1, -1)(N+4, \mu-1, -1) \cdots (N+\mu, \mu-1, -1) \\
 & \dots\dots\dots \\
 & (N+\mu-1, 1, -1)(N+\mu, 2, -1)(N+\mu, 1, -1)S_0 \\
 & (N+\mu, \mu+r_1, 1)(N+\mu-1, \mu+r_1, 1) \cdots (N+1, \mu+r_1, 1)S_1 \\
 & (N+\mu, \mu+r_2, 1)(N+\mu-1, \mu+r_2, 1) \cdots (N+1, \mu+r_2, 1)S_2 \\
 & \dots\dots\dots \\
 & (N+\mu, \mu+r_n, 1)(N+\mu-1, \mu+r_n, 1) \cdots (N+1, \mu+r_n, 1)S_n,
 \end{aligned} \tag{1}$$

where  $n \geq \mu$  and  $S_0, S_1, \dots, S_n$  are the sets of letters belonging to rays  $\tilde{\alpha}_0, \alpha_1, \dots, \alpha_N$ . Select from list (1) all letters belonging to  $\bigcup_{i=1}^{\mu} \langle \gamma_{k_i} \rangle \cap \langle \tilde{\alpha}_0 \rangle$ . Let  $r_{n_1}, r_{n_2}, \dots, r_{n_\mu}$  be the second coordinates of these letters arranged in the order as they are encountered in word (1). Replace the numbering of arcs  $\gamma_{k_1}, \dots, \gamma_{k_\mu}$  in the following way. The arc containing the letter with the second coordinate  $r_{n_i}$  will have a new number  $\mu+i$  and be denoted by  $\gamma_{\mu+i}^*$ . Then we replace the second coordinates of letters belonging to  $\gamma_{\mu+i}^*$  by  $\mu+i$  and mark the letters with changed second coordinates by the superscript  $*$ .

Consider the following list of sets:

$$\begin{aligned}
 & \{(N+1, \mu, -1), (N+1, 2\mu, 1)^*\}, \{(N+2, \mu, -1), (N+2, 2\mu, 1)^*\}, \dots \\
 & \quad \{(N+\mu, \mu, -1), (N+\mu, 2\mu, 1)^*\}, \\
 & \{(N+2, \mu-1, -1), (N+2, 2\mu-1, -1)^*\}, \dots \\
 & \{(N+\mu, \mu-1, -1), (N+\mu, 2\mu-1, 1)^*\}, \dots,
 \end{aligned}$$

$$\begin{aligned}
& \{(N+k, \mu - (k-1), -1), (N+k, 2\mu - k + 1, 1)^*\}, \quad (2) \\
& \{(N+k+1, \mu - (k-1), -1), (N+k+1, 2\mu - k + 1, 1)^*\}, \dots, \\
& \{(N+\mu, \mu - (k-1), -1), (N+\mu, 2\mu - k + 1, 1)^*\}, \dots \\
& \{(N+\mu-1, 2, -1), (N+\mu-1, \mu+2, 1)^*\}, \\
& \{(N+\mu, 2, -1), (N+\mu, \mu+2, 1)^*\}, \{(N+\mu, 1, -1), (N+\mu, \mu+1, 1)^*\}.
\end{aligned}$$

Include all sets of list (2) as well as all sets of  $\mathcal{G}(\gamma)$  into  $\mathcal{G}(\gamma^*)$ .  $\mathcal{B}(\gamma^*)$  consists of all sets of  $\mathcal{B}(\gamma)$  except  $B_0$ . We claim that  $\mathcal{G}(\gamma^*) \cup \mathcal{B}(\gamma^*)$  is an assemblage for  $\gamma^*$ . It is evident that conditions (a), (c) (d), (e) are fulfilled.

Recall that  $B = \{0, k_0, 1), (0, \mu+1, 1)^*, (0, \mu+2, 1)^*, \dots, (0, 2\mu, 1)^*\}$  was included in  $\mathcal{A}(\gamma)$  and hence the set  $[\theta_0, \theta_{2\mu+1}] \cup (\gamma^*)^{-1}(\bigcup_{j=1}^{\mu} \langle \gamma_{\mu+j}^* \rangle)$  does not separate, on  $S^1$ , any  $(\gamma^*)^{-1}(A)$  for  $A \in \mathcal{A}(\gamma)$ ,  $A \neq B$ . Thus we have to check that for any two sets  $A_1$  and  $A_2$  from the (2)  $(\gamma^*)^{-1}(A_1)$  does not separate, on  $S^1$ , the set  $(\gamma^*)^{-1}(A_2)$ .

Consider two sets  $A_i = \{(N+k_i, m_i, -1), (N+k_i, \mu+m_i, 1)\}$ ,  $i = 1, 2$ . Suppose that  $(\gamma^*)^{-1}(N+k_1, m_1, -1)$  precedes  $(\gamma^*)^{-1}(N+k_2, m_2, -1)$ . Then either  $m_1 > m_2$ , or  $m_1 = m_2$  and  $k_1 < k_2$ . In the former case  $\mu+m_2 < \mu+m_1$  and hence  $(N+k_1, \mu+m_1, 1)^*$  follows  $(N+k_2, \mu+m_2, 1)^*$ . In the latter case the crossing points  $(N+k_1, \mu+m_1, 1)^*$  and  $(N+k_2, \mu+m_1, 1)^*$  lie on the same  $\gamma_{\mu+m_1}^*$  and as  $k_1 < k_2$ ,  $(N+k_2, \mu+m_1, 1)^*$  precedes  $(N+k_1, \mu+m_1, 1)^*$  on  $\gamma_{\mu+m_1}^*$ . In both cases  $(\gamma^*)^{-1}(A_1)$  does not separate  $(\gamma^*)^{-1}(A_2)$  on  $S^1$ . Thus  $\mathcal{A}(\gamma^*)$  is an assemblage for  $\gamma^*$ . Moreover,  $\mathcal{A}(\gamma^*)$  is a maximal assemblage since

$$\sum_{B \in \mathcal{B}(\gamma^*)} (\text{card } B - 1) = \sum_{B \in \mathcal{B}(\gamma) \setminus B_0} (\text{card } B - 1) = \tau(\gamma) - \mu = \tau(\gamma^*).$$

Denote by  $f^*(\zeta)$ ,  $\zeta \in \overline{D}^2$ , the interior extension of  $\gamma^*$  and let  $\ell$  be the arc of  $S^1$  corresponding to  $g$  under the mapping  $f^*$ .

Now we are going to construct some bordered Riemann surface using the procedure of identification of boundary arcs of the domains  $D_k$ ,  $k = \overline{1, 2\mu+1}$ . Write boundaries of  $D_1$  and  $D_2$  in the form

$$\begin{aligned}
\partial D_1 &= \gamma[\zeta_0, \tilde{\zeta}] \cdot \gamma[\tilde{\zeta}, \zeta_1] g_1^{-1}, \\
\partial D_2 &= g_2^{-1} \cdot \gamma[\zeta_1, \tilde{\zeta}] \cdot \gamma[\tilde{\zeta}, \zeta_2]
\end{aligned}$$

and paste  $\overline{D}_1$  and  $\overline{D}_2$  along  $\langle \gamma[\zeta_1, \tilde{\zeta}] \rangle$ . Denote by  $\widetilde{D}_2$  the obtained domain. We have

$$\partial \widetilde{D}_2 = \gamma[\zeta_0, \tilde{\zeta}] \cdot \gamma[\tilde{\zeta}, \zeta_2] \cdot g_2^{-1} \cdot g_1^{-1}$$

and

$$\partial D_3 = \gamma[\zeta_2, \tilde{\zeta}] \cdot \gamma[\tilde{\zeta}, \zeta_3] \cdot g_3^{-1}$$

After pasting  $\widetilde{D}_2$  and  $\overline{D}_3$  along  $\langle \gamma[\zeta_2, \widetilde{\zeta}] \rangle$ , we obtain the domain  $\widetilde{D}_3$  which is no longer one-sheeted and

$$\partial \widetilde{D}_3 = \gamma[\zeta_0, \widetilde{\zeta}] \cdot \gamma[\widetilde{\zeta}, \zeta_3] \cdot g_3^{-1} \cdot g_2^{-1} \cdot g_1^{-1}.$$

After  $2\mu$  steps of pasting, we obtain the many-sheeted domain  $D_{2\mu+1}$  and

$$\partial \widetilde{D}_{2\mu+1} = \gamma[\zeta_0, \widetilde{\zeta}] \cdot \gamma[\widetilde{\zeta}, \zeta_{2\mu+1}] \cdot g_{2\mu+1}^{-1} \cdot g_{2\mu}^{-1} \cdots g_1^{-1} = \gamma[\zeta_0, \zeta_{2\mu+1}] \cdot g^{-1}.$$

Denote by  $G$  the image of  $D_{2\mu+1}$  under the mapping  $h(w) = \sqrt[\mu+1]{w - \gamma(\widetilde{\zeta})}$ .  $G$  is a simply-connected domain bounded by a simple closed curve.

Let  $D'$  be a simply connected domain such that  $D' \subset C\overline{D}^2$  and  $\overline{D}' \cap \overline{D}^2 = \langle \ell \rangle$ . Consider the function  $h(f^*(e^{i\theta}))$ ,  $e^{i\theta} \in \langle \ell \rangle$ , where we mean that the points  $f^*(e^{i\theta})$  belonging to  $\langle g \rangle$  are located on the boundary of the many-sheeted domain  $\widetilde{D}_{2\mu+1}$ . Then  $e^{i\theta} \rightarrow h(f^*(e^{i\theta}))$  is a homeomorphism of  $\langle \ell \rangle$  onto some arc of the boundary of the domain  $G$ . Let  $H$  be a homeomorphism of  $\overline{D}'$  onto  $\overline{G}$  such that

$$H|_{\langle \ell \rangle} = h \circ f^*. \quad (3)$$

The existence of such a homeomorphism  $H$  is evident (it follows from Lemma 6.1 of [11]). Consider the function

$$\widetilde{f}(z) = \begin{cases} f^*(z), & \text{if } z \in \overline{D}^2, \\ h^{-1} \circ H, & \text{if } z \in \overline{D}'. \end{cases}$$

By (3)  $\widetilde{f}(z)$  is continuous in  $\overline{D}^2 \cup \overline{D}' \cup \langle \ell \rangle$ . As  $f^*(z)$  and  $h^{-1} \circ H$  are interior mappings in  $D^2$  and  $D'$ , respectively, and  $H$  and  $h$  are homeomorphisms on  $\langle \ell \rangle$  and  $\langle g \rangle$ , respectively,  $\widetilde{f}(z)$  is an interior function in  $D^2 \cup D' \cup \langle \ell \rangle$ . The behavior of  $\widetilde{f}$  in a vicinity of the point  $m \in \langle \partial(D^2 \cup D' \cup \langle \ell \rangle) \rangle$ ,  $\widetilde{f}(m) = \gamma(\widetilde{\zeta})$  is defined by the behavior of  $h$  at the point  $\gamma(\widetilde{\zeta})$  and hence  $m$  is a partial critical point of  $\widetilde{f}$  of multiplicity  $\mu$ .  $\square$

Now we will prove the inverse statement.

**Theorem 2.** *If a normal curve  $\gamma$  has an interior extension possessing partial critical points, then there exists an interior extension of  $\gamma$  without partial critical points.*

Let  $\widetilde{f}$  be an interior extension of  $\gamma$ . Denote by  $\zeta_j$ ,  $j = \overline{1, m}$ , partial critical points of  $\widetilde{f}$ , and by  $\mu_j$  their multiplicities. Let  $U_0(\zeta_j)$  be a neighborhood of  $\zeta_j$  such that  $\widetilde{f}|_{\overline{U_0(\zeta_j)} \cap S^1}$  is a simple arc. Let  $\ell_j = [\zeta'_j, \zeta''_j]$  be an arc of  $S^1$  such that  $\zeta_j \in \langle \ell_j \rangle \subset U_0(\zeta_j)$  and let  $U(\zeta_j)$  be a neighborhood of  $\zeta_j$  such that  $\partial U^-(\zeta_j) = \ell'_j \cdot \ell_j^{-1}$ , where  $\ell'_j$  is a simple arc  $\langle \ell'_j \rangle \cap \overline{D}^2 = \{\zeta'_j, \zeta''_j\}$ . Denote  $L_j = \widetilde{f} \circ \ell_j$ , and consider a neighborhood  $U(\gamma(\zeta_j))$  such that  $\partial U(\gamma(\zeta_j))$  is a simple arc and  $\partial U(\gamma(\zeta_j)) \cap \langle \gamma \rangle = \{\widetilde{f}(\zeta'_j), \widetilde{f}(\zeta''_j)\}$ . Denote  $\partial U^-(\gamma(\zeta_j)) = L'_j \cdot L_j^{-1}$  and let  $\varphi :$

$\langle \ell'_j \rangle \rightarrow \langle L'_j \rangle$  be an arbitrary homeomorphism  $\varphi(\zeta'_j) = \tilde{f}(\zeta'_j)$ ,  $\varphi(\zeta''_j) = \tilde{f}(\zeta''_j)$ . Denote by  $\Phi$  the homeomorphism

$$\Phi : \overline{U}^-(\zeta_j) \rightarrow \overline{U}^-(\gamma(\zeta_j))$$

coinciding with  $\varphi$  on  $\partial U^-(\zeta_j)$ .

The mapping

$$f_j = \begin{cases} \tilde{f}(z), & z \in \overline{D}^2, \\ \Phi(z), & z \in \overline{U}^-(\zeta_j) \end{cases}$$

is an interior mapping of the simply connected domain  $D^2 \cup U^-(\zeta_j) \cup \langle \ell_j \rangle$  and from the definition of a partial critical point it follows that  $\zeta_j$  is a (interior) critical point of  $f_j$  of multiplicity  $\mu_j$ .

Repeat the above described procedure for every partial critical point of  $\tilde{f}$ . Finally, we obtain a function  $f$  without partial critical points. The restriction of  $f$  to the boundary of the domain of definition is a representation of a normal curve  $\Gamma$  which differs from  $\gamma$ . Applying the theorem of Section 2 to  $\Gamma$  we conclude that there exists an assemblage  $\mathcal{A}(\Gamma)$ . As the partial critical points  $\zeta_j$ ,  $j = \overline{1, m}$ , of  $\tilde{f}$  have become the interior critical points of  $f$ , each  $\zeta_j$ ,  $j = \overline{1, m}$ , must be the initial point for a ray  $\alpha_j$ ,  $j = \overline{1, m}$ , included into the raying  $R(\Gamma)$  generating  $\mathcal{A}(\Gamma)$ .

Take  $m$  points  $z_j$ ,  $z_j \in U^+(\gamma(\zeta_j))$ ,  $j = \overline{1, m}$ , and connect  $z_j$  with  $\zeta_j$  by a simple arc  $\lambda_j$  in  $U^+(\gamma(\zeta_j))$ . Consider a new system  $R'$  of rays obtained from  $R(\Gamma)$  by replacing of  $\alpha_j$  by  $\alpha'_j = \lambda_j \cdot \alpha_j$ . It is clear that  $R'$  satisfies all conditions of raying for  $\gamma$ .

Let  $B \in \mathcal{B}(\Gamma)$ , and let  $B$  consist of points belonging to  $\alpha'_j$ . If the crossing point of  $\alpha_j$  with  $L'_j$  does not belong to  $B$  we include  $B$  into  $\mathcal{B}(\gamma)$ . If the crossing point of  $\alpha_j$  with  $L'_j$  belongs to  $B$ , we replace this point by the crossing point of  $\alpha'_j$  with  $\ell$  and include obtained set  $B'$  into  $\mathcal{B}(\gamma)$ . All other sets of  $\mathcal{B}(\Gamma)$  are included into  $\mathcal{A}(\gamma)$  unchanged as well as all sets of  $\mathcal{G}(\Gamma)$ . It is evident that the set  $\mathcal{B}(\gamma) \cup \mathcal{G}(\Gamma)$  is an assemblage for  $\gamma$ . As  $\text{card } B' = \text{card } B$  the multiplicity of every  $z_j$  is equal to  $\mu_j$ ,  $j = \overline{1, m}$ . Hence by Theorem 24.2 of [11]  $\mathcal{A}(\gamma)$  is a maximal assemblage for  $\gamma$ .

#### 4. THE EXISTENCE OF AN INTERIOR EXTENSION OF A NONNORMAL CURVE

Let  $\gamma(\zeta) \equiv \gamma(e^{i\theta})$  be a parametrization of a closed curve  $\gamma$ . The point  $\hat{\zeta} = e^{i\hat{\theta}}$  is called a reversing point if there exist  $\varepsilon' > 0$ ,  $\varepsilon'' > 0$  and a sense reversing homeomorphism  $\psi_{\hat{\zeta}} : [e^{i(\hat{\theta}-\varepsilon')}, e^{i\hat{\theta}}] \rightarrow [e^{i\hat{\theta}}, e^{i(\hat{\theta}+\varepsilon'')}]$  such that

$$\gamma(\psi_{\hat{\zeta}}(e^{i\theta})) = \gamma(e^{i\theta}), \quad \theta \in [\hat{\theta} - \varepsilon', \hat{\theta}].$$

Let  $f$  be the interior extension of  $\gamma$  and let  $\hat{\zeta}$  be a reversing point of  $\gamma$ . We say that  $\hat{\zeta}$  is a reversing critical point of  $f(z)$  (or  $f(\hat{\zeta})$  is a reversing branch point of  $f^{-1}$ ) of multiplicity  $\mu(f, \hat{\zeta}) \geq 0$  if there exist some  $U(\hat{\zeta})$ ,  $U(f(\hat{\zeta}))$  and  $\mu(f, \hat{\zeta})$  simple arcs  $\ell_j$  passing through  $U^+(\hat{\zeta})$  and such that  $\langle \ell_j \rangle$  have a unique



common point at  $\widehat{\zeta}$ ,  $f(z)$  is schlicht in every component  $v_j$ ,  $j = \overline{1, \mu(f, \widehat{\zeta}) + 1}$ , of  $U(\widehat{\zeta}) \setminus \cup \langle \ell_j \rangle$  and  $f(z)$  takes every  $V_j$  onto  $U(f(\widehat{\zeta})) \setminus \langle \gamma[e^{i(\theta-\varepsilon')}, e^{i(\theta+\varepsilon'')}] \rangle$ .

Let  $Q = (q_1, q_2, \dots, q_m)$  be an ordered  $m$ -tuple of points of  $\langle \gamma \rangle$  distinct from the chosen initial point of  $\gamma$ . It is allowed that there are coinciding points among  $q_j$  but we require that every pair of points with successive indices consists of distinct points. Denote by  $\gamma_1$  the arc of  $\gamma$  such that  $i(\gamma_1) = i(\gamma)$  and  $t(\gamma_1) = q_1$ , by  $\gamma_2$  the arc of  $\gamma$  or  $\gamma^{-1}$ ,  $i(\gamma_2) = q_1$ ,  $t(\gamma_2) = q_2$  etc,  $i(\gamma_{m+1}) = q_m$ ,  $t(\gamma_{m+1}) = t(\gamma)$ . Consider the curve  $\gamma_Q = \gamma_1 \cdot \gamma_2 \cdots \gamma_{m+1}$ .

The curve coincides with  $\gamma$  iff every  $q_{j-1}$  precedes  $q_j$  on  $\gamma$ ,  $j = \overline{2, m}$ . Denote by  $Q'$  the subset of  $Q$  consisting of points  $q_k$  not separating  $q_{k-1}$  and  $q_{k+1}$  on  $\gamma$ ,  $k = \overline{2, m-1}$ . Denote  $n = \text{card } Q'$ . It is clear that  $n$  is even. Let  $\bar{\mu} = (\mu_1, \mu_2, \dots, \mu_m)$  be a vector where  $\mu_j$ ,  $j = \overline{1, m}$ , are positive integers. Denote  $|\bar{\mu}| = \sum_{j=1}^m \mu_j$  and  $\nu = n/2$ .

**Theorem 3.** *Let  $\gamma$  be a normal curve and let  $f$  be an interior extension of  $\gamma$  with a partial critical point  $\widehat{\zeta}$  of multiplicity  $\tilde{\mu}$ . Then for any  $Q \subset U(f(\widehat{\zeta}))$  and  $\bar{\mu}$  such that  $|\bar{\mu}| + \nu = \tilde{\mu}$  there exists an interior extension  $\widehat{f}$  of  $\gamma_Q$  such that every  $q_j \in Q \setminus Q'$  is a partial branch point and  $q_j \in Q'$  is a reversing branch point of  $\widehat{f}^{-1}$  of multiplicity  $\mu_j$ ,  $j = \overline{1, m}$ .*

Denote  $n_k = \text{card}(Q' \cap \{q_1, q_2, \dots, q_k\})$  and by  $\widehat{\zeta}_j$ ,  $j = \overline{1, m}$ , the points corresponding to  $q_j$ ,  $j = \overline{1, m}$ , under the mapping  $f$ . Let the set  $E$ , points  $\zeta_k$ ,  $k = 0, 2\mu + 1$ , and curves  $g_k$ ,  $k = \overline{1, 2\tilde{\mu} + 1}$ , be chosen as in Section 3. We assume that  $U(f(\widehat{\zeta}))$  is chosen so that  $\langle g_k \rangle \cap U(f(\widehat{\zeta})) = \emptyset$ ,  $k = \overline{1, 2\tilde{\mu} + 1}$ . Denote  $f|_{S^1} = \gamma(\zeta)$ . Then

$$\partial D_j = \gamma[\zeta_{j-1}, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}, \zeta_j] g_j^{-1}, \quad j = 1, 2\mu_1 + 1 + n_1.$$

Paste  $\overline{D}_1$  and  $\overline{D}_2$  along  $\langle \gamma[\widehat{\zeta}_1, \zeta_1] \rangle$  and denote the obtained domain by  $\overline{\overline{D}}_2$ . We have

$$\partial \widehat{D}_2 = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}_1, \zeta_2] \cdot g_2^{-1} \cdot g_1^{-1}.$$

After pasting  $\overline{\overline{D}}_2$  and  $\overline{D}_3$  along  $\langle \gamma[\zeta_2, \widehat{\zeta}_1] \rangle$ , we obtain  $\overline{\overline{D}}_3$  and

$$\partial \widehat{D}_3 = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}_1, \zeta_3] \cdot g_3^{-1} \cdot g_2^{-1} \cdot g_1^{-1}.$$

After  $2\mu_1 + n_1$  steps of pasting we obtain the many-sheeted domain  $\widehat{D}_{2\mu_1+1+n_1}$  with the boundary

$$\partial \widehat{D}_{2\mu_1+1+n_1} = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}_1, \zeta_{2\mu_1+1+n_1}] \cdot g_{2\mu_1+1+n_1}^{-1} \cdots g_1^{-1}.$$

The next domain  $\overline{\overline{D}}_{2\mu_1+n_1+2}$  is pasted with  $\overline{\overline{D}}_{2\mu_1+n_1+1}$  along  $\langle \gamma[\zeta_{2\mu_1+n_1+1}, \widehat{\zeta}_2] \rangle$ . The procedure is continued by pasting  $\overline{\overline{D}}_{2\mu_1+n_1+k}$  with  $\overline{\overline{D}}_{2\mu_1+n_1+k-1}$  along  $\langle \gamma[\zeta_{2\mu_1+n_1+k}, \widehat{\zeta}_2] \rangle$  for  $k \leq 2\mu_1 + n_2 + 1$ . The domain  $\overline{\overline{D}}_{2\mu_1+2\mu_2+n_2+2}$  is pasted with  $\overline{\overline{D}}_{2\mu_1+2\mu_2+n_2+1}$  along  $\langle \gamma[\zeta_{2\mu_1+2\mu_2+n_2+1}, \widehat{\zeta}_3] \rangle$ . The general rule of pasting

is the following: the domain  $\overline{D}_{2 \sum_{j=1}^{s-1} \mu_j + n_{s-1} + k}$  is pasted with  $\overline{D}_{2 \sum_{j=1}^{s-1} \mu_j + n_{s-1} + k - 1}$  along  $\langle \gamma[\zeta_{2 \sum_{j=1}^{s-1} \mu_j + k}, \widehat{\zeta}_s] \rangle$  for  $2 \leq k \leq 2\mu_s + n_s + 1$ . The domain  $\overline{D}_{2 \sum_{j=1}^s \mu_j + n_{s-1} + 2}$  is pasted with  $\overline{D}_{2 \sum_{j=1}^s \mu_j + n_{s-1} + 1}$  along  $\langle \gamma[\zeta_{2 \sum_{j=1}^s \mu_j + n_s + 1}, \widehat{\zeta}_{s+1}] \rangle$ ,  $1 \leq s \leq m-1$ ,  $n_0 = 0$  and the domain  $\overline{D}_{2 \sum_{j=1}^m n_m} = \overline{D}_{2 \left( \sum_{j=1}^m \mu_j + \nu \right)} = \overline{D}_{2\tilde{\mu}}$  is pasted with  $\overline{D}_{2\tilde{\mu}+1}$  along  $\langle \gamma[\widehat{\zeta}_m, \zeta_{2\tilde{\mu}+1}] \rangle$ . We have

$$\partial \widehat{D}_{2 \sum_{j=1}^{s-1} \mu_j + n_{s-1} + k} = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}_1, \zeta_{2 \sum_{j=1}^{s-1} \mu_j + n_{s-1} + k}] \cdot g_{2 \sum_{j=1}^{s-1} \mu_j + n_{s-1} + k}^{-1} \cdots g_1^{-1}.$$

and

$$\partial \widehat{D}_{2\tilde{\mu}+1} = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}_1, \widehat{\zeta}_2] \cdots \gamma[\widehat{\zeta}_m, \zeta_{2\tilde{\mu}+1}] \cdot g_{2\tilde{\mu}+1}^{-1} \cdot g_{2\tilde{\mu}}^{-1} \cdot g_1^{-1}. \quad (4)$$

We claim that  $\widehat{D}_{2\tilde{\mu}+1}$  is a simply connected domain. Indeed, all domains involved in the pasting procedure are simply connected and each domain is pasted with only one other domain along only one boundary arc.

Let  $\widehat{h} : \widehat{D}_{2\tilde{\mu}+1} \rightarrow G$  be a homeomorphism onto a one-sheeted simply connected domain. Let the domain  $D'$  be the same as in the proof of Theorem 1, and let  $\widehat{H}$  be a homeomorphism of  $D'$  onto  $G$  such that

$$\widehat{H}|_{\langle \ell \rangle} = \widehat{h}(f(e^{i\theta})), \quad e^{i\theta} \in \langle \ell \rangle.$$

From (4) it follows that the function

$$\widehat{f} = \begin{cases} f(z), & z \in \overline{D}^2, \\ h^{-1} \circ \widehat{H}, & z \in \overline{D}' \end{cases}$$

is an interior extension of  $\gamma_Q$ .

To complete the proof, we have to show that the point  $q_j$  is a branch point (partial or reversing) of multiplicity  $\mu_j$ ,  $j = \overline{1, m}$ . It is clear that without loss of generality we can assume that  $\widehat{f}$  is defined in  $\overline{D}^2$ , and the preimages of  $q_j$  under  $\widehat{f}$  are the same points  $\widehat{\zeta}_j$ ,  $j = \overline{1, m}$ .

Take some  $\widehat{\zeta}_s$  and calculate the number of domains  $D_k$  such that  $q_s$  belongs to  $\overline{D}_k$ . From the construction of  $\widehat{D}_k$  it follows that  $q_s$  belongs to  $\overline{D}_{2 \sum_{j=1}^{s-1} \mu_j + n_{s-1} + k}$ ,

$1 \leq k \leq 2\mu_s + n_s + 1$ . Hence  $q_s$  belongs to  $2\mu_s + 1$  domains  $D_k$  if  $n_s = n_{s-1}$  and to  $2\mu_s + 2$  domains  $D_k$  if  $n_s = n_{s-1} + 1$ . Thus  $\widehat{\zeta}_s$  is a partial critical point of multiplicity  $\mu_s$  if  $n_s = n_{s-1}$  ( $q_s$  separates  $q_{s-1}$  and  $q_{s+1}$ ) and  $\widehat{\zeta}_s$  is a reversing critical point of multiplicity  $\mu_s$  if  $n_s = n_{s-1} + 1$  ( $q_s$  does not separate  $q_{s-1}$  and  $q_{s+1}$ ).

## REFERENCES

1. C. J. TITUS, The combinatorial topology of analytic functions on the boundary of a disk. *Acta Mathematica* **106**(1961), No. 1–2, 45–64.
2. S. J. BLANK, Extending immersions of the circle. *Brandeis University, Waltham, Mass*, 1967.
3. M. L. MARX, Normal curves arising from light open mappings of the annulus. *Trans. Amer. Math. Soc.* **120**(1965), 46–56.
4. M. L. MARX, The branch point structure of extensions of interior boundaries. *Trans. Amer. Math. Soc.* **131**(1968), 79–97.
5. M. L. MARX, Extensions of normal immersions of  $S$  into  $R$ . *Trans. Amer. Math. Soc.* **187**(1974), 309–325.
6. G. K. FRANCIS, Assembling compact Riemann surfaces with given boundary curves and branch points on the sphere. *Illionis J. Math.* **20**(1976), 198–217.
7. R. F. VERHEY, Diffeomorphic invariants of immersed circles. *Trans. Amer. Math. Soc.* **163**(1973), 47–63.
8. C. L. EZELL and M. L. MARX, Branched extensions of curves in orientable surfaces. *Trans. Amer. Math. Soc.* **259**(1980), 515–547.
9. H. WHITNEY, On regular closed curves in the plane. *Compositio Mathematica* **4**(1937) 276–284.
10. S. STOÏLOW, Leçons sur les principes topologiques de la théorie des fonctions analytiques. *Gauthier–Villars, Paris*, 1956.
11. M. MORSE, Topological methods in the theory of functions of a complex variable. *Princeton University Press, Princeton*, 1947.

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