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ON THE PRANDTL EQUATION

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Abstract. The unique solvability of the airfoil (Prandtl) integro-differential equation on the semi-axis $\mathbb{R}^+ = [0,\infty)$ is proved in the Sobolev space W_p^1 and Bessel potential spaces H_p^s under certain restrictions on p and s.

§ 0. Introduction

The purpose of this paper is to investigate the integro-differential equation

$$A\nu(t) = \nu(t) - \frac{\lambda}{\pi}\int_0^\infty \frac{\nu'(\tau)}{\tau - t}\,d\tau = 0, \quad t \in \mathbb{R}_+, \quad \lambda = \text{const} > 0, \quad (0.1)$$

which is known as the Prandtl equation.

Such equations occur, for instance, in elasticity theory (see [1] and § 2 below), hydrodynamics (aircraft wing motion, see [2]–[5]).

In elasticity theory, a solution $\nu(t)$ of (0.1) is sought for in the Sobolev space $W_p^1(\mathbb{R}_+)$ and satisfies the boundary condition

$$\nu(0) = c_0 \neq 0, \quad (0.2)$$

where the constant c_0 is defined by elastic constants (see (2.12) below).

Theorem 0.1. *Equation* (0.1) *with the boundary condition* (0.2) *has a unique solution in Sobolev spaces W_p^1 if and only if* $1 < p < 2$.

The proof of the theorem is given in § 4.

We shall consider the nonhomogeneous equation corresponding to (0.1)

$$A\nu(t) = f(t) \quad (0.3)$$

which will be treated as a pseudodifferential equation in the Bessel potential spaces, namely, A maps $\widetilde{H}_p^s(\mathbb{R}_+)$ into the space $H_p^{s-1}(\mathbb{R}_+)$, $s\in\mathbb{R}$, $1<p<\infty$.

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The necessary and sufficient conditions for equation (0.1) to be Fredholm are given and the index formula is derived (Theorem 3.1).

In [1, § 32] the boundary value problem (0.1), (0.2) is solved by means of the Wiener–Hopf method. Applying the Fourier transform, equation (0.1) is reduced to a boundary value problem of function theory (BVPFTh) which is solved by standard procedures (see [3]).

As is known, an equivalent reduction of problem (0.1), (0.2) to the corresponding BVPFTh is possible only for Hilbert spaces $H_2^s(\mathbb{R}_+)$, whereas for spaces $H_p^s(\mathbb{R}_+)$, $p \neq 2$, the BVPFTh should be considered in the complicated space $\mathcal{F}H_p^s(\mathbb{R}_+)$ that is not described exactly. Theorem 0.1 clearly implies that the case $p = 2$ is not suitable for considering problem (0.1), (0.2), whereas the case $1 < p < 2$ can be treated directly, without applying the Fourier transform.

In this paper we develop a precise theory of the boundary value problem (0.1), (0.2) in the spaces $H_p^s(\mathbb{R}_+)$ and $W_p^1(\mathbb{R}_+)$ and suggest criteria (necessary and sufficient conditions) for its solvability.

Remark. By the results of [3], [6], the solution of equation (0.1) has the asymptotics

$$\nu(t) = c_0 + c_1 t^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \quad \text{as} \quad t \to 0, \quad c_1 = const \neq 0.$$

A full asymptotic expression of the solution can be derived, but this makes the subject of a separate investigation.

§ 1. Basic Notation and Spaces

Let us recall some standard notation:

$\mathbb{R}$ is the one-dimensional Euclidean space.

$L_p(\mathbb{R})$ $(1 < p < \infty)$ is the Lebesgue space.

$S(\mathbb{R})$ is the Schwarz space of infinitely smooth functions rapidly vanishing at infinity.

$S'(\mathbb{R})$ is the dual Schwarz space of tempered distributions.

The Fourier transform

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{i\xi x}\varphi(x)\,dx, \quad x \in \mathbb{R}, \tag{1.1}$$

and the inverse Fourier transform

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\varphi(\xi)\,d\xi, \quad \xi \in \mathbb{R}, \tag{1.2}$$

are the bounded operators in both spaces $S(\mathbb{R})$ and $S'(\mathbb{R})$. Hence the convolution operator

$$a(D)\varphi = W_a^0\varphi := \mathcal{F}^{-1}a\mathcal{F}\varphi \quad \text{with} \quad a \in S'(\mathbb{R}), \quad \varphi \in S(\mathbb{R}), \tag{1.3}$$

is the bounded transformation from $S(\mathbb{R})$ into $S'(\mathbb{R})$ (see [7]).

The Bessel potential space $H_p^s(\mathbb{R})$ ($s \in \mathbb{R}$, $1 < p < \infty$) is defined as a subset of $S'(\mathbb{R})$ endowed with the norm

$$\big\|\varphi|\, H_p^s(\mathbb{R})\big\| := \big\|\langle D\rangle^s \varphi|\, L_p(\mathbb{R})\big\|, \quad \text{where} \quad \langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}. \quad (1.4)$$

For a non-negative integer $s \in N_0 = \{0, 1, \dots\}$ the space $H_p^s(\mathbb{R})$ coincides with the Sobolev space $W_p^s(\mathbb{R})$, and in that case the equivalent norm is defined as follows:

$$\big\|\varphi|\, H_p^s(\mathbb{R})\big\| \simeq \sum_{k=0}^{s} \big\|\partial^k \varphi|\, L_p(\mathbb{R})\big\| \quad \text{provided} \quad s \in N_0, \qquad (1.5)$$

where ∂ denotes a (generalized) derivative.

The space $\widetilde{H}_p^s(\mathbb{R}_+)$ is defined as a subspace of $H_p^s(\mathbb{R})$ of the functions $\varphi \in H_p^s(\mathbb{R})$ supported in the half-space $\operatorname{supp}\varphi \subset \overline{\mathbb{R}}_+$, where $H_p^s(\mathbb{R}_+)$ denotes distributions φ on $\mathbb{R}_+$ which admit an extension $l_+\varphi \in H_p^s(\mathbb{R})$. Therefore $r_+ H_p^s(\mathbb{R}) = H_p^s(\mathbb{R}_+)$.

If the convolution operator (1.3) has the bounded extension

$$W_a^0 : L_p(\mathbb{R}) \to L_p(\mathbb{R}),$$

then we write $a \in M_p(\mathbb{R})$. For $\mu \in \mathbb{R}$ let

$$M_p^{(\mu)}(\mathbb{R}) = \big\{\langle\xi\rangle^\mu a(\xi) : \quad a \in M_p(\mathbb{R})\big\}. \qquad (1.6)$$

The following fact is valid:

The operator

$$W_a^0 : H_p^s(\mathbb{R}) \to H_p^{s-\mu}(\mathbb{R})$$

is bounded if and only if $a \in M_p^{(\mu)}(\mathbb{R})$.

$PC_p(\mathbb{R})$ will denote the closure of an algebra of piecewise-constant functions by the norm

$$\|a\|_p^0 = \|W_a^0|\, L_p\|.$$

Note that for $1 < p < \infty$ all functions of bounded variation belong to $PC_p(\mathbb{R})$.

$S_{\mathbb{R}}$ denotes the Cauchy singular integral operator

$$S_{\mathbb{R}}\nu(t) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\nu(\tau)}{\tau - t}\, d\tau, \qquad (1.7)$$

where the integral is understood in a sense of the Cauchy principal value:

$$S_{\mathbb{R}}\nu(t) = \frac{1}{\pi i} \lim_{N\to\infty} \lim_{\varepsilon\to 0} \left(\int_{-N}^{t-\varepsilon} + \int_{t+\varepsilon}^{N} \right) \frac{\nu(\tau)}{\tau - t}\, d\tau.$$

§ 2. Half-Plane with a Semi-Infinite Stringer along the Border [1]

Let us consider an elastic plate lying in the complex lower half-plane $z = x + iy$, $y < 0$. Superpose the stringer axis on the positive part of the real axis so that one stringer end would take its origin at 0 and the other would tend to infinity.

It is assumed that the stringer is an elastic line to which tensile force is applied. It is also assumed that stresses within the plate and the stringer are produced by a single axial force applied to the stringer origin 0 and directed along the negative x-axis.

Let E be the elastic constant of the plate, E_0 the elastic modulus of the stringer, h the plate thickness, and S_0 the stringer cross-section; h and S_0 are assumed to be constant values.

According to the condition, the part of the half-plane border (on the left-hand side of the origin) is free from load. Therefore the boundary conditions are written as

$$\sigma_y = \tau_{xy} = 0 \quad \text{for} \quad x < 0, \tag{2.1}$$

where σ_x, σ_y, σ_{xy} are the stress components. On the other part of the border, where the plate is reinforced by the stringer, forces are in the state of equilibrium and there is no bending moment, the boundary conditions read as

$$p_0 - h\int_0^x \tau_{xy}\,dt + k\sigma_x = 0, \quad -h\int_0^x \sigma_y\,dt = 0 \quad \text{for} \quad x > 0, \tag{2.2}$$

where $k = \frac{E_0 S_0}{E}$. Combined together, the latter conditions acquire the form

$$p_0 - h\int_0^x (\tau_{xy} + i\sigma_y)\,dt + k\sigma_x = 0 \quad (x > 0). \tag{2.3}$$

Let us recall the well-known Kolosov–Muskhelishvili representation

$$\sigma_x + \sigma_y = 2\big[\varphi'(z) + \overline{\varphi'(z)}\big], \quad \sigma_y - \sigma_x + 2i\tau_{xy} = 2\big[\overline{z}\varphi''(z) + \psi'(z)\big] \tag{2.4}$$

(see [3]) and the Muskhelishvili formula

$$-i\int_0^t (\tau_{xy} + i\sigma_y)\,d\tau = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \text{const}\,. \tag{2.5}$$

By virtue of (2.4) and (2.5) we can rewrite (2.1) and (2.3) as

$$\begin{gathered}
\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0 \quad (t < 0),\\
ip_0 + h\big[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}\big] +\\
+ik\,\mathrm{Re}\left[\varphi'(t) + \overline{\varphi'(t)} - t\overline{\varphi''(t)} - \overline{\psi'(t)}\right] = 0 \quad (t > 0),
\end{gathered} \tag{2.6}$$

with some nonessential constants omitted.

To solve problem (2.6), we are to find a function $w(t) = \mu(t) + i\nu(t)$ on $[0, \infty]$ which is related to the complex potentials $\varphi(z)$, $\psi(z)$ by the formulae

$$\psi(z) = -\overline{\varphi(z)} - z\varphi'(z), \quad y < 0 \quad (z = x + iy), \tag{2.7}$$

$$\varphi(z) = -\frac{p_0}{2\pi h} \ln z + \varphi_0(z), \tag{2.8}$$

$$\varphi_0(z) = -\frac{1}{2\pi i} \int_0^\infty \frac{\omega(\tau)}{\tau - z} \, d\tau, \tag{2.9}$$

where under $\ln z$ we mean any fixed branch, say $\arg z = 0$ when $x > 0$, $y = 0$.

For the function $w(t)$ we assume that $w(t) \in L_p(\mathbb{R}_+)$ for some $p > 1$, $w'(t) \in L_1(\mathbb{R}_+)$.

For the function $w(t)$ we get

$$\mu(t) = 0, \tag{2.10}$$

$$\nu(t) - \frac{\lambda}{\pi} \int_0^\infty \frac{\nu'(\tau)}{\tau - t} \, d\tau = 0 \quad (t > 0) \tag{2.11}$$

where

$$\lambda = \frac{2E_0 S_0}{Eh}$$

and

$$\nu(0) = -\frac{p_0}{h}\,. \tag{2.12}$$

Thus for the density of integral (2.9) we have obtained the Prandtl equation (2.11) and the boundary condition (2.12).

One can readily obtain equation (2.11) by considering the problem of an infinite plane with a half-infinite stringer attached along the half-axes $\mathbb{R}_+$.

§ 3. A Nonhomogeneous Equation in Bessel Potential Spaces

Lemma 3.1. *The Prandtl operator*

$$A\nu(t) = \nu(t) - \frac{\lambda}{\pi} \int_0^\infty \frac{\nu'(\tau)}{\tau - t}\,, \quad \lambda > 0, \tag{3.1}$$

emerging in equation (2.11) *is a convolution operator*

$$A\nu(t) = \mathcal{F}^{-1}(1 + \lambda|x|)\mathcal{F}\nu(t)$$

with the symbol $1 + \lambda|x| \in M_p^{(1)}(\mathbb{R})$ *of first order* (*see* [8]).

Proof. Note that

$$\mathcal{F}S_{\mathbb{R}}\nu(t) = \mathcal{F}\left(\frac{1}{\pi i}\int_{\mathbb{R}} \frac{\nu(\tau)}{\tau - t}\,d\tau\right) = -\operatorname{sgn} x\mathcal{F}\nu(t)$$

[8, § 1] and

$$\mathcal{F}(\nu'(t)) = -ix\mathcal{F}\nu(t).$$

Therefore

$$\mathcal{F}A\nu(t) = \big(1 - i\lambda(-ix)(-\operatorname{sgn} x)\big)\mathcal{F}\nu(t) = (1 + \lambda|x|)\mathcal{F}\nu(t)$$

and the operator

$$r_+A : \widetilde{H}^s_p(\mathbb{R}_+) \to H^{s-1}_p(\mathbb{R}_+), \quad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{3.2}$$

is bounded [8, § 5]. □

Let us investigate operator (3.1).

Theorem 3.1. *Let $s \in \mathbb{R}$ and $s = [s] + \{s\}$, $[s] = 0, \pm 1, \pm 2, \dots$, $0 \leq \{s\} < 1$, be the decomposition of s into the integer part and the fractional one. The operator r_+A in (3.2) is Fredholm if and only if $|\{s\} - \frac{1}{p}| \neq \frac{1}{2}$. When the latter condition is fulfilled, the operator r_+A is invertible, invertible from the left or invertible from the right provided that $\varkappa$ is zero, positive or negative, respectively.*

Here

$$\begin{aligned} &\varkappa = [s] \quad \text{if} \quad \left|\{s\} - \frac{1}{p}\right| < \frac{1}{2}, \\ &\varkappa = [s] + 1 \quad \text{if} \quad \{s\} - \frac{1}{p} > \frac{1}{2}, \\ &\varkappa = [s] - 1 \quad \text{if} \quad \{s\} - \frac{1}{p} < -\frac{1}{2}, \end{aligned} \tag{3.3}$$

and

$$\operatorname{Ind} r_+A = -\varkappa.$$

We need the following lemma from [8, § 5].

Lemma 3.2. *The operators*

$$\Lambda^s_+ = (D + i)^s l_+, \quad \Lambda^s_- = r_+(D - i)^s,$$
$$(D \pm i)^{\pm s}\varphi = \mathcal{F}^{-1}(x \pm i)^{\pm s}\mathcal{F}\varphi, \quad \varphi \in C^\infty_0(\mathbb{R}_+),$$

arrange the isomorphisms of the spaces

$$\Lambda^s_+ : \widetilde{H}^s_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+), \quad \Lambda^{-s}_- : L_p(\mathbb{R}_+) \to H^s_p(\mathbb{R}_+). \tag{3.4}$$

Proof of Theorem 3.1. Consider the lifted operator $B = \Lambda_-^{s-1} r_+ A \Lambda_+^{-s}$

$$\begin{tikzcd} \widetilde{H}_p^s(\mathbb{R}_+) \arrow[r, "r_+A"] \arrow[d, "\Lambda_+^{-s}"] & H_p^{s-1}(\mathbb{R}_+) \arrow[d, "\Lambda_-^{s-1}"] \\ L_p(\mathbb{R}_+) \arrow[r, "B"] & L_p(\mathbb{R}_+) \end{tikzcd} . \tag{3.5}$$

Due to Lemma 3.2 the operators r_+A and B are isometrically equivalent and therefore it suffices to study the operator B in the space $L_p(\mathbb{R}_+)$ (see diagram (3.5)).

The presymbol $b(x)$ of B equals

$$b(x) = \frac{1+\lambda|x|}{(x+i)^s}\,(x-i)^{s-1} = \Big(\frac{x-i}{x+i}\Big)^s \frac{1+\lambda|x|}{x-i} \tag{3.6}$$

belonging to the class $PC_p(\mathbb{R})$, $1 < p < \infty$ [8].

The corresponding p-symbol reads as

$$\begin{aligned} b_p(x,\xi) = \frac{1}{2}\left[\widetilde{b}(x-0) + \widetilde{b}(x+0)\right] + \\ + \frac{1}{2}\left[\widetilde{b}(x-0) - \widetilde{b}(x+0)\right] \coth \pi\Big(\frac{i}{p} + \xi\Big) \end{aligned} \tag{3.7}$$

[8, § 4], where $\widetilde{b}(x \pm 0) = b(x \pm 0)$, $x \in \mathbb{R}$, $\widetilde{b}(\infty \pm 0) = b(\pm\infty)$.

When s is not an integer, $s \neq 0, \pm 1, \dots$, the function $(\frac{x-i}{x+i})^s$ has a jump on $\overset{\circ}{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and we fix this jump at infinity, i. e., $b(-\infty) \neq b(+\infty)$.

Since $s = [s] + \{s\}$, where $[s] = 0, \pm 1, \pm 2, \dots$, $0 \leq \{s\} < 1$, we can rewrite $b(x)$ as follows:

$$\begin{aligned} b(x) &= \Big(\frac{x-i}{x+i}\Big)^{[s]} \Big(\frac{x-i}{x+i}\Big)^{\{s\}} \frac{1+\lambda|x|}{x-i} = \\ &= -\Big(\frac{x-i}{x+i}\Big)^{[s]} \frac{1+\lambda|x|}{(x^2+1)^{\frac{1}{2}}} \Big(\frac{x-i}{x+i}\Big)^{\{s\}-\frac{1}{2}} \equiv g(x) b^0(x), \end{aligned} \tag{3.8}$$

where

$$g(x) = -\Big(\frac{x-i}{x+i}\Big)^{[s]} \frac{1+\lambda|x|}{(x^2+1)^{\frac{1}{2}}}, \quad b^0(x) = \Big(\frac{x-i}{x+i}\Big)^{\{s\}-\frac{1}{2}}, \tag{3.9}$$

$g(x)$ is a continuous function and $\operatorname{ind} g = [s]$.

Now let us investigate the p-symbol of $b^0(x)$. We shall consider three cases.

I. $\{s\} = \frac{1}{2}$. It is easy to show that $b^0(x)$ is the continuous function $b^0(-\infty) = b^0(+\infty)$ and $\operatorname{ind} b_p^0 = 0$.

II. $0 \leq \{s\} < \frac{1}{2}$. This situation is shown in Fig. 1, where the image of $b(x)$ is plotted on the complex plane and the answer depends on the connecting function $\coth\pi(\frac{i}{p}+\xi)$ $(\coth z=\frac{e^z+e^{-z}}{e^z-e^{-z}})$ which fills up the gap between $b^0(\pm\infty)$.

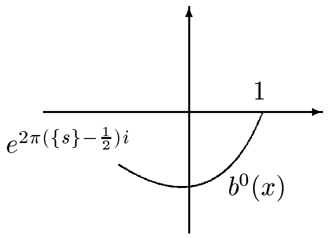

Fig. 1

Let us define the image $\operatorname{Im} b^0_p(\infty,0)$.
Since

$$b^0_p(\infty,0)=\frac{1}{2}\big(1+e^{2\pi(\{s\}-\frac{1}{2})i}\big)-\frac{1}{2}\big(1-e^{2\pi(\{s\}-\frac{1}{2})i}\big)i\operatorname{ctg}\frac{\pi}{\rho}\,,$$

we obtain

$$\begin{aligned}\operatorname{Im} b^0_p(\infty,0)&=i\Big[\sin(2\pi\{s\}-\pi)-\operatorname{ctg}\frac{\pi}{p}+\operatorname{ctg}\frac{\pi}{p}\cos(2\pi\{s\}-\pi)\Big]=\\&=\Big[-\sin 2\pi\{s\}-\operatorname{ctg}\frac{\pi}{p}-\cos\frac{\pi}{p}\cos 2\pi\{s\}\Big]i=\\&=-2\cos\pi\{s\}\Big[\sin\pi\{s\}+\operatorname{ctg}\frac{\pi}{p}\cos\pi\{s\}\Big]i=\\&=-\frac{2\cos\pi\{s\}}{\sin\frac{\pi}{p}}\cos\Big(\frac{\pi}{p}-\pi\{s\}\Big)i.\end{aligned}$$

If $-\frac{2\cos\pi\{s\}}{\sin\frac{\pi}{p}}\cos(\frac{\pi}{p}-\pi\{s\})>0$, then $\operatorname{ind} b^0_p=-1$ and this inequality implies

$$\cos\Big(\frac{\pi}{p}-\pi\{s\}\Big)<0\Longrightarrow 2\pi k+\frac{\pi}{2}<\frac{\pi}{p}-\pi\{s\}<\frac{3\pi}{2}+2\pi k\Longrightarrow\frac{1}{2}<\frac{1}{p}-\{s\}$$

because $\cos\pi\{s\}>0$, $0\le\{s\}<\frac{1}{2}$, $-\frac{1}{2}<\frac{1}{p}-\{s\}<1$.

In a similar manner, $\frac{1}{i}\operatorname{Im} b^0_p(\infty,0)<0$ implies $\frac{1}{p}-\{s\}<\frac{1}{2}$, $\operatorname{ind} b^0_p=0$ and if $\frac{1}{p}-\{s\}=\frac{1}{2}$, then $\inf|b^0_p|=0$.

III. When $\frac{1}{2}<\{s\}<1$, we can proceed as in the foregoing case and obtain

$$\operatorname{ind} b^0_p=1\quad\text{if}\quad\frac{1}{p}-\{s\}<-\frac{1}{2}\,,\quad\operatorname{ind} b^0_p=0\quad\text{if}\quad\frac{1}{p}-\{s\}>-\frac{1}{2},$$
$$\text{and}\quad\inf|b^0_p|=0\quad\text{if}\quad\frac{1}{p}-\{s\}=-\frac{1}{2}\,.$$

Hence by virtue of the equality $\operatorname{ind} b_p = \operatorname{ind} g + \operatorname{ind} b_p^0$ we have

$$\begin{aligned} \operatorname{ind} b_p^0 &= [s] - 1 \quad \text{if} \quad \{s\} - \frac{1}{p} < -\frac{1}{2}\,, \\ \inf |b_p| &= 0 \quad \text{if} \quad \Big|\{s\} - \frac{1}{p}\Big| = \frac{1}{2}\,, \\ \operatorname{ind} |b_p| &= [s] \quad \text{if} \quad \Big|\{s\} - \frac{1}{p}\Big| < \frac{1}{2}, \\ \text{and} \quad \operatorname{ind} b_p &= [s] + 1 \quad \text{if} \quad \{s\} - \frac{1}{p} > \frac{1}{2}\,. \quad \square \end{aligned} \tag{3.10}$$

Lemma 3.3. *The function* b (*see* (3.6)) *has the* p'*-factorization*

$$b(\xi) = b_-(\xi)\Big(\frac{\xi - i}{\xi + i}\Big)^{\varkappa} b_+(\xi)$$

(*see Definition* 1.22 *and Theorem* 4.4 *in* [8]). *Here*

I. $\varkappa = [s], \quad b_\pm(\xi) = g_\pm(\xi)\Big(\dfrac{-2i}{\xi \mp i}\Big)^{\mp(\{s\}-\frac{1}{2})}, \quad$ *when* $\quad \Big|\{s\} - \dfrac{1}{p}\Big| < \dfrac{1}{2}\,,$

II. $\varkappa = [s] + 1, \quad b_\pm(\xi) = g_\pm(\xi)\Big(\dfrac{-2i}{\xi \mp i}\Big)^{\mp(\{s\}-\frac{3}{2})}, \quad$ *when* $\quad \{s\} - \dfrac{1}{p} > \dfrac{1}{2}\,,$

III. $\varkappa = [s] - 1, \quad b_\pm(\xi) = g_\pm(\xi)\Big(\dfrac{-2i}{\xi \mp i}\Big)^{\mp(\{s\}+\frac{1}{2})}, \quad$ *when* $\quad \{s\} - \dfrac{1}{p} < \dfrac{1}{2}\,,$

where

$$g_\pm(\xi) = \pm \exp \frac{1}{2}(I \pm S_{\mathbb{R}}) \ln \frac{1 + \lambda|\xi|}{(\xi^2 + 1)^{\frac{1}{2}}}.$$

Proof. This fact is valid since $g(\xi) = -(\frac{\xi - i}{\xi + i})^{[s]} \frac{1+\lambda|\xi|}{(\xi^2+1)^{\frac{1}{2}}}$ is a nonvanishing continuous function and has the following general p'-factorization which is the same for all $1 < p < \infty$:

$$g(\xi) = g_-(\xi)\Big(\frac{\xi - i}{\xi + i}\Big)^{[s]} g_+(\xi)$$

with

$$g_\pm(\xi) = \pm \exp \frac{1}{2}\,(I \pm S_{\mathbb{R}}) \ln \frac{1 + \lambda|\xi|}{(\xi^2 + 1)^{\frac{1}{2}}}\,.$$

For $b^0(\xi)$ we have

$$b^0(\xi) = \Big(\frac{-2i}{\xi + i}\Big)^{\{s\}-\frac{1}{2}} \Big(\frac{-2i}{\xi - i}\Big)^{\frac{1}{2}-\{s\}},$$
$$-\frac{1}{p} < \frac{1}{2} - \{s\} < 1 - \frac{1}{p} \quad \text{or} \quad \Big|\{s\} - \frac{1}{p}\Big| < \frac{1}{2}\,,$$

$$b^0(\xi)=\Big(\frac{-2i}{\xi+i}\Big)^{\{s\}-\frac{3}{2}}\Big(\frac{\xi-i}{\xi+i}\Big)\Big(\frac{-2i}{\xi-i}\Big)^{\frac{3}{2}-\{s\}},$$

$$-\frac{1}{p}<\frac{3}{2}-\{s\}<1-\frac{1}{p}\quad\text{or}\quad \{s\}-\frac{1}{p}>\frac{1}{2},$$

$$b^0(\xi)=\Big(\frac{-2i}{\xi+i}\Big)^{\{s\}+\frac{1}{2}}\Big(\frac{\xi-i}{\xi+i}\Big)^{-1}\Big(\frac{-2i}{\xi-i}\Big)^{-\frac{1}{2}-\{s\}},$$

$$-\frac{1}{p}<-\frac{1}{2}-\{s\}<1-\frac{1}{p}\quad\text{or}\quad \{s\}-\frac{1}{p}<-\frac{1}{2}. \qquad \square$$

§ 4. A Homogeneous Equation in the Bessel Potential Spaces

Theorem 4.1. *Let*

$$1\le s<\frac{1}{p}+\frac{1}{2} \tag{4.1}$$

and A be the operator defined by equation (3.1). *Then the operator*

$$r_+A: H^s_p(\mathbb{R}_+)\to H^{s-1}_p(\mathbb{R}_+) \tag{4.2}$$

is Fredholm and

$$\operatorname{Ind} r_+A=1. \tag{4.3}$$

Proof. Since $0\le s-1<\frac{1}{p}$, the spaces $\widetilde{H}^{s-1}_p(\mathbb{R}_+)$ and $H^{s-1}_p(\mathbb{R}_+)$ can be identified (see [9, Theorem 2.10.3c]); thus

$$\partial: H^s_p(\mathbb{R}_+)\to H^{s-1}_p(\mathbb{R}_+)=\widetilde{H}^{s-1}_p(\mathbb{R}_+),\quad \partial u(x):=\frac{du(x)}{dx},$$

is a bounded operator. Now

$$r_+A=I-2i\lambda S_{\mathbb{R}_+}\partial: H^s_p(\mathbb{R}_+)\to H^{s-1}_p(\mathbb{R}_+),\quad S_{\mathbb{R}_+}u(x)=\frac{1}{\pi i}\int_0^\infty\frac{u(y)\,dy}{y-x}$$

is bounded because $S_{\mathbb{R}_+}:\widetilde{H}^\theta_p(\mathbb{R}_+)\to H^\theta_p(\mathbb{R}_+)$ is bounded for arbitrary $\theta\in\mathbb{R}$ [8, § 5] and the embedding $H^s_p(\mathbb{R}_+)\subset H^{s-1}_p(\mathbb{R}_+)$ is continuous [9, § 2.8].

Next we have to show that $\dim\operatorname{Ker} r_+A=1$.

Let us fix arbitrary $u_0\in H^s_p(\mathbb{R}_+)$ with $u_0(0)=1$ (note that $u(0)$ exists due to the embedding $H^s_p(\mathbb{R}_+)\subset C(\mathbb{R}_+)$ [9, § 2.8]). Then

$$H^s_p(\mathbb{R}_+)=\widetilde{H}^s_p(\mathbb{R}_+)+\{\lambda u_0\}_{\lambda\in\mathbb{C}} \tag{4.4}$$

because an arbitrary function $v\in H^s_p(\mathbb{R}_+)$ can be represented as

$$v=v_0+v(0)u_0,\quad v_0=v-v(0)u_0\in\widetilde{H}^s_p(\mathbb{R}_+).$$

Since $u_0 \in H_p^s(\mathbb{R}_+)$, we have $r_+Au_0 \in H_p^{s-1}(\mathbb{R}_+)$ and due to Theorem 3.1 (r_+A is invertible) there exists a function $\varphi_0 \in \widetilde{H}_p^s(\mathbb{R}_+)$ such that $\varphi_0 = -r_+A(r_+Au_0)$. Then $\varphi_0 + u_0 \in \operatorname{Ker} r_+A$ because $r_+A(\varphi_0 + u_0) = 0$.

Now let $v_1, v_2 \in \operatorname{Ker} r_+A$. Due to Theorem 3.1 $v_k(0) \neq 0$ because if $v_k \in \widetilde{H}_p^s(\mathbb{R}_+) \cap \operatorname{Ker} r_+A$, then $v_k = 0$ $(k = 1, 2)$. For the same reason $v = v_1 - \frac{v_1(0)}{v_2(0)} v_2 = 0$, because $v(0) = 0$ and $v \in \operatorname{Ker} r_+A$. Thus $\dim \operatorname{Ker} r_+A = 1$.

From (4.4) we obtain $H_p^s(\mathbb{R}_+) = \widetilde{H}_p^s(\mathbb{R}_+) + \operatorname{Ker} r_+A$ and by Theorem 3.1 we conclude that $r_+AH_p^s(\mathbb{R}_+) = H_p^{s-1}(\mathbb{R}_+)$, i.e., $\dim \operatorname{Co}\operatorname{Ker} r_+A = 0$. The results obtained imply that (4.2) is Fredholm and (4.3) holds. □

Proof of Theorem 0.1. We know that

$$H_p^s(\mathbb{R}_+) \subset W_{p_1}^1(\mathbb{R}_+) \text{ provided that } 1 < p \leq p_1 < \infty, \quad s - \frac{1}{p} \geq 1 - \frac{1}{p_1} \tag{4.5}$$

[9, § 2.8]. On the other hand, for any $1 < p_1 < 2$ we can find s and p which satisfy conditions (4.1) and (4.5). Therefore by Theorem 4.1 the solutions of the Prandtl homogeneous equations can be written as

$$v = v_0 + c_0 u_0, \quad v_0 \in \widetilde{H}_p^s(\mathbb{R}_+). \tag{4.6}$$

Obviously, $v(0) = c_0$ (see (0.2)) while v_0 in (4.6) is the unique solution of the equation $r_+Av_0 = -c_0 r_+Au_0 \in H_p^{s-1}(\mathbb{R}_+)$ provided that $1 < p < 2$ (see Theorem 3.1).

If $p_1 \geq 2$, from (4.5) we obtain

$$s - 1/p \geq 1/2\,. \tag{4.7}$$

Note that for $\frac{1}{p} < s < \frac{1}{p} + 1$ operator (4.2) is bounded and representation (4.4) holds (see the proof of Theorem 4.1). Therefore for

$$1/p + 1/2 \leq s < 1/p + 1 \tag{4.8}$$

we obtain either $[s] = 0$ and $1/2 \leq \{s\} - 1/p < 1$, or $[s] = 1$ and $-1/2 \leq \{s\} - 1/p < 0$.

By Theorem 3.1 we conclude that $\operatorname{Ind} r_+A = -1$ provided that $|\{s\} - \frac{1}{p}| \neq \frac{1}{2}$ (for $|\{s\} - \frac{1}{p}| = \frac{1}{2}$ operator (3.2) is not normally solvable as proved in [8, § 4]). Hence $\dim \operatorname{Ker} r_+A = 0$ (including the case $|\{s\} - \frac{1}{p}| = \frac{1}{2}$) and $\dim \operatorname{Co}\operatorname{Ker} r_+A = 1$.

Now let us show that operator (4.2) has the trivial kernel $\operatorname{Ker} r_+A = \{0\}$. For this we consider the function $u_0 \in H_p^s(\mathbb{R}_+)$, $u_0(0) = 1$, from the proof of Theorem 4.1. Then we cannot find $\varphi \in \widetilde{H}_p^s(\mathbb{R}_+)$ such that $r_+A\varphi = -c_0 r_+Au_0$, $c_0 \neq 0$. Be it otherwise, we would have

$$c_0u_0 + \varphi = I(c_0u_0 + \varphi) = 2\lambda i S_{\mathbb{R}_+}\partial(c_0u_0 + \varphi) \in H_p^{s-1}(\mathbb{R}_+) = \widetilde{H}_p^{s-1}(\mathbb{R}_+), \tag{4.9}$$

since the spaces $H_p^{s-1}(\mathbb{R}_+)$ and $\widetilde{H}_p^{s-1}(\mathbb{R}_+)$ can be identified for $\frac{1}{p} - 1 < s - 1 < \frac{1}{p}$. Thus $c_0 = c_0 u_0 + \varphi(0) = 0$, which is a contradiction. Therefore under condition (4.8) operator (4.2) is invertible and equation (0.1) would have only a trivial solution in $W_p^1(\mathbb{R}_+)$ $(p \geq 2)$.

If $s > 1 + \frac{1}{p}$, operator (4.2) is unbounded, since there exists a function $u \in H_p^s(\mathbb{R}_+)$ with the property $u'(0) \neq 0$, $u' \in H_p^{s-1}(\mathbb{R}_+) \subset C(\overline{\mathbb{R}}_+)$. Thus $S_{\mathbb{R}} u'$ has a logarithmic singularity at 0 and the inclusion $u' \in H_p^{s-1}(\mathbb{R}_+) \subset C(\overline{\mathbb{R}}_+)$ fails to hold.

If $s = 1 + \frac{1}{p}$, operator (4.2) is unbounded. Otherwise, due to the boundedness, for $s = 1$, $1 < p_0 < 2$, the complex interpolation theorem will imply that operator (4.2) is bounded for $\frac{1}{p} + \frac{1}{2} \leq s < \frac{1}{p} + 1$, which contradict the proved part of the theorem. $\square$

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