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ON SOME ENTIRE MODULAR FORMS OF AN INTEGRAL WEIGHT FOR THE CONGRUENCE SUBGROUP $\Gamma_0(4N)$

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Abstract. Four types of entire modular forms of weight $\frac{s}{2}$ are constructed for the congruence subgroup $\Gamma_0(4N)$ when s is even. One can find these forms helpful in revealing the arithmetical meaning of additional terms in the formulas for the number of representations of positive integers by positive quadratic forms with integral coefficients in an even number of variables.

All the notations, definitions and lemmas used in this paper are taken from [1].

To make the notations shorter throughout the paper we put

$$\Psi_j(\tau;g_l,h_l,c_l,N_l)=$$
$$=\Psi_j\big(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2;\dots;g_{s-8},h_{s-8},c_{s-8},N_{s-8}\big)$$
$$(j=1,2,3,4).$$

1.

Lemma 1. *For even $s\geq 12$ and given N let*

$$\Psi_1(\tau;g_l,h_l,c_l,N_l)=\bigg\{\frac{1}{N_1}\,\vartheta'''_{g_1h_1}(\tau;c_1,2N_1)\vartheta'_{g_2h_2}(\tau;c_2,2N_2)-$$
$$-\frac{1}{N_2}\,\vartheta'''_{g_2h_2}(\tau;c_2,2N_2)\vartheta'_{g_1h_1}(\tau;c_1,2N_1)\bigg\}\prod_{k=3}^{s-8}\vartheta_{g_kh_k}(\tau;c_k,2N_k)\quad(1.1)$$

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and

$$\Psi_2(\tau;g_l,h_l,c_l,N_l)=\left\{\frac{1}{N_1^2}\,\vartheta^{(4)}_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)+\right.$$
$$+\frac{1}{N_2^2}\,\vartheta^{(4)}_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)-$$
$$\left.-\frac{6}{N_1N_2}\,\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\right\}\prod_{k=3}^{s-8}\vartheta_{g_kh_k}(\tau;c_k,2N_k), \quad (1.2)$$

where

$$2|g_k, \quad N_k|N \ \ (k=1,2,\dots,s-8), \quad 4\Big|N\sum_{k=1}^{s-8}\frac{h_k}{N_k}\,. \quad (1.3)$$

Then we have

$$(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l)=\sum_{n=0}^{\infty}B_n^{(j)}e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \quad (j=1,2) \quad (1.4)$$

for all substitutions from Γ *in the neighborhood of each rational point* $\tau=-\frac{\delta}{\gamma}$ $(\gamma\neq 0,\ (\gamma,\delta)=1)$.

In [2] this lemma is proved for arbitrary $s>9$ (even and odd).

Lemma 2. *For even* s *we have*

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}=\begin{cases}1 & \text{if } s\equiv 0 \pmod 4,\\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \text{if } s\equiv 2 \pmod 4.\end{cases}$$

Proof. (a) Let $s\equiv 0 \pmod 4$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}=i^{2h\eta(\gamma)(\operatorname{sgn}\delta-1)}\cdot i^{(2h-4)(1-|\delta|)}=$$
$$=(-1)^{h\eta(\gamma)(\operatorname{sgn}\delta-1)}\cdot(-1)^{(h-2)(1-|\delta|)}=1.$$

(b) Let $s\equiv 2 \pmod 4$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}=i^{(4h+2)\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(4h-6)(1-|\delta|)/2}=$$
$$=i^{(2h+1)\eta(\gamma)(\operatorname{sgn}\delta-1)}\cdot(-1)^{(2h-3)(1-|\delta|)/2}=\operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big). \qquad \square$$

Theorem 1. *For even* s *and given* N *the functions* $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ *are entire modular forms of weight* $\frac{s}{2}$ *and the character*

$$\chi(\delta)=\begin{cases}\Big(\dfrac{\Delta}{|\delta|}\Big) & \text{if } s\equiv 0 \pmod 4,\\ \operatorname{sgn}\delta\Big(\dfrac{-\Delta}{|\delta|}\Big) & \text{if } s\equiv 2 \pmod 4\end{cases} \quad (1.5)$$

(Δ *is the determinant of the positive quadratic form with integral coefficients in* s *variables for which the function* Ψ_j *will be used) for the congruence subgroup* $\Gamma_0(4N)$ *if the following conditions hold:*

$$1)\ \ 2|g_k,\ \ N_k|N\ \ (k=1,2,\dots,s-8), \tag{1.6}$$

$$2)\ \ 4\Big|N\sum_{k=1}^{s-8}\frac{h_k^2}{N_k},\quad 4\Big|\sum_{k=1}^{s-8}\frac{g_k^2}{4N_k}, \tag{1.7}$$

$$3)\ \ \left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)\Psi_j(\tau;\alpha g_l,h_l,0,N_l)=\left(\frac{\Delta}{|\delta|}\right)\Psi_j(\tau;g_l,h_l,0,N_l)\ \ (j=1,2) \tag{1.8}$$

for all α *and* δ *with* $\alpha\delta\equiv 1 \pmod{4N}$.

Proof. I. As is well known, the theta-series which contained in the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ are regular on $\mathcal{H}$. Therefore these functions satisfy condition 1) and by Lemma 1 also condition 4) of the Definition from [1] (p. 53).

II. From (1.6) it follows that

$$\Gamma_0(4N)\subset\Gamma_0(4N_k)\quad (k=1,2,\dots,s-8). \tag{1.9}$$

It is easy to verify that (1.7) implies

$$4\Big|N\delta^2\sum_{k=1}^{s-8}\frac{h_k^2}{N_k},\quad 4\Big|\sum_{k=1}^{s-8}\frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2}, \tag{1.10}$$

since $2\nmid\delta$ by virtue of $\alpha\delta\equiv 1 \pmod{4N}$.

Using (1.9), (1.10) and Lemma 3 from [1] (p. 58), for all substitutions from $\Gamma_0(4N)$ we get:

1) if $n=3$, $n=1$ and $n=0$,

$$\begin{aligned}
&\vartheta'''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta'_{g_th_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_t\Big)\times\\
&\qquad\times\prod_{k=3}^{s-8}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=\\
&=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times\\
&\times\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\prod_{k=3}^{s-8}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k)
\end{aligned} \tag{1.11}$$

for $r=1$, $t=2$ and $r=2$, $t=1$;

2) if $n=4$ and $n=0$,

$$\vartheta^{(4)}_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times$$
$$\times\prod_{k=3}^{s-8}\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)=$$
$$=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times$$
$$\times\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\prod_{k=3}^{s-8}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k)\qquad(1.12)$$

for $r=1$, $t=2$ and $r=2$, $t=1$;

3) if $n=2$ and $n=0$,

$$\vartheta''_{g_1 h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_1\Big)\vartheta''_{g_2 h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_2\Big)\times$$
$$\times\prod_{k=3}^{s-8}\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)=$$
$$=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times$$
$$\times\vartheta''_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta''_{\alpha g_2,h_2}(\tau;0,2N_2)\prod_{k=3}^{s-8}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k).\qquad(1.13)$$

Thus by (1.1), (1.11) and (1.2), (1.12), (1.13) we have

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\times$$
$$\times\left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\quad(j=1,2),$$

which by Lemma 2 implies respectively for $j=1$ and $j=2$

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)$$
$$\text{if}\quad s\equiv 0\pmod 4$$

and

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=$$

$$=\operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Bigg(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\quad\text{if}\quad s\equiv 2\pmod 4.$$

Hence by (1.5) and (1.8) we obtain

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\chi(\delta)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l).$$

Thus the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ satisfy condition (2) of the Definition from [1].

III. The functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ also satisfy condition 3) of the Definition from [1]. For the proof see [2] (p. 109). □

2.

To simplify the notations of this section we put

$$\begin{aligned}\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)&=\\=&\frac{1}{N_1}\,\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)-\\-&\frac{1}{N_2}\,\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)\end{aligned}\tag{2.1}$$

and occasionally

$$(s-2)/3=\tilde{s},\quad (s-1)/3=s'.\tag{2.2}$$

Lemma 3. *For even $s\geq 10$ and given N let*

$$\Psi_3(\tau;g_l,h_l,c_l,N_l)=\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)\times$$
$$\times\prod_{k=3}^{s/3}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\quad\textit{if}\quad s\equiv 0\pmod 6,\tag{2.3}$$

$$\Psi_3(\tau;g_l,h_l,c_l,N_l)=\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)\times$$
$$\times\prod_{k=3}^{\tilde{s}}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\prod_{k=\tilde{s}+1}^{\tilde{s}+2}\vartheta_{g_kh_k}(\tau;c_k,2N_k)\quad\textit{if}\quad s\equiv 2\pmod 6,\tag{2.4}$$

$$\Psi_3(\tau;g_l,h_l,c_l,N_l)=\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)\times$$
$$\times\prod_{k=3}^{s'}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\vartheta_{g_{s'+1}h_{s'+1}}(\tau;c_{s'+1},2N_{s'+1})\quad\textit{if}\quad s\equiv 4\pmod 6\tag{2.5}$$

and

$$\Psi_4(\tau; g_l, h_l, c_l, N_l) = \prod_{k=1}^{s/3} \vartheta'_{g_k h_k}(\tau; c_k, 2N_k) \quad \textit{if} \quad s \equiv 0 \pmod 6, \tag{2.6}$$

$$\Psi_4(\tau; g_l, h_l, c_l, N_l) = \prod_{k=1}^{\tilde{s}} \vartheta'_{g_k h_k}(\tau; c_k, 2N_k)\vartheta_{g_{\tilde{s}+1} h_{\tilde{s}+1}}(\tau; c_{\tilde{s}+1}, 2N_{\tilde{s}+1}) \times$$
$$\times \vartheta_{g_{\tilde{s}+2} h_{\tilde{s}+2}}(\tau; c_{\tilde{s}+2}, 2N_{\tilde{s}+2}) \quad \textit{if} \quad s \equiv 2 \pmod 6, \tag{2.7}$$

$$\Psi_4(\tau; g_l, h_l, c_l, N_l) = \prod_{k=3}^{s'} \vartheta'_{g_k h_k}(\tau; c_k, 2N_k)\vartheta_{g_{s'+1} h_{s'+1}}(\tau; c_{s'+1}, 2N_{s'+1}) \tag{2.8}$$
$$\textit{if} \quad s \equiv 4 \pmod 6,$$

where

$$2|g_k, \quad N_k|N \quad (k = 1, 2, \ldots, s/3), \quad 4\Big|N\sum_{k=1}^{s/3} \frac{h_k}{N_k} \quad \textit{if} \quad s \equiv 0 \pmod 6, \tag{2.9}$$

$$2|g_k, \quad N_k|N \quad (k = 1, 2, \ldots, \widetilde{s}+2), \quad 4\Big|N\sum_{k=1}^{\tilde{s}+2} \frac{h_k}{N_k} \quad \textit{if} \quad s \equiv 2 \pmod 6, \tag{2.10}$$

$$2|g_k, \quad N_k|N \quad (k = 1, 2, \ldots, s'+1), \quad 4\Big|N\sum_{k=1}^{s'+1} \frac{h_k}{N_k} \quad \textit{if} \quad s \equiv 4 \pmod 6, \tag{2.11}$$

Then we have

$$(\gamma\tau + \delta)^{s/2}\Psi_j(\tau; g_l, h_l, 0, N_l) = \sum_{n=0}^{\infty} C_n^{(j)} e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \quad (j = 3, 4) \tag{2.12}$$

for all substitutions from Γ *in the neighborhood of each point* $\tau = -\frac{\delta}{\gamma}$ $(\gamma \neq 0,$ $(\gamma, \delta) = 1)$.

Proof. I. It is shown in [3] (formula (1.7)) that

$$(\gamma\tau + \delta)^3\Omega(\tau; g_1, h_1, 0, N_1; g_2, h_2, 0, N_2) =$$
$$= -e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(4N_1N_2\gamma^2)^{-1/2} \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \prod_{k=1}^{2} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$
$$\times \Omega\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g'_1 h'_1, H_1, N_1; g'_2, h'_2, H_2, 2N_2\Big). \tag{2.13}$$

For $n = 1$ and $n = 0$ it follows by Lemma 4 from [1] (p. 61) that:

1) $$(\gamma\tau+\delta)^{(s-6)/2}\prod_{k=3}^{s/3}\vartheta'_{g_kh_k}(\tau;0,2N_k)=$$
$$=e\big((s-6)(\operatorname{sgn}\gamma)/8\big)(-i\operatorname{sgn}\gamma)^{(s-6)/3}\times$$
$$\times\Big(2^{(s-6)/3}\prod_{k=3}^{s/3}N_k|\gamma|^{(s-6)/3}\Big)^{-1/2}\sum_{\substack{H_3 \mod 2N_3\\ \cdots\cdots\cdots\\ H_{s/3} \mod 2N_{s/3}}}\prod_{k=3}^{s/3}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)\quad\text{if}\quad s\equiv 0 \pmod 6, \tag{2.14}$$

2) $$(\gamma\tau+\delta)^{(s-6)/2}\prod_{k=3}^{\tilde s}\vartheta'_{g_kh_k}(\tau;0,2N_k)\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\times$$
$$\times\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2})=e\big((s-6)(\operatorname{sgn}\gamma)/8\big)\times$$
$$\times\Big(2^{\tilde s}\sum_{k=3}^{\tilde s+2}N_k|\gamma|^{\tilde s}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{(s-8)/3}\times$$
$$\times\sum_{\substack{H_3 \mod 2N_3\\ \cdots\cdots\cdots\\ H_{\tilde s+2} \mod 2N_{\tilde s+2}}}\prod_{k=3}^{\tilde s}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta'_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)\times$$
$$\times\prod_{k=\tilde s+1}^{\tilde s+2}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)\tag{2.15}$$
$$\text{if}\quad s\equiv 2 \pmod 6,$$

3) $$(\gamma\tau+\delta)^{(s-6)/2}\prod_{k=3}^{s'}\vartheta'_{g_kh_k}(\tau;0,2N_k)\vartheta_{g_{s'+1}h_{s'+1}}(\tau;0,2N_{s'+1})=$$
$$=e\big((s-6)(\operatorname{sgn}\gamma)/8\big)\Big(2^{s'-1}\sum_{k=3}^{s'+1}N_k|\gamma|^{s'-1}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{(s-7)/3}\times$$
$$\times\sum_{\substack{H_3 \mod 2N_3\\ \cdots\cdots\cdots\\ H_{s'+1} \mod 2N_{s'+1}}}\prod_{k=3}^{s'}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta'_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\times$$
$$\times\varphi_{g'_{s'+1}g_{s'+1}h_{s'+1}}(0,H_{s'+1};2N_{s'+1})\times$$
$$\times\vartheta_{g_{s'+1}h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_{s'+1},2N_{s'+1}\Big)\quad\text{if}\quad s\equiv 4 \pmod 6.\tag{2.16}$$

After multiplying (2.13) successively by (2.14), (2.15) and (2.16), by (2.3) we obtain

1) $(\gamma\tau+\delta)^{s/2}\Psi_3(\tau;g_l,h_l,0,N_l)=(-1)^{s/6}e\big(s(\operatorname{sgn}\gamma)/8\big)\times$

$$\times\Big(2^{s/3}\prod_{k=1}^{s/3}N_k|\gamma|^{s/3}\Big)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\cdots\\ H_{s/3} \bmod 2N_{s/3}}}\prod_{k=3}^{s/3}\varphi_{g'_k g_k h_k}(0,H_k;2N_k)\times$$

$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big)\quad\text{if}\quad s\equiv 0 \pmod 6, \tag{2.17}$$

2) $(\gamma\tau+\delta)^{s/2}\Psi_3(\tau;g_l,h_l,0,N_l)=(-1)^{(s-2)/6}e\big(s(\operatorname{sgn}\gamma)/8\big)\times$

$$\times\Big(2^{\tilde{s}+2}\prod_{k=1}^{\tilde{s}+2}N_k|\gamma|^{\tilde{s}+2}\Big)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\cdots\\ H_{\tilde{s}+2} \bmod 2N_{\tilde{s}+2}}}\prod_{k=3}^{\tilde{s}+2}\varphi_{g'_k g_k h_k}(0,H_k;2N_k)\times$$

$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big)\quad\text{if}\quad s\equiv 2 \pmod 6, \tag{2.18}$$

3) $(\gamma\tau+\delta)^{s/2}\Psi_3(\tau;g_l,h_l,0,N_l)=(-1)^{(s+2)/6}(i\operatorname{sgn}\gamma)e\big(s(\operatorname{sgn}\gamma)/8\big)\times$

$$\times\Big(2^{s'+1}\prod_{k=1}^{s'+1}N_k|\gamma|^{s'+1}\Big)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\cdots\\ H_{s'+1} \bmod 2N_{s'+1}}}\prod_{k=3}^{s'+1}\varphi_{g'_k g_k h_k}(0,H_k;2N_k)\times$$

$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big)\quad\text{if}\quad s\equiv 4 \pmod 6. \tag{2.19}$$

Further, reasoning as in [1] (Lemma 5, the bottom part of p. 64 and p. 65), we obtain (2.12) if $j=3$.

II. By Lemma 4 from [1], for $n=1$ and $n=0$ it respectively follows by (2.6), (2.7) and (2.8) that

$$(\gamma\tau+\delta)^{s/2}\Psi_4(\tau;g_l,h_l,0,N_l)=(\gamma\tau+\delta)^{s/2}\prod_{k=1}^{s/3}\vartheta'_{g_k h_k}(\tau;0,2N_k)=$$

$$=e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{s/3}\prod_{k=1}^{s/3}N_k|\gamma|^{s/3}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{s/3}\times$$

$$\times\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\cdots\\ H_{s/3} \bmod 2N_{s/3}}}\prod_{k=1}^{s/3}\Big(\varphi_{g'_k g_k h_k}(0,H_k;2N_k)\vartheta'_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)=$$

$$= (-1)^{s/6} e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{s/3}\prod_{k=1}^{s/3} N_k|\gamma|^{s/3}\Big)^{-1/2} \times$$

$$\times \sum_{\substack{H_1 \mod 2N_1 \\ \cdots\cdots\cdots \\ H_{s/3} \mod 2N_{s/3}}} \prod_{k=1}^{s/3} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g'_l, h'_l, H_l, N_l\Big) \quad \text{if} \quad s \equiv 0 \pmod 6, \tag{2.20}$$

$$(\gamma\tau+\delta)^{s/2}\Psi_4(\tau; g_l, h_l, 0, N_l) = (\gamma\tau+\delta)^{s/2}\prod_{k=1}^{\tilde{s}} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \times$$

$$\times \vartheta_{g_{\tilde{s}+1} h_{\tilde{s}+1}}(\tau; 0, 2N_{\tilde{s}+1})\vartheta_{g_{\tilde{s}+2} h_{\tilde{s}+2}}(\tau; 0, 2N_{\tilde{s}+2}) =$$

$$= e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{\tilde{s}+2}\prod_{k=1}^{\tilde{s}+2} N_k|\gamma|^{\tilde{s}+2}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{\tilde{s}} \times$$

$$\times \sum_{\substack{H_1 \mod 2N_1 \\ \cdots\cdots\cdots \\ H_{\tilde{s}} \mod 2N_{\tilde{s}}}} \prod_{k=1}^{\tilde{s}} \Big(\varphi_{g'_k g_k h_k}(0, H_k; 2N_k)\vartheta'_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big)\Big) \times$$

$$\times \sum_{\substack{H_{\tilde{s}+1} \mod 2N_{\tilde{s}+1} \\ H_{\tilde{s}+2} \mod 2N_{\tilde{s}+2}}} \prod_{k=\tilde{s}+1}^{\tilde{s}+2} \Big(\varphi_{g'_k g_k h_k}(0, H_k; 2N_k)\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big)\Big) =$$

$$= (-1)^{(s-2)/6} e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{\tilde{s}+2}\prod_{k=1}^{\tilde{s}+2} N_k|\gamma|^{\tilde{s}+2}\Big)^{-1/2} \times$$

$$\times \sum_{\substack{H_1 \mod 2N_1 \\ \cdots\cdots\cdots \\ H_{\tilde{s}+2} \mod 2N_{\tilde{s}+2}}} \prod_{k=1}^{\tilde{s}+2} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g'_l, h'_l, H_l, N_l\Big) \quad \text{if} \quad s \equiv 2 \pmod 6 \tag{2.21}$$

and

$$(\gamma\tau+\delta)^{s/2}\Psi_4(\tau; g_l, h_l, 0, N_l) = (\gamma\tau+\delta)^{s/2}\prod_{k=1}^{s'} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \times$$

$$\times \vartheta_{g_{s'+1} h_{s'+1}}(\tau; 0, 2N_{s'+1}) e\big(s(\operatorname{sgn}\gamma)/8\big) \times$$

$$\times \Big(2^{s'+1}\prod_{k=1}^{s'+1} N_k|\gamma|^{s'+1}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{s'} \times$$

$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ \cdots\cdots\cdots \\ H_{s'} \bmod 2N_{s'}}} \prod_{k=1}^{s'} \Big(\varphi_{g'_k g_k h_k}(0, H_k; 2N_k)\vartheta'_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big)\Big) \times$$

$$\times \sum_{H_{s'+1} \bmod 2N_{s'+1}} \varphi_{g'_{s+1} g_{s'+1} h_{s'+1}}(0, H_{s'+1}; 2N_{s'+1}) \times$$

$$\times \vartheta_{g_{s'+1} h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_{s'+1}, 2N_{s'+1}\Big) =$$

$$= (-1)^{(s+2)/6}(i \operatorname{sgn}\gamma) e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{s'+1}\prod_{k=1}^{s'+1} N_k |\gamma|^{s'+1}\Big)^{-1/2} \times$$

$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ \cdots\cdots\cdots \\ H_{s'+1} \bmod 2N_{s'+1}}} \prod_{k=1}^{s'+1} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g'_l, h'_l, H_l, N_l\Big) \quad \text{if} \quad s \equiv 4 \pmod 6. \tag{2.22}$$

Further, reasoning as in [1] (Lemma 5), we obtain (2.12) if $j = 4$. □

Lemma 4. *For even s we have*

(a) $$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{(s/3)(1-|\delta|)/2} = \begin{cases} 1 & \textit{if} \quad s \equiv 0 \pmod{12}, \\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \textit{if} \quad s \equiv 6 \pmod{12}; \end{cases}$$

(b) $$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+4)/3)(1-|\delta|)/2} = \begin{cases} 1 & \textit{if} \quad s \equiv 8 \pmod{12}, \\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \textit{if} \quad s \equiv 2 \pmod{12}; \end{cases}$$

(c) $$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+2)/3)(1-|\delta|)/2} = \begin{cases} \Big(\dfrac{-1}{|\delta|}\Big) & \textit{if} \quad s \equiv 4 \pmod{12}, \\ \operatorname{sgn}\delta & \textit{if} \quad s \equiv 10 \pmod{12}. \end{cases}$$

Proof. (a) Let $s \equiv 0 \pmod 6$, i.e., $s \equiv 0 \pmod{12}$ or $s \equiv 6 \pmod{12}$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{(s/3)(1-|\delta|)/2} = i^{3h\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot i^{2h(1-|\delta|)/2} =$$

$$= \begin{cases} 1 & \text{if} \quad 2|h, \ \text{ i.e., if } \ s \equiv 0 \pmod{12}, \\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \text{if} \quad 2\nmid h, \ \text{ i.e., if } \ s \equiv 6 \pmod{12}. \end{cases}$$

(b) Let $s \equiv 2 \pmod 6$, i.e., $s \equiv 2 \pmod{12}$ or $s \equiv 8 \pmod{12}$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+4)/3)(1-|\delta|)/2} = i^{(3h+1)\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot i^{2(h+1)(1-|\delta|)/2} =$$
$$= i^{(h+1)\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot (-1)^{(h+1)(1-|\delta|)/2} =$$
$$= \begin{cases} \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \text{if } \ 2|h, \ \text{ i.e., if } \ s \equiv 2 \pmod{12}, \\ 1 & \text{if } \ 2 \nmid h, \ \text{ i.e., if } \ s \equiv 8 \pmod{12}. \end{cases}$$

(c) Let $s \equiv 4 \pmod 6$, i.e., $s \equiv 4 \pmod{12}$ or $s \equiv 10 \pmod{12}$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+2)/3)(1-|\delta|)/2} = i^{(3h+2)\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot i^{2(h+1)(1-|\delta|)/2} =$$
$$= \begin{cases} \Big(\dfrac{-1}{|\delta|}\Big) & \text{if } \ 2|h, \ \text{ i.e., if } \ s \equiv 4 \pmod{12}, \\ \operatorname{sgn}\delta & \text{if } \ 2 \nmid h, \ \text{ i.e., if } \ s \equiv 10 \pmod{12}. \end{cases} \qquad \square$$

Theorem 2. *For even s and given N the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 3, 4)$ are entire modular forms of weight $\frac{s}{2}$ and the character*

$$\chi(\delta) = \begin{cases} \Big(\dfrac{\Delta}{|\delta|}\Big) & \text{if } \ s \equiv 0 \pmod 4, \\ \operatorname{sgn}\delta\Big(\dfrac{-\Delta}{|\delta|}\Big) & \text{if } \ s \equiv 2 \pmod 4 \end{cases} \tag{2.23}$$

(Δ is the determinant of the positive quadratic form with integral coefficients in s variables for which the function Ψ_j will be used) for the congruence subgroup $\Gamma_0(4N)$ if the following conditions hold:

(a) *when* $s \equiv 0 \pmod 6$

$$1) \ \ 2|g_k, \ \ N_k|N \ \ (k = 1, 2, \ldots, s/3), \tag{2.24}$$

$$2) \ \ 4\Big|N\sum_{k=1}^{s/3}\frac{h_k^2}{N_k}, \quad 4\Big|\sum_{k=1}^{s/3}\frac{g_k^2}{4N_k}, \tag{2.25}$$

3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\Bigg(\frac{\prod\limits_{k=1}^{s/3} N_k}{|\delta|}\Bigg)\Psi_j(\tau; \alpha g_l, h_l, 0, N_l) = \Big(\frac{\Delta}{|\delta|}\Big)\Psi_j(\tau; g_l, h_l, 0, N_l); \tag{2.26}$$

(b) *when* $s \equiv 2 \pmod 6$

$$1) \ \ 2|g_k, \ \ N_k|N \ \ (k = 1, 2, \ldots, (s+4)/3), \tag{2.27}$$

$$2) \ \ 4\Big|N\sum_{k=1}^{(s+4)/3}\frac{h_k^2}{N_k}, \quad 4\Big|\sum_{k=1}^{(s+4)/3}\frac{g_k^2}{4N_k}, \tag{2.28}$$

3) *for all* α *and* δ *with* $\alpha\delta \equiv 1 \pmod{4N}$

$$\Bigg(\frac{\prod\limits_{k=1}^{(s+4)/3} N_k}{|\delta|}\Bigg)\Psi_j(\tau;\alpha g_l, h_l, 0, N_l) = \Big(\frac{\Delta}{|\delta|}\Big)\Psi_j(\tau; g_l, h_l, 0, N_l); \qquad (2.29)$$

(c) *when* $s \equiv 4 \pmod 6$

1) $2|g_k,\ \ N_k|N\ \ (k = 1, 2, \dots, (s+2)/3)$, (2.30)

2) $4\Big|N \sum\limits_{k=1}^{(s+2)/3} \dfrac{h_k^2}{N_k},\quad 4\Big|\sum\limits_{k=1}^{(s+2)/3} \dfrac{g_k^2}{4N_k}$, (2.31)

3) *for all* α *and* δ *with* $\alpha\delta \equiv 1 \pmod{4N}$

$$\Bigg(\frac{\prod\limits_{k=1}^{(s+2)/3} N_k}{|\delta|}\Bigg)\Psi_j(\tau;\alpha g_l, h_l, 0, N_l) = \operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)\Psi_j(\tau; g_l, h_l, 0, N_l). \quad (2.32)$$

Proof. I. As in the case of Theorem 1, the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ ($j = 3, 4$) satisfy condition 1) and by Lemma 3 also condition 4) of the Definition from [1].

II. From (2.24), (2.27) and (2.30) we obtain

$$\Gamma_0(4N) \subset \Gamma_0(4N_k). \qquad (2.33)$$

Obviously, (2.25), (2.28) and (2.31) respectively imply

$$4\Big|N\delta^2 \sum_{k=1}^{s/3} \frac{h_k^2}{N_k},\quad 4\Big|\sum_{k=1}^{s/3} \frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2} \quad \text{if } s \equiv 0 \pmod 6, \qquad (2.34)$$

$$4\Big|N\delta^2 \sum_{k=1}^{(s+4)/3} \frac{h_k^2}{N_k},\quad 4\Big|\sum_{k=1}^{(s+4)/3} \frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2} \quad \text{if } s \equiv 2 \pmod 6, \qquad (2.35)$$

and

$$4\Big|N\delta^2 \sum_{k=1}^{(s+2)/3} \frac{h_k^2}{N_k},\quad 4\Big|\sum_{k=1}^{(s+2)/3} \frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2} \quad \text{if } s \equiv 4 \pmod 6. \qquad (2.36)$$

Using (2.33) and Lemma 3 from [1], for all substitutions from $\Gamma_0(4N)$ we obtain:

1) for $n = 2$, $n = 1$ and $n = 0$

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; 0, 2N_r\Big)\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; 0, 2N_t\Big) \times$$

$$\times \prod_{k=3}^{s/3} \vartheta'_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big) =$$

$$= i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s/3)(1-|\delta|)/2}\Bigg(\frac{\prod\limits_{k=1}^{s/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\times$$
$$\times\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\times$$
$$\times\prod_{k=3}^{s/3}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\quad\text{if}\quad s\equiv 0 \pmod 6, \tag{2.37}$$

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times$$
$$\times\prod_{k=3}^{\tilde{s}}\vartheta'_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)\times$$
$$\times\vartheta_{g_{\tilde{s}+1}h_{\tilde{s}+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_{\tilde{s}+1}\Big)\vartheta_{g_{\tilde{s}+2}h_{\tilde{s}+2}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_{\tilde{s}+2}\Big)=$$
$$= i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(\tilde{s}+2)(1-|\delta|)/2}\Bigg(\frac{\prod\limits_{k=1}^{\tilde{s}+2} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\times$$
$$\times\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\prod_{k=3}^{\tilde{s}}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\times$$
$$\times\vartheta_{\alpha g_{\tilde{s}+1},h_{\tilde{s}+1}}(\tau;0,2N_{\tilde{s}+1})\vartheta_{\alpha g_{\tilde{s}+2},h_{\tilde{s}+2}}(\tau;0,2N_{\tilde{s}+2}) \tag{2.38}$$
$$\text{if}\quad s\equiv 2 \pmod 6,$$

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times$$
$$\times\prod_{k=3}^{s'}\vartheta'_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)\vartheta_{g_{s'+1}h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_{s'+1}\Big)=$$
$$= (\operatorname{sgn}\delta) i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s'+1)(1-|\delta|)/2}\times$$
$$\times\Bigg(\frac{\prod\limits_{k=1}^{s'} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\times$$
$$\times\prod_{k=3}^{s'}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k)\vartheta_{\alpha g_{s'+1},h_{s'+1}}(\tau;0,2N_{s'+1}) \tag{2.39}$$
$$\text{if}\quad s\equiv 4 \pmod 6,$$

where $r=2$, $t=1$ and $r=1$, $t=2$;

2) for $n=1$ and $n=0$

$$\prod_{k=1}^{s/3}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_k\Big)=$$

$$=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s/3)(1-|\delta|)/2}\left(\frac{\prod\limits_{k=1}^{s/3}N_k}{|\delta|}\right)\times$$

$$\times(\gamma\tau+\delta)^{s/2}\prod_{k=1}^{s/3}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\quad\text{if}\quad s\equiv 0\pmod 6,\tag{2.40}$$

$$\prod_{k=1}^{\tilde s}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_k\Big)\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_{\tilde s+1}\Big)\times$$

$$\times\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_{\tilde s+2}\Big)=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(\tilde s+2)(1-|\delta|)/2}\times$$

$$\times\left(\frac{\prod\limits_{k=1}^{\tilde s+2}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\prod_{k=1}^{\tilde s}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\times$$

$$\times\vartheta_{\alpha g_{\tilde s+1},h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\vartheta_{\alpha g_{\tilde s+2},h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2})\tag{2.41}$$

$$\text{if}\quad s\equiv 2\pmod 6,$$

$$\prod_{k=1}^{s'}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_k\Big)\vartheta_{g_{s'+1}h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_{s'+1}\Big)=$$

$$=\operatorname{sgn}\delta\, i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s'+1)(1-|\delta|)/2}\left(\frac{\prod\limits_{k=1}^{s'+1}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times$$

$$\times\prod_{k=1}^{s'}\vartheta'_{\alpha g_kh_k}(\tau;0,2N_k)\vartheta_{\alpha g_{s'+1},h_{s'+1}}(\tau;0,2N_{s'+1})\tag{2.42}$$

$$\text{if}\quad s\equiv 4\pmod 6.$$

By (2.3), (2.37) and (2.6), (2.40), for $j=3$ and $j=4$ we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s/3)(1-|\delta|)/2}\times$$

$$\times\Bigg(\frac{\prod\limits_{k=1}^{s/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 0 \pmod 6. \quad (2.43)$$

By (2.4), (2.38) and (2.7), (2.41), for $j=3$ and $j=4$ we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big) = i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{((s+4)/3)(1-|\delta|)/2}\,\times$$
$$\times\Bigg(\frac{\prod\limits_{k=1}^{(s+4)/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 2 \pmod 6. \quad (2.44)$$

By (2.5), (2.39) and (2.8), (2.42), for $j=3$ and $j=4$ we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big) = i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{((s+2)/3)(1-|\delta|)/2}\,\times$$
$$\times\Bigg(\frac{\prod\limits_{k=1}^{(s+2)/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 4 \pmod 6. \quad (2.45)$$

Further, by Lemma 4, for $j=3,4$ realtions (2.43), (2.44), and (2.45) respectively imply:

$$\begin{aligned}\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big) &= \Bigg(\frac{\prod\limits_{k=1}^{s/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\\ &\qquad\qquad \text{if} \quad s\equiv 0 \pmod{12},\\ &= \operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Bigg(\frac{\prod\limits_{k=1}^{s/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\,\times\\ &\quad\times\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 6 \pmod{12};\end{aligned} \quad (2.46)$$

$$\begin{aligned}\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big) &= \operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Bigg(\frac{\prod\limits_{k=1}^{(s+4)/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\,\times\\ &\quad\times\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 2 \pmod{12},\\ &= \Bigg(\frac{\prod\limits_{k=1}^{(s+4)/3} N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\\ &\qquad\qquad \text{if} \quad s\equiv 8 \pmod{12};\end{aligned} \quad (2.47)$$

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\bigg(\frac{\prod\limits_{k=1}^{(s+2)/3}N_k}{|\delta|}\bigg)(\gamma\tau+\delta)^{s/2}\times$$
$$\times\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\quad\text{if}\quad s\equiv 4 \pmod{12},$$
$$=\bigg(\frac{\prod\limits_{k=1}^{(s+2)/3}N_k}{|\delta|}\bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \tag{2.48}$$
$$\text{if}\quad s\equiv 10 \pmod{12}.$$

From (2.46)–(2.48), (2.26), (2.29), (2.32), and (2.23) it follows that

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\chi(\delta)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l)\quad(j=3,4).$$

Thus the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=3,4)$ satisfy condition 2) of the Definition from [1].

III. By [1] (formula 1.3), for $r=1$, $t=2$ and $r=2$, $t=1$ we obtain:

1) If $s\equiv 0 \pmod 6$,

$$\vartheta''_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_th_t}(\tau;0,2N_t)\prod_{k=3}^{s/3}\vartheta'_{g_kh_k}(\tau;0,2N_k)=$$
$$=(\pi i)^{s/3}\sum_{m_r,m_t,m_3,\dots,m_{s/3}=-\infty}^{\infty}(-1)^{h_rm_r+h_tm_t+\sum\limits_{k=3}^{s/3}h_km_k}(4N_rm_r+g_r)^2\times$$
$$\times\prod_{k=3}^{s/3}(4N_km_k+g_k)e\bigg(\sum_{k=1}^{s/3}\frac{1}{4N_k}\Big(2N_km_k+\frac{g_k}{2}\Big)^2\bigg)=\sum_{n=0}^{\infty}B_n^{(r,t)}e(n\tau),$$

since by (2.24) and (2.25),

$$n=\sum_{k=1}^{s/3}\frac{1}{4N_k}\Big(2N_km_k+\frac{g_k}{2}\Big)^2=\sum_{k=1}^{s/3}\big(N_km_k^2+m_kg_k/2\big)+\frac14\sum_{k=1}^{s/3}g_k^2/4N_k$$

is a non-negative integer;

2) If $s\equiv 2 \pmod 6$,

$$\vartheta''_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_th_t}(\tau;0,2N_t)\prod_{k=3}^{\tilde s}\vartheta'_{g_kh_k}(\tau;0,2N_k)\times$$
$$\times\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2})=$$

$$= (\pi i)^{\tilde{s}} \sum_{m_r,m_t,m_3,\ldots,m_{\tilde{s}+2}=-\infty}^{\infty} (-1)^{h_r m_r + h_t m_t + \sum_{k=3}^{\tilde{s}+2} h_k m_k} (4N_r m_r + g_r)^2 \times$$
$$\times \prod_{k=3}^{\tilde{s}} (4N_k m_k + g_k) e\Big(\sum_{k=1}^{\tilde{s}+2} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 \Big) = \sum_{n=0}^{\infty} C_n^{(r,t)} e(n\tau),$$

since by (2.27) and (2.28),

$$n = \sum_{k=1}^{\tilde{s}+2} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 = \sum_{k=1}^{\tilde{s}+2} \big(N_k m_k^2 + m_k g_k/2\big) + \frac{1}{4} \sum_{k=1}^{\tilde{s}+2} g_k^2/4N_k$$

is a non-negative integer;

3) If $s \equiv 4 \pmod 6$,

$$\vartheta''_{g_r h_r}(\tau; 0, 2N_r) \vartheta_{g_t h_t}(\tau; 0, 2N_t) \times$$
$$\times \prod_{k=3}^{s'} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \vartheta_{g_{s'+1} h_{s'+1}}(\tau; 0, 2N_{s'+1}) \times$$
$$= (\pi i)^{s'} \sum_{m_r,m_t,m_3,\ldots,m_{s'+1}=-\infty}^{\infty} (-1)^{h_r m_r + h_t m_t + \sum_{k=3}^{s'+1} h_k m_k} (4N_r m_r + g_r)^2 \times$$
$$\times \prod_{k=3}^{s'} (4N_k m_k + g_k) e\Big(\sum_{k=1}^{s'+1} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 \Big) = \sum_{n=0}^{\infty} D_n^{(r,t)} e(n\tau),$$

since By (2.30) and (2.31)

$$n = \sum_{k=1}^{s'+1} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 = \sum_{k=1}^{s'+1} \big(N_k m_k^2 + m_k g_k/2\big) + \frac{1}{4} \sum_{k=1}^{s'+2} g_k^2/4N_k$$

is a non-negative integer.

Analogously,

$$\prod_{k=1}^{s/3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) =$$
$$= (\pi i)^{s/3} \sum_{m_1,m_2,\ldots,m_{s/3}=-\infty}^{\infty} (-1)^{\sum_{k=1}^{s/3} h_k m_k} \prod_{k=1}^{s/3} (4N_k m_k + g_k) \times$$
$$\times e\Big(\sum_{k=1}^{s/3} \frac{1}{4N_k} \big(2N_k m_k + g_k/2\big)^2 \tau \Big) = \sum_{n=0}^{\infty} B_n e(n\tau) \quad \text{if} \quad s \equiv 0 \pmod 6,$$

$$\prod_{k=1}^{\tilde{s}} \vartheta'_{g_k h_k}(\tau;0,2N_k)\vartheta_{g_{\tilde{s}+1} h_{\tilde{s}+1}}(\tau;0,2N_{\tilde{s}+1})\vartheta_{g_{\tilde{s}+2} h_{\tilde{s}+2}}(\tau;0,2N_{\tilde{s}+2}) =$$

$$= (\pi i)^{\tilde{s}} \sum_{m_1,m_2,\dots,m_{\tilde{s}+2}=-\infty}^{\infty} (-1)^{\sum\limits_{k=1}^{\tilde{s}+2} h_k m_k} \prod_{k=1}^{\tilde{s}} (4N_k m_k + g_k) \times$$

$$\times e\Big(\sum_{k=1}^{\tilde{s}+2} \frac{1}{4N_k}\left(2N_k m_k + g_k/2\right)^2 \tau\Big) = \sum_{n=0}^{\infty} C_n e(n\tau) \quad \text{if} \quad s \equiv 2 \pmod 6,$$

$$\prod_{k=1}^{s'} \vartheta'_{g_k h_k}(\tau;0,2N_k)\vartheta_{g_{s'+1} h_{s'+1}}(\tau;0,2N_{s'+1}) =$$

$$= (\pi i)^{s'} \sum_{m_1,m_2,\dots,m_{s'+1}=-\infty}^{\infty} (-1)^{\sum\limits_{k=1}^{s'+1} h_k m_k} \prod_{k=1}^{s'} (4N_k m_k + g_k) \times$$

$$\times e\Big(\sum_{k=1}^{s'+1} \frac{1}{4N_k}\left(2N_k m_k + g_k/2\right)^2 \tau\Big) = \sum_{n=0}^{\infty} D_n e(n\tau) \quad \text{if} \quad s \equiv 4 \pmod 6.$$

Thus by (2.3)–(2.8) the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 3,4)$ satisfy condition 3) of the Definition from [1]. $\square$

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