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ON SOME ENTIRE MODULAR FORMS OF AN INTEGRAL WEIGHT FOR THE CONGRUENCE SUBGROUP $\Gamma_0(4N)$

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ABSTRACT. Four types of entire modular forms of weight $\frac{s}{2}$ are constructed for the congruence subgroup $\Gamma_0(4N)$ when s is even. One can find these forms helpful in revealing the arithmetical meaning of additional terms in the formulas for the number of representations of positive integers by positive quadratic forms with integral coefficients in an even number of variables.

All the notations, definitions and lemmas used in this paper are taken from [1].

To make the notations shorter throughout the paper we put

$$\Psi_j(\tau; g_l, h_l, c_l, N_l) = \\ = \Psi_j\big(\tau; g_1, h_1, c_1, N_1; g_2, h_2, c_2, N_2; \dots; g_{s-8}, h_{s-8}, c_{s-8}, N_{s-8}\big) \\ (j = 1, 2, 3, 4).$$

1.

Lemma 1. *For even $s \geq 12$ and given N let*

$$\Psi_1(\tau; g_l, h_l, c_l, N_l) = \bigg\{\frac{1}{N_1}\,\vartheta'''_{g_1h_1}(\tau; c_1, 2N_1)\vartheta'_{g_2h_2}(\tau; c_2, 2N_2) - \\ -\frac{1}{N_2}\,\vartheta'''_{g_2h_2}(\tau; c_2, 2N_2)\vartheta'_{g_1h_1}(\tau; c_1, 2N_1)\bigg\}\prod_{k=3}^{s-8}\vartheta_{g_kh_k}(\tau; c_k, 2N_k) \quad (1.1)$$

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and

$$\Psi_2(\tau;g_l,h_l,c_l,N_l)=\bigg\{\frac{1}{N_1^2}\,\vartheta^{(4)}_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)+ + \frac{1}{N_2^2}\,\vartheta^{(4)}_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)- - \frac{6}{N_1N_2}\,\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\bigg\}\prod_{k=3}^{s-8}\vartheta_{g_kh_k}(\tau;c_k,2N_k), \quad (1.2)$$

where

$$2|g_k,\quad N_k|N\ \ (k=1,2,\dots,s-8),\quad 4\Big|N\sum_{k=1}^{s-8}\frac{h_k}{N_k}\,. \quad (1.3)$$

Then we have

$$(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l)=\sum_{n=0}^{\infty}B_n^{(j)}e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\quad (j=1,2) \quad (1.4)$$

for all substitutions from Γ *in the neighborhood of each rational point* $\tau=-\frac{\delta}{\gamma}$ $(\gamma\neq 0,\ (\gamma,\delta)=1)$.

In [2] this lemma is proved for arbitrary $s>9$ (even and odd).

Lemma 2. *For even* s *we have*

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}=\begin{cases}1 & \text{if } s\equiv 0 \pmod 4,\\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \text{if } s\equiv 2 \pmod 4.\end{cases}$$

Proof. (a) Let $s\equiv 0 \pmod 4$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}=i^{2h\eta(\gamma)(\operatorname{sgn}\delta-1)}\cdot i^{(2h-4)(1-|\delta|)}= =(-1)^{h\eta(\gamma)(\operatorname{sgn}\delta-1)}\cdot(-1)^{(h-2)(1-|\delta|)}=1.$$

(b) Let $s\equiv 2 \pmod 4$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}=i^{(4h+2)\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(4h-6)(1-|\delta|)/2}= =i^{(2h+1)\eta(\gamma)(\operatorname{sgn}\delta-1)}\cdot(-1)^{(2h-3)(1-|\delta|)/2}=\operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big). \qquad \square$$

Theorem 1. *For even* s *and given* N *the functions* $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ *are entire modular forms of weight* $\frac{s}{2}$ *and the character*

$$\chi(\delta)=\begin{cases}\Big(\dfrac{\Delta}{|\delta|}\Big) & \text{if } s\equiv 0 \pmod 4,\\ \operatorname{sgn}\delta\Big(\dfrac{-\Delta}{|\delta|}\Big) & \text{if } s\equiv 2 \pmod 4\end{cases} \quad (1.5)$$

(Δ *is the determinant of the positive quadratic form with integral coefficients in s variables for which the function Ψ_j will be used) for the congruence subgroup $\Gamma_0(4N)$ if the following conditions hold:*

$$1)\ \ 2|g_k,\ \ N_k|N\ \ (k=1,2,\dots,s-8), \tag{1.6}$$

$$2)\ \ 4\Big|N\sum_{k=1}^{s-8}\frac{h_k^2}{N_k}\,,\quad 4\Big|\sum_{k=1}^{s-8}\frac{g_k^2}{4N_k}\,, \tag{1.7}$$

$$3)\ \ \left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)\Psi_j(\tau;\alpha g_l,h_l,0,N_l)=\left(\frac{\Delta}{|\delta|}\right)\Psi_j(\tau;g_l,h_l,0,N_l)\ \ (j=1,2) \tag{1.8}$$

for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$.

Proof. I. As is well known, the theta-series which contained in the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ are regular on $\mathcal{H}$. Therefore these functions satisfy condition 1) and by Lemma 1 also condition 4) of the Definition from [1] (p. 53).

II. From (1.6) it follows that

$$\Gamma_0(4N)\subset\Gamma_0(4N_k)\quad (k=1,2,\dots,s-8). \tag{1.9}$$

It is easy to verify that (1.7) implies

$$4\Big|N\delta^2\sum_{k=1}^{s-8}\frac{h_k^2}{N_k}\,,\quad 4\Big|\sum_{k=1}^{s-8}\frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2}, \tag{1.10}$$

since $2\nmid\delta$ by virtue of $\alpha\delta\equiv 1 \pmod{4N}$.

Using (1.9), (1.10) and Lemma 3 from [1] (p. 58), for all substitutions from $\Gamma_0(4N)$ we get:

1) if $n=3$, $n=1$ and $n=0$,

$$\begin{aligned}\vartheta'''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta'_{g_th_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times\\ \times\prod_{k=3}^{s-8}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)=\\ =i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\left(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times\\ \times\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\prod_{k=3}^{s-8}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k)\end{aligned} \tag{1.11}$$

for $r=1$, $t=2$ and $r=2$, $t=1$;

2) if $n=4$ and $n=0$,

$$\vartheta^{(4)}_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_th_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times\times\prod_{k=3}^{s-8}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)=$$

$$=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\bigg(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\bigg)(\gamma\tau+\delta)^{s/2}\times$$

$$\times\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\prod_{k=3}^{s-8}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k) \tag{1.12}$$

for $r=1$, $t=2$ and $r=2$, $t=1$;

3) if $n=2$ and $n=0$,

$$\vartheta''_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_1\Big)\vartheta''_{g_2h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_2\Big)\times\times\prod_{k=3}^{s-8}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)=$$

$$=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\bigg(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\bigg)(\gamma\tau+\delta)^{s/2}\times$$

$$\times\vartheta''_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta''_{\alpha g_2,h_2}(\tau;0,2N_2)\prod_{k=3}^{s-8}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k). \tag{1.13}$$

Thus by (1.1), (1.11) and (1.2), (1.12), (1.13) we have

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s-8)(1-|\delta|)/2}\times$$

$$\times\bigg(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\quad(j=1,2),$$

which by Lemma 2 implies respectively for $j=1$ and $j=2$

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\bigg(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)$$

$$\text{if}\quad s\equiv 0 \pmod 4$$

and

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=$$

$$=\operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Bigg(\frac{\prod\limits_{k=1}^{s-8}N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\quad\text{if}\quad s\equiv 2\pmod 4.$$

Hence by (1.5) and (1.8) we obtain

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\chi(\delta)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l).$$

Thus the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ satisfy condition (2) of the Definition from [1].

III. The functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=1,2)$ also satisfy condition 3) of the Definition from [1]. For the proof see [2] (p. 109). □

2.

To simplify the notations of this section we put

$$\begin{aligned}&\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,h_2,c_2,N_2)=\\&=\frac{1}{N_1}\,\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)-\\&-\frac{1}{N_2}\,\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)\end{aligned}\tag{2.1}$$

and occasionally

$$(s-2)/3=\widetilde{s},\quad (s-1)/3=s'.\tag{2.2}$$

Lemma 3. *For even $s\ge 10$ and given N let*

$$\begin{aligned}&\Psi_3(\tau;g_l,h_l,c_l,N_l)=\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)\times\\&\times\prod_{k=3}^{s/3}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\quad\textit{if}\quad s\equiv 0\pmod 6,\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\Psi_3(\tau;g_l,h_l,c_l,N_l)=\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)\times\\&\times\prod_{k=3}^{\tilde{s}}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\prod_{k=\tilde{s}+1}^{\tilde{s}+2}\vartheta_{g_kh_k}(\tau;c_k,2N_k)\quad\textit{if}\quad s\equiv 2\pmod 6,\end{aligned}\tag{2.4}$$

$$\begin{aligned}&\Psi_3(\tau;g_l,h_l,c_l,N_l)=\Omega(\tau;g_1,h_1,c_1,N_1;g_2,h_2,c_2,N_2)\times\\&\times\prod_{k=3}^{s'}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\vartheta_{g_{s'+1}h_{s'+1}}(\tau;c_{s'+1},2N_{s'+1})\quad\textit{if}\quad s\equiv 4\pmod 6\end{aligned}\tag{2.5}$$

and

$$\Psi_4(\tau;g_l,h_l,c_l,N_l)=\prod_{k=1}^{s/3}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\quad \textit{if}\quad s\equiv 0 \pmod 6, \tag{2.6}$$

$$\Psi_4(\tau;g_l,h_l,c_l,N_l)=\prod_{k=1}^{\tilde{s}}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\vartheta_{g_{\tilde{s}+1}h_{\tilde{s}+1}}(\tau;c_{\tilde{s}+1},2N_{\tilde{s}+1})\times$$
$$\times\vartheta_{g_{\tilde{s}+2}h_{\tilde{s}+2}}(\tau;c_{\tilde{s}+2},2N_{\tilde{s}+2})\quad \textit{if}\quad s\equiv 2 \pmod 6, \tag{2.7}$$

$$\Psi_4(\tau;g_l,h_l,c_l,N_l)=\prod_{k=3}^{s'}\vartheta'_{g_kh_k}(\tau;c_k,2N_k)\vartheta_{g_{s'+1}h_{s'+1}}(\tau;c_{s'+1},2N_{s'+1}) \tag{2.8}$$
$$\textit{if}\quad s\equiv 4 \pmod 6,$$

where

$$2|g_k,\quad N_k|N\ \ (k=1,2,\dots,s/3),\quad 4\Big|N\sum_{k=1}^{s/3}\frac{h_k}{N_k}\quad \textit{if}\ \ s\equiv 0 \pmod 6, \tag{2.9}$$

$$2|g_k,\quad N_k|N\ \ (k=1,2,\dots,\widetilde{s}+2),\quad 4\Big|N\sum_{k=1}^{\tilde{s}+2}\frac{h_k}{N_k}\quad \textit{if}\ \ s\equiv 2 \pmod 6, \tag{2.10}$$

$$2|g_k,\quad N_k|N\ \ (k=1,2,\dots,s'+1),\quad 4\Big|N\sum_{k=1}^{s'+1}\frac{h_k}{N_k}\quad \textit{if}\ \ s\equiv 4 \pmod 6, \tag{2.11}$$

Then we have

$$(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l)=\sum_{n=0}^{\infty}C_n^{(j)}e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\quad (j=3,4) \tag{2.12}$$

for all substitutions from Γ *in the neighborhood of each point* $\tau=-\frac{\delta}{\gamma}$ $(\gamma\neq 0,$ $(\gamma,\delta)=1)$.

Proof. I. It is shown in [3] (formula (1.7)) that

$$(\gamma\tau+\delta)^3\Omega(\tau;g_1,h_1,0,N_1;g_2,h_2,0,N_2)=$$
$$=-e\Big(\frac{3}{4}\ \mathrm{sgn}\,\gamma\Big)(4N_1N_2\gamma^2)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\prod_{k=1}^{2}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\Omega\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_1h'_1,H_1,N_1;g'_2,h'_2,H_2,2N_2\Big). \tag{2.13}$$

For $n=1$ and $n=0$ it follows by Lemma 4 from [1] (p. 61) that:

1)
$$(\gamma\tau+\delta)^{(s-6)/2}\prod_{k=3}^{s/3}\vartheta'_{g_kh_k}(\tau;0,2N_k)=$$
$$=e\big((s-6)(\operatorname{sgn}\gamma)/8\big)(-i\operatorname{sgn}\gamma)^{(s-6)/3}\times$$
$$\times\Big(2^{(s-6)/3}\prod_{k=3}^{s/3}N_k|\gamma|^{(s-6)/3}\Big)^{-1/2}\sum_{\substack{H_3 \bmod 2N_3\\ \cdots\cdots\cdots\\ H_{s/3} \bmod 2N_{s/3}}}\prod_{k=3}^{s/3}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)\quad\text{if}\quad s\equiv 0 \pmod 6, \tag{2.14}$$

2)
$$(\gamma\tau+\delta)^{(s-6)/2}\prod_{k=3}^{\tilde s}\vartheta'_{g_kh_k}(\tau;0,2N_k)\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\times$$
$$\times\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2})=e\big((s-6)(\operatorname{sgn}\gamma)/8\big)\times$$
$$\times\Big(2^{\tilde s}\sum_{k=3}^{\tilde s+2}N_k|\gamma|^{\tilde s}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{(s-8)/3}\times$$
$$\times\sum_{\substack{H_3 \bmod 2N_3\\ \cdots\cdots\cdots\\ H_{\tilde s+2} \bmod 2N_{\tilde s+2}}}\prod_{k=3}^{\tilde s}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta'_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)\times$$
$$\times\prod_{k=\tilde s+1}^{\tilde s+2}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big) \tag{2.15}$$
$$\text{if}\quad s\equiv 2 \pmod 6,$$

3)
$$(\gamma\tau+\delta)^{(s-6)/2}\prod_{k=3}^{s'}\vartheta'_{g_kh_k}(\tau;0,2N_k)\vartheta_{g_{s'+1}h_{s'+1}}(\tau;0,2N_{s'+1})=$$
$$=e\big((s-6)(\operatorname{sgn}\gamma)/8\big)\Big(2^{s'-1}\sum_{k=3}^{s'+1}N_k|\gamma|^{s'-1}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{(s-7)/3}\times$$
$$\times\sum_{\substack{H_3 \bmod 2N_3\\ \cdots\cdots\cdots\\ H_{s'+1} \bmod 2N_{s'+1}}}\prod_{k=3}^{s'}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta'_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\times$$
$$\times\varphi_{g'_{s'+1}g_{s'+1}h_{s'+1}}(0,H_{s'+1};2N_{s'+1})\times$$
$$\times\vartheta_{g_{s'+1}h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_{s'+1},2N_{s'+1}\Big)\quad\text{if}\quad s\equiv 4 \pmod 6. \tag{2.16}$$

After multiplying (2.13) successively by (2.14), (2.15) and (2.16), by (2.3) we obtain

1) $$(\gamma\tau+\delta)^{s/2}\Psi_3(\tau;g_l,h_l,0,N_l)=(-1)^{s/6}e\big(s(\operatorname{sgn}\gamma)/8\big)\times$$
$$\times\Big(2^{s/3}\prod_{k=1}^{s/3}N_k|\gamma|^{s/3}\Big)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\\ H_{s/3} \bmod 2N_{s/3}}}\prod_{k=3}^{s/3}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big)\quad\text{if}\quad s\equiv 0 \pmod 6, \tag{2.17}$$

2) $$(\gamma\tau+\delta)^{s/2}\Psi_3(\tau;g_l,h_l,0,N_l)=(-1)^{(s-2)/6}e\big(s(\operatorname{sgn}\gamma)/8\big)\times$$
$$\times\Big(2^{\tilde{s}+2}\prod_{k=1}^{\tilde{s}+2}N_k|\gamma|^{\tilde{s}+2}\Big)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\\ H_{\tilde{s}+2} \bmod 2N_{\tilde{s}+2}}}\prod_{k=3}^{\tilde{s}+2}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big)\quad\text{if}\quad s\equiv 2 \pmod 6, \tag{2.18}$$

3) $$(\gamma\tau+\delta)^{s/2}\Psi_3(\tau;g_l,h_l,0,N_l)=(-1)^{(s+2)/6}(i\operatorname{sgn}\gamma)e\big(s(\operatorname{sgn}\gamma)/8\big)\times$$
$$\times\Big(2^{s'+1}\prod_{k=1}^{s'+1}N_k|\gamma|^{s'+1}\Big)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\\ H_{s'+1} \bmod 2N_{s'+1}}}\prod_{k=3}^{s'+1}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$
$$\times\Psi_3\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big)\quad\text{if}\quad s\equiv 4 \pmod 6. \tag{2.19}$$

Further, reasoning as in [1] (Lemma 5, the bottom part of p. 64 and p. 65), we obtain (2.12) if $j=3$.

II. By Lemma 4 from [1], for $n=1$ and $n=0$ it respectively follows by (2.6), (2.7) and (2.8) that

$$(\gamma\tau+\delta)^{s/2}\Psi_4(\tau;g_l,h_l,0,N_l)=(\gamma\tau+\delta)^{s/2}\prod_{k=1}^{s/3}\vartheta'_{g_kh_k}(\tau;0,2N_k)=$$
$$=e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{s/3}\prod_{k=1}^{s/3}N_k|\gamma|^{s/3}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{s/3}\times$$
$$\times\sum_{\substack{H_1 \bmod 2N_1\\ \cdots\cdots\\ H_{s/3} \bmod 2N_{s/3}}}\prod_{k=1}^{s/3}\Big(\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\vartheta'_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big)=$$

$$= (-1)^{s/6} e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{s/3}\prod_{k=1}^{s/3} N_k|\gamma|^{s/3}\Big)^{-1/2} \times$$

$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ \cdots\cdots \\ H_{s/3} \bmod 2N_{s/3}}} \prod_{k=1}^{s/3} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; g'_l, h'_l, H_l, N_l\Big) \quad \text{if} \quad s \equiv 0 \pmod 6, \tag{2.20}$$

$$(\gamma\tau+\delta)^{s/2}\Psi_4(\tau; g_l, h_l, 0, N_l) = (\gamma\tau+\delta)^{s/2}\prod_{k=1}^{\tilde{s}} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \times$$

$$\times \vartheta_{g_{\tilde{s}+1} h_{\tilde{s}+1}}(\tau; 0, 2N_{\tilde{s}+1})\vartheta_{g_{\tilde{s}+2} h_{\tilde{s}+2}}(\tau; 0, 2N_{\tilde{s}+2}) =$$

$$= e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{\tilde{s}+2}\prod_{k=1}^{\tilde{s}+2} N_k|\gamma|^{\tilde{s}+2}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{\tilde{s}} \times$$

$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ \cdots\cdots \\ H_{\tilde{s}} \bmod 2N_{\tilde{s}}}} \prod_{k=1}^{\tilde{s}} \bigg(\varphi_{g'_k g_k h_k}(0, H_k; 2N_k)\vartheta'_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; H_k, 2N_k\Big)\bigg) \times$$

$$\times \sum_{\substack{H_{\tilde{s}+1} \bmod 2N_{\tilde{s}+1} \\ H_{\tilde{s}+2} \bmod 2N_{\tilde{s}+2}}} \prod_{k=\tilde{s}+1}^{\tilde{s}+2} \bigg(\varphi_{g'_k g_k h_k}(0, H_k; 2N_k)\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; H_k, 2N_k\Big)\bigg) =$$

$$= (-1)^{(s-2)/6} e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{\tilde{s}+2}\prod_{k=1}^{\tilde{s}+2} N_k|\gamma|^{\tilde{s}+2}\Big)^{-1/2} \times$$

$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ \cdots\cdots \\ H_{\tilde{s}+2} \bmod 2N_{\tilde{s}+2}}} \prod_{k=1}^{\tilde{s}+2} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; g'_l, h'_l, H_l, N_l\Big) \quad \text{if} \quad s \equiv 2 \pmod 6 \tag{2.21}$$

and

$$(\gamma\tau+\delta)^{s/2}\Psi_4(\tau; g_l, h_l, 0, N_l) = (\gamma\tau+\delta)^{s/2}\prod_{k=1}^{s'} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \times$$

$$\times \vartheta_{g_{s'+1} h_{s'+1}}(\tau; 0, 2N_{s'+1}) e\big(s(\operatorname{sgn}\gamma)/8\big) \times$$

$$\times \Big(2^{s'+1}\prod_{k=1}^{s'+1} N_k|\gamma|^{s'+1}\Big)^{-1/2}(-i\operatorname{sgn}\gamma)^{s'} \times$$

$$\times \sum_{\substack{H_1 \mod 2N_1 \\ \cdots\cdots\cdots \\ H_{s'} \mod 2N_{s'}}} \prod_{k=1}^{s'} \Big(\varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \vartheta'_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; H_k, 2N_k\Big)\Big) \times$$

$$\times \sum_{H_{s'+1} \mod 2N_{s'+1}} \varphi_{g'_{s+1} g_{s'+1} h_{s'+1}}(0, H_{s'+1}; 2N_{s'+1}) \times$$

$$\times \vartheta_{g_{s'+1} h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; H_{s'+1}, 2N_{s'+1}\Big) =$$

$$= (-1)^{(s+2)/6}(i \operatorname{sgn}\gamma) e\big(s(\operatorname{sgn}\gamma)/8\big)\Big(2^{s'+1}\prod_{k=1}^{s'+1} N_k |\gamma|^{s'+1}\Big)^{-1/2} \times$$

$$\times \sum_{\substack{H_1 \mod 2N_1 \\ \cdots\cdots\cdots \\ H_{s'+1} \mod 2N_{s'+1}}} \prod_{k=1}^{s'+1} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$

$$\times \Psi_4\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,; g'_l, h'_l, H_l, N_l\Big) \quad \text{if} \quad s \equiv 4 \pmod 6. \tag{2.22}$$

Further, reasoning as in [1] (Lemma 5), we obtain (2.12) if $j = 4$. □

Lemma 4. *For even s we have*

(a) $$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{(s/3)(1-|\delta|)/2} = \begin{cases} 1 & \textit{if} \quad s \equiv 0 \pmod{12}, \\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \textit{if} \quad s \equiv 6 \pmod{12}; \end{cases}$$

(b) $$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+4)/3)(1-|\delta|)/2} = \begin{cases} 1 & \textit{if} \quad s \equiv 8 \pmod{12}, \\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \textit{if} \quad s \equiv 2 \pmod{12}; \end{cases}$$

(c) $$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+2)/3)(1-|\delta|)/2} = \begin{cases} \Big(\dfrac{-1}{|\delta|}\Big) & \textit{if} \quad s \equiv 4 \pmod{12}, \\ \operatorname{sgn}\delta & \textit{if} \quad s \equiv 10 \pmod{12}. \end{cases}$$

Proof. (a) Let $s \equiv 0 \pmod 6$, i.e., $s \equiv 0 \pmod{12}$ or $s \equiv 6 \pmod{12}$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{(s/3)(1-|\delta|)/2} = i^{3h\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot i^{2h(1-|\delta|)/2} =$$

$$= \begin{cases} 1 & \text{if} \quad 2|h, \quad \text{i.e., if} \quad s \equiv 0 \pmod{12}, \\ \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \text{if} \quad 2 \nmid h, \quad \text{i.e., if} \quad s \equiv 6 \pmod{12}. \end{cases}$$

(b) Let $s \equiv 2 \pmod 6$, i.e., $s \equiv 2 \pmod{12}$ or $s \equiv 8 \pmod{12}$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+4)/3)(1-|\delta|)/2} = i^{(3h+1)\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot i^{2(h+1)(1-|\delta|)/2} =$$
$$= i^{(h+1)\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot (-1)^{(h+1)(1-|\delta|)/2} =$$
$$= \begin{cases} \operatorname{sgn}\delta\Big(\dfrac{-1}{|\delta|}\Big) & \text{if } \ 2|h, \ \text{ i.e., if } \ s \equiv 2 \pmod{12}, \\ 1 & \text{if } \ 2 \nmid h, \ \text{ i.e., if } \ s \equiv 8 \pmod{12}. \end{cases}$$

(c) Let $s \equiv 4 \pmod 6$, i.e., $s \equiv 4 \pmod{12}$ or $s \equiv 10 \pmod{12}$. Then

$$i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2} \cdot i^{((s+2)/3)(1-|\delta|)/2} = i^{(3h+2)\eta(\gamma)(\operatorname{sgn}\delta-1)} \cdot i^{2(h+1)(1-|\delta|)/2} =$$
$$= \begin{cases} \Big(\dfrac{-1}{|\delta|}\Big) & \text{if } \ 2|h, \ \text{ i.e., if } \ s \equiv 4 \pmod{12}, \\ \operatorname{sgn}\delta & \text{if } \ 2 \nmid h, \ \text{ i.e., if } \ s \equiv 10 \pmod{12}. \end{cases} \qquad \square$$

Theorem 2. *For even s and given N the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 3, 4)$ are entire modular forms of weight $\frac{s}{2}$ and the character*

$$\chi(\delta) = \begin{cases} \Big(\dfrac{\Delta}{|\delta|}\Big) & \text{if } \ s \equiv 0 \pmod 4, \\ \operatorname{sgn}\delta\Big(\dfrac{-\Delta}{|\delta|}\Big) & \text{if } \ s \equiv 2 \pmod 4 \end{cases} \tag{2.23}$$

(Δ *is the determinant of the positive quadratic form with integral coefficients in s variables for which the function Ψ_j will be used*) *for the congruence subgroup $\Gamma_0(4N)$ if the following conditions hold:*

(a) *when $s \equiv 0 \pmod 6$*

$$1) \ \ 2|g_k, \ \ N_k|N \ \ (k = 1, 2, \ldots, s/3), \tag{2.24}$$

$$2) \ \ 4\Big|N\sum_{k=1}^{s/3}\frac{h_k^2}{N_k}\,, \quad 4\Big|\sum_{k=1}^{s/3}\frac{g_k^2}{4N_k}\,, \tag{2.25}$$

3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\Bigg(\frac{\prod\limits_{k=1}^{s/3} N_k}{|\delta|}\Bigg)\Psi_j(\tau; \alpha g_l, h_l, 0, N_l) = \Big(\frac{\Delta}{|\delta|}\Big)\Psi_j(\tau; g_l, h_l, 0, N_l); \tag{2.26}$$

(b) *when $s \equiv 2 \pmod 6$*

$$1) \ \ 2|g_k, \ \ N_k|N \ \ (k = 1, 2, \ldots, (s+4)/3), \tag{2.27}$$

$$2) \ \ 4\Big|N\sum_{k=1}^{(s+4)/3}\frac{h_k^2}{N_k}\,, \quad 4\Big|\sum_{k=1}^{(s+4)/3}\frac{g_k^2}{4N_k}\,, \tag{2.28}$$

3) *for all* α *and* δ *with* $\alpha\delta \equiv 1 \pmod{4N}$

$$\Bigg(\frac{\prod\limits_{k=1}^{(s+4)/3} N_k}{|\delta|}\Bigg)\Psi_j(\tau;\alpha g_l,h_l,0,N_l) = \Big(\frac{\Delta}{|\delta|}\Big)\Psi_j(\tau;g_l,h_l,0,N_l); \qquad (2.29)$$

(c) *when* $s \equiv 4 \pmod 6$

1) $2|g_k,\quad N_k|N \quad (k=1,2,\dots,(s+2)/3),$ (2.30)

2) $4\Big|N\sum\limits_{k=1}^{(s+2)/3}\dfrac{h_k^2}{N_k}\,,\quad 4\Big|\sum\limits_{k=1}^{(s+2)/3}\dfrac{g_k^2}{4N_k}\,,$ (2.31)

3) *for all* α *and* δ *with* $\alpha\delta \equiv 1 \pmod{4N}$

$$\Bigg(\frac{\prod\limits_{k=1}^{(s+2)/3} N_k}{|\delta|}\Bigg)\Psi_j(\tau;\alpha g_l,h_l,0,N_l) = \operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)\Psi_j(\tau;g_l,h_l,0,N_l). \qquad (2.32)$$

Proof. I. As in the case of Theorem 1, the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ ($j = 3,4$) satisfy condition 1) and by Lemma 3 also condition 4) of the Definition from [1].

II. From (2.24), (2.27) and (2.30) we obtain

$$\Gamma_0(4N) \subset \Gamma_0(4N_k). \qquad (2.33)$$

Obviously, (2.25), (2.28) and (2.31) respectively imply

$$4\Big|N\delta^2\sum_{k=1}^{s/3}\frac{h_k^2}{N_k}\,,\quad 4\Big|\sum_{k=1}^{s/3}\frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2} \quad \text{if } s\equiv 0 \pmod 6, \qquad (2.34)$$

$$4\Big|N\delta^2\sum_{k=1}^{(s+4)/3}\frac{h_k^2}{N_k}\,,\quad 4\Big|\sum_{k=1}^{(s+4)/3}\frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2} \quad \text{if } s\equiv 2 \pmod 6, \qquad (2.35)$$

and

$$4\Big|N\delta^2\sum_{k=1}^{(s+2)/3}\frac{h_k^2}{N_k}\,,\quad 4\Big|\sum_{k=1}^{(s+2)/3}\frac{g_k^2}{4N_k}\,\delta^{2\varphi(2N_k)-2} \quad \text{if } s\equiv 4 \pmod 6. \qquad (2.36)$$

Using (2.33) and Lemma 3 from [1], for all substitutions from $\Gamma_0(4N)$ we obtain:

1) for $n=2$, $n=1$ and $n=0$

$$\vartheta''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_th_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times$$

$$\times \prod_{k=3}^{s/3} \vartheta'_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big) =$$

$$= i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s/3)(1-|\delta|)/2}\left(\frac{\prod_{k=1}^{s/3} N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times$$

$$\times\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\times$$

$$\times\prod_{k=3}^{s/3}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k) \quad \text{if} \quad s\equiv 0 \pmod 6, \tag{2.37}$$

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times$$

$$\times \prod_{k=3}^{\tilde s} \vartheta'_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)\times$$

$$\times\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_{\tilde s+1}\Big)\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_{\tilde s+2}\Big) =$$

$$= i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(\tilde s+2)(1-|\delta|)/2}\left(\frac{\prod_{k=1}^{\tilde s+2} N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\times$$

$$\times\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\prod_{k=3}^{\tilde s}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\times$$

$$\times\vartheta_{\alpha g_{\tilde s+1},h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\vartheta_{\alpha g_{\tilde s+2},h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2}) \tag{2.38}$$

$$\text{if} \quad s\equiv 2 \pmod 6,$$

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_r\Big)\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_t\Big)\times$$

$$\times \prod_{k=3}^{s'} \vartheta'_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_k\Big)\vartheta_{g_{s'+1}h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;0,2N_{s'+1}\Big) =$$

$$= (\operatorname{sgn}\delta) i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s'+1)(1-|\delta|)/2}\times$$

$$\times\left(\frac{\prod_{k=1}^{s'} N_k}{|\delta|}\right)(\gamma\tau+\delta)^{s/2}\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\times$$

$$\times\prod_{k=3}^{s'}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k)\vartheta_{\alpha g_{s'+1},h_{s'+1}}(\tau;0,2N_{s'+1}) \tag{2.39}$$

$$\text{if} \quad s\equiv 4 \pmod 6,$$

where $r=2$, $t=1$ and $r=1$, $t=2$;

2) for $n=1$ and $n=0$

$$\prod_{k=1}^{s/3}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_k\Big)=$$

$$=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s/3)(1-|\delta|)/2}\Bigg(\frac{\prod\limits_{k=1}^{s/3}N_k}{|\delta|}\Bigg)\times$$

$$\times(\gamma\tau+\delta)^{s/2}\prod_{k=1}^{s/3}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\quad\text{if}\quad s\equiv 0\pmod 6,\tag{2.40}$$

$$\prod_{k=1}^{\tilde s}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_k\Big)\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_{\tilde s+1}\Big)\times$$

$$\times\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_{\tilde s+2}\Big)=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(\tilde s+2)(1-|\delta|)/2}\times$$

$$\times\Bigg(\frac{\prod\limits_{k=1}^{\tilde s+2}N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\prod_{k=1}^{\tilde s}\vartheta'_{\alpha g_k,h_k}(\tau;0,2N_k)\times$$

$$\times\vartheta_{\alpha g_{\tilde s+1},h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\vartheta_{\alpha g_{\tilde s+2},h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2})\tag{2.41}$$

$$\text{if}\quad s\equiv 2\pmod 6,$$

$$\prod_{k=1}^{s'}\vartheta'_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_k\Big)\vartheta_{g_{s'+1}h_{s'+1}}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,,0,2N_{s'+1}\Big)=$$

$$=\operatorname{sgn}\delta\, i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s'+1)(1-|\delta|)/2}\Bigg(\frac{\prod\limits_{k=1}^{s'+1}N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\times$$

$$\times\prod_{k=1}^{s'}\vartheta'_{\alpha g_kh_k}(\tau;0,2N_k)\vartheta_{\alpha g_{s'+1},h_{s'+1}}(\tau;0,2N_{s'+1})\tag{2.42}$$

$$\text{if}\quad s\equiv 4\pmod 6.$$

By (2.3), (2.37) and (2.6), (2.40), for $j=3$ and $j=4$ we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(s/3)(1-|\delta|)/2}\times$$

$$\times\Big(\frac{\prod_{k=1}^{s/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 0 \pmod 6. \tag{2.43}$$

By (2.4), (2.38) and (2.7), (2.41), for $j=3$ and $j=4$ we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big) = i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{((s+4)/3)(1-|\delta|)/2}\times$$
$$\times\Big(\frac{\prod_{k=1}^{(s+4)/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 2 \pmod 6. \tag{2.44}$$

By (2.5), (2.39) and (2.8), (2.42), for $j=3$ and $j=4$ we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big) = i^{s\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{((s+2)/3)(1-|\delta|)/2}\times$$
$$\times\Big(\frac{\prod_{k=1}^{(s+2)/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 4 \pmod 6. \tag{2.45}$$

Further, by Lemma 4, for $j=3,4$ realtions (2.43), (2.44), and (2.45) respectively imply:

$$\begin{aligned}\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big) &= \Big(\frac{\prod_{k=1}^{s/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\\ &\qquad \text{if} \quad s\equiv 0 \pmod{12},\\ &= \operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Big(\frac{\prod_{k=1}^{s/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\times\\ &\quad\times\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 6 \pmod{12};\end{aligned} \tag{2.46}$$

$$\begin{aligned}\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big) &= \operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Big(\frac{\prod_{k=1}^{(s+4)/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\times\\ &\quad\times\Psi_j(\tau;\alpha g_l,h_l,0,N_l) \quad \text{if} \quad s\equiv 2 \pmod{12},\\ &= \Big(\frac{\prod_{k=1}^{(s+4)/3} N_k}{|\delta|}\Big)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\\ &\qquad \text{if} \quad s\equiv 8 \pmod{12};\end{aligned} \tag{2.47}$$

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)\Bigg(\frac{\prod\limits_{k=1}^{(s+2)/3}N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\times$$
$$\times\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\quad\text{if}\quad s\equiv 4\pmod{12},$$
$$=\Bigg(\frac{\prod\limits_{k=1}^{(s+2)/3}N_k}{|\delta|}\Bigg)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;\alpha g_l,h_l,0,N_l)\tag{2.48}$$
$$\text{if}\quad s\equiv 10\pmod{12}.$$

From (2.46)–(2.48), (2.26), (2.29), (2.32), and (2.23) it follows that

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,;g_l,h_l,0,N_l\Big)=\chi(\delta)(\gamma\tau+\delta)^{s/2}\Psi_j(\tau;g_l,h_l,0,N_l)\quad(j=3,4).$$

Thus the functions $\Psi_j(\tau;g_l,h_l,0,N_l)$ $(j=3,4)$ satisfy condition 2) of the Definition from [1].

III. By [1] (formula 1.3), for $r=1$, $t=2$ and $r=2$, $t=1$ we obtain:

1) If $s\equiv 0\pmod 6$,

$$\vartheta''_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_th_t}(\tau;0,2N_t)\prod_{k=3}^{s/3}\vartheta'_{g_kh_k}(\tau;0,2N_k)=$$
$$=(\pi i)^{s/3}\sum_{m_r,m_t,m_3,\dots,m_{s/3}=-\infty}^{\infty}(-1)^{h_rm_r+h_tm_t+\sum\limits_{k=3}^{s/3}h_km_k}(4N_rm_r+g_r)^2\times$$
$$\times\prod_{k=3}^{s/3}(4N_km_k+g_k)e\Big(\sum_{k=1}^{s/3}\frac{1}{4N_k}\Big(2N_km_k+\frac{g_k}{2}\Big)^2\Big)=\sum_{n=0}^{\infty}B_n^{(r,t)}e(n\tau),$$

since by (2.24) and (2.25),

$$n=\sum_{k=1}^{s/3}\frac{1}{4N_k}\Big(2N_km_k+\frac{g_k}{2}\Big)^2=\sum_{k=1}^{s/3}\big(N_km_k^2+m_kg_k/2\big)+\frac14\sum_{k=1}^{s/3}g_k^2/4N_k$$

is a non-negative integer;

2) If $s\equiv 2\pmod 6$,

$$\vartheta''_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_th_t}(\tau;0,2N_t)\prod_{k=3}^{\tilde s}\vartheta'_{g_kh_k}(\tau;0,2N_k)\times$$
$$\times\vartheta_{g_{\tilde s+1}h_{\tilde s+1}}(\tau;0,2N_{\tilde s+1})\vartheta_{g_{\tilde s+2}h_{\tilde s+2}}(\tau;0,2N_{\tilde s+2})=$$

$$= (\pi i)^{\tilde{s}} \sum_{m_r, m_t, m_3, \dots, m_{\tilde{s}+2}=-\infty}^{\infty} (-1)^{h_r m_r + h_t m_t + \sum\limits_{k=3}^{\tilde{s}+2} h_k m_k} (4N_r m_r + g_r)^2 \times$$

$$\times \prod_{k=3}^{\tilde{s}} (4N_k m_k + g_k) e\Big(\sum_{k=1}^{\tilde{s}+2} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 \Big) = \sum_{n=0}^{\infty} C_n^{(r,t)} e(n\tau),$$

since by (2.27) and (2.28),

$$n = \sum_{k=1}^{\tilde{s}+2} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 = \sum_{k=1}^{\tilde{s}+2} \big(N_k m_k^2 + m_k g_k/2\big) + \frac{1}{4} \sum_{k=1}^{\tilde{s}+2} g_k^2/4N_k$$

is a non-negative integer;

3) If $s \equiv 4 \pmod 6$,

$$\vartheta''_{g_r h_r}(\tau; 0, 2N_r) \vartheta_{g_t h_t}(\tau; 0, 2N_t) \times$$

$$\times \prod_{k=3}^{s'} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \vartheta_{g_{s'+1} h_{s'+1}}(\tau; 0, 2N_{s'+1}) \times$$

$$= (\pi i)^{s'} \sum_{m_r, m_t, m_3, \dots, m_{s'+1}=-\infty}^{\infty} (-1)^{h_r m_r + h_t m_t + \sum\limits_{k=3}^{s'+1} h_k m_k} (4N_r m_r + g_r)^2 \times$$

$$\times \prod_{k=3}^{s'} (4N_k m_k + g_k) e\Big(\sum_{k=1}^{s'+1} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 \Big) = \sum_{n=0}^{\infty} D_n^{(r,t)} e(n\tau),$$

since By (2.30) and (2.31)

$$n = \sum_{k=1}^{s'+1} \frac{1}{4N_k} \Big(2N_k m_k + \frac{g_k}{2}\Big)^2 = \sum_{k=1}^{s'+1} \big(N_k m_k^2 + m_k g_k/2\big) + \frac{1}{4} \sum_{k=1}^{s'+2} g_k^2/4N_k$$

is a non-negative integer.

Analogously,

$$\prod_{k=1}^{s/3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k) =$$

$$= (\pi i)^{s/3} \sum_{m_1, m_2, \dots, m_{s/3}=-\infty}^{\infty} (-1)^{\sum\limits_{k=1}^{s/3} h_k m_k} \prod_{k=1}^{s/3} (4N_k m_k + g_k) \times$$

$$\times e\Big(\sum_{k=1}^{s/3} \frac{1}{4N_k} \big(2N_k m_k + g_k/2\big)^2 \tau \Big) = \sum_{n=0}^{\infty} B_n e(n\tau) \quad \text{if} \quad s \equiv 0 \pmod 6,$$

$$\prod_{k=1}^{\tilde{s}} \vartheta'_{g_k h_k}(\tau;0,2N_k)\vartheta_{g_{\tilde{s}+1}h_{\tilde{s}+1}}(\tau;0,2N_{\tilde{s}+1})\vartheta_{g_{\tilde{s}+2}h_{\tilde{s}+2}}(\tau;0,2N_{\tilde{s}+2}) =$$

$$= (\pi i)^{\tilde{s}} \sum_{m_1,m_2,\dots,m_{\tilde{s}+2}=-\infty}^{\infty} (-1)^{\sum\limits_{k=1}^{\tilde{s}+2} h_k m_k} \prod_{k=1}^{\tilde{s}} (4N_k m_k + g_k) \times$$

$$\times e\Big(\sum_{k=1}^{\tilde{s}+2} \frac{1}{4N_k} \big(2N_k m_k + g_k/2\big)^2 \tau\Big) = \sum_{n=0}^{\infty} C_n e(n\tau) \quad \text{if} \quad s \equiv 2 \pmod 6,$$

$$\prod_{k=1}^{s'} \vartheta'_{g_k h_k}(\tau;0,2N_k)\vartheta_{g_{s'+1}h_{s'+1}}(\tau;0,2N_{s'+1}) =$$

$$= (\pi i)^{s'} \sum_{m_1,m_2,\dots,m_{s'+1}=-\infty}^{\infty} (-1)^{\sum\limits_{k=1}^{s'+1} h_k m_k} \prod_{k=1}^{s'} (4N_k m_k + g_k) \times$$

$$\times e\Big(\sum_{k=1}^{s'+1} \frac{1}{4N_k} \big(2N_k m_k + g_k/2\big)^2 \tau\Big) = \sum_{n=0}^{\infty} D_n e(n\tau) \quad \text{if} \quad s \equiv 4 \pmod 6.$$

Thus by (2.3)–(2.8) the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 3, 4)$ satisfy condition 3) of the Definition from [1]. $\square$

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