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A PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS FOR A QUASILINEAR PARABOLIC EQUATION

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ABSTRACT. The solvability of the nonlocal boundary value problem

$$u_t = a(t,x,u,u_x)u_{xx} + b(t,x,u,u_x), \quad 0 \le t \le T, \quad |x| \le l,$$
$$u(0,x) = 0, \quad u(t,-l) = u(t,l), \quad u_x(t,-l) = u_x(t,l)$$

in a class of functions is investigated for a quasilinear parabolic equation. The solution uniqueness follows from the maximum principle.

The problem of solving the basic boundary value problems and Cauchy's problem has been thoroughly investigated for a wide class of nonlinear parabolic equations of second order. Local (see, for example, [1]) and global ([2]) existence theorems under different assumptions on the character of nonlinearity of equations have been proved by different methods. Local solvability takes place for equations with smooth coefficients without any essential restrictions on the nonlinearity character of coefficients. Such restrictions become necessary when constructing a global solution.

Problems for linear equations with nonlocal (initial or boundary) conditions have been studied in many papers, but boundary value problems for nonlinear equations with nonlocal conditions have so far remained nearly uninvestigated. In this paper we shall consider a problem on the rectangle $Q = \{(t,x) : 0 \le t \le T,\ |x| \le l\}$ for the quasilinear parabolic equation

$$u_t = a(t,x,u,u_x)u_{xx} + b(t,x,u,u_x) \tag{1}$$

under the conditions

$$u(0,x) = 0, \tag{2}$$
$$u(t,-l) = u(t,l), \quad u_x(t,-l) = u_x(t,l). \tag{3}$$

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It is assumed that the functions $a(t,x,u,p)$ and $b(t,x,u,p)$ are defined for $(t,x) \in Q$ and arbitrary (u,p) and bounded on every compactum.

Note the problem under consideration appears when studying the birth process of separate population in the one-dimensional biological reactor having the shape of a long tube and closed like a ring [3]. In that case $u(t,x)$ is the population density, l is the reactor length.

When constructing the theory of nonlocal boundary value problems and Cauchy's problem for parabolic equations of form (1), a major difficulty is to obtain a priori estimates for the absolute value of the derivative u_x and its Hölder constant.

We shall use the notation from [2]:

$$Q_0^\delta = \{(t,x) : \quad 0 \le t \le T, \quad |x| \le l-\delta\},$$
$$Q^\delta = \{(t,x) : \quad \delta \le t \le T, \quad |x| \le l-\delta\},$$

where $0 < \delta < \min(l,T)$.

Some of S. N. Kruzhkov's results [2, 4] will also be used in the investigation of the formulated problem.

Let the function $u(t,x)$ be defined on some set D; for every number $\gamma \in (0,1)$ let

$$|u|_\gamma^D = \sup_D |u(t,x)| + \sup_{(t,x)\in D,\ (\tau,y)\in D} \frac{|u(t,x)-u(\tau,y)|}{(|t-\tau|+|x-y|^2)^{\gamma/2}},$$
$$|u|_{1+\gamma}^D = |u|_\gamma^D + |u_x|_\gamma^D, \quad |u|_{2+\gamma}^D = |u|_{1+\gamma}^D + |u_{xx}|_\gamma^D + |u_t|_\gamma^D.$$

Throughout the paper it is assumed that the following basic conditions are fulfilled for equation (1):

A. $b_u(t,x,u,0) \le b_0$ and $|b(t,x,0,0)| \le b_1$ for $(t,x) \in Q$ and arbitrary $u(t,x)$;

B. $a(t,x,u,p) \ge a_0 = const > 0$ for $(t,x) \in Q$ and arbitrary $u(t,x)$ and $p(t,x)$.

Further we denote by C, C_i, N_j any constants which depend only on known data.

1. A Priori Estimates for Solving the Problem.

Theorem 1. *Let the function $u(t,x)$ and its derivative u_x be continuous on Q and satisfy conditions* (2), (3) *and equation* (1) *on Q except perhaps the points $(t,x) : 0 \le t \le T$, $x = 0$. Further, let $\max_Q |u| = M$, and for $(t,x) \in Q$, $|u| \le M$ and arbitrary p the continuous functions a and b satisfy the conditions*

$$a(t,-l,u,p) = a(t,l,u,p), \quad b(t,-l,u,p) = b(t,l,u,p), \tag{4}$$

$$\frac{|b(t,x,u,p)|}{a(t,x,u,p)} \le K(p^2+1).$$

Then for $(t,x) \in Q$ $|u_x(t,x)| \le C(M,a_0,K) = M_1$. *If* $a_1 = \max a(t,x,u,p)$, $b_2 = \max|b(t,x,u,p)|$ *in the domain* $\{(t,x) \in Q, \ \ |u| \le M, \ \ |p| \le M_1\}$, *then* (*see* [4], *Lemma* 4.5)

$$|u(t_1,x) - u(t_2,x)| \le C(M,a_1,b_2,K,M_1)|t_1-t_2|^{1/2}. \tag{5}$$

Let $u(t,x)$ *have the generalized derivatives* $u_{xx}, u_{tx} \in L_2(Q)$. *Then there exists* $\gamma = \gamma(M,a_0,a_1,K)$ *such that*

$$|u|^Q_{1+\gamma} \le C(M,a_0,a_1,K), \quad 0<\gamma<1. \tag{6}$$

Proof. The estimate $|u_x| \le C$ for Q_0^δ immediately follows from Theorem 2 of [2]. Here we use the method of introducing additional space variables. To complete the proof, it is necessary to establish the correctness of the estimate up to lateral sides of the rectangle Q. In the well-known work [2], as $u|_{x=\pm l} = 0$, S. N. Kruzhkov has used the method of continuing the solution through the lateral sides which leads to the odd function. We propose to continue the function $u(t,x)$ through the lateral sides of Q according to the rule

$$u(t,x) = u(t,2l+x) \quad \text{for} \quad -3l \le x \le -l, \tag{7}$$

$$u(t,x) = u(t,x-2l) \quad \text{for} \quad l \le x \le 3l. \tag{8}$$

It is assumed that the coefficients of equation (1) are continued in x according to (7), (8). The new function (also denoted by $u(t,x)$) has a continuous derivative u_x at every point of rectangles $R_\pm = \{(t,x) : 0 \le t \le T, \ |x \pm \frac{3}{2}l| \le \frac{3}{2}l\}$ and satisfies a "continued" equation of form (1) (for example, under $l<x<3l$, $u_t = a(t,x-2l,u,u_x)u_{xx} + b(t,x-2l,u,u_x)$) with the same properties as in Theorem 1. Using the well-known "intrinsic" results, we get an estimate of $|u_x|$ in rectangles whose union contains Q. Since the obtaining of intrinsic estimates is based on the maximum principle, the statements of the theorem remain true when the function $u(t,x)$ is continuous in Q, has a continuous derivative u_x and satisfies (1) throughout Q except the points of a finite number of straight lines $x = const$ (see the proof of Theorem 4.3 in [4]).

Now, using the obtained estimates of $|u|$ and $|u_x|$ in Q, by virtue of [4, Theorem 4.5] we get estimate (5).

The intrinsic estimate

$$|u|^{Q^{2\delta}}_{1+\gamma} \le C(M,a_0,a_1,K,\delta) \tag{9}$$

follows from Theorem 3 of [2]. By virtue of the obtained results the function $u(t,x)$ can be considered as a solution of the linear equation $u_t = \overline{a}(t,x)u_{xx} + \overline{b}(t,x)$ with bounded and Hölder continuous coefficients. To get an estimate up to the boundary, as in the first statement of Theorem 1, we shall continue $u(t,x)$ by rule (7), (8). As it is known (see the proof of Theorem 4 in [2]), for the solution of the "intrinsic" equation intrinsic a priori estimates of form (11) hold in the rectangles containing Q. Hence the results of [2] (Theorem 3 in [2]) on Hölder continuity of the generalized solution can be used. Thus we obtain estimate (6). $\square$

Lemma. *For the solution $u(t,x)$ of problem (1)–(3) the estimate*

$$|u(t,x)| \le \frac{b_1 \exp\{\beta T\}}{\beta - b_0} = M, \tag{10}$$

is true. Here β satisfies the condition $\beta - b_0 > 0$.

The proof is conducted according to the following scheme. The function $v = e^{-\beta t}u(t,x)$ is introduced and the equation for $v(t,x)$ is considered at the intrinsic points of positive maximum and negative minimum of the solution. With regard for A and B and using the boundary point of extremum (see [5], Lemma 5) and also the absolute value estimate of the solution of the linear equation ([5], Lemma 6), we get (10).

2. A priori Estimates of Higher Derivatives.

Theorem 2. *In addition to all the assumptions of Theorem 1, let the inequality $|b_x| \le B_1$ hold and the function $a(t,x,u,p)$ have continuous bounded derivatives $|a_x|, |a_u|, |a_p| \le K_1$; let the function $u(t,x)$, satisfying conditions (2), (3) and equation (1) in Q, be continuous in Q together with its derivatives u_t, u_x, u_{xx}, and on the compact subsets of Q together with u_{tx}, u_{xxx}. Further, let $|u|_{2+\gamma}^{Q_0^\delta} < +\infty$. Then $|u|_{2+\gamma}^{Q_0^{2\delta}} \le C(M, a_0, a_1, K, \delta)$. Moreover, if $|u|_{2+\gamma}^{Q} < \infty$, then $|u|_{2+\gamma}^{Q} \le C(M_1, a_0, a_1, K_1, K)$.*

Proof. Differentiating equation (1) with respect to x on the compact subsets of Q, we get, for the function $p = u_x(t,x)$, the equation

$$p_t = a p_{xx} + \bar{b}(t,x,u,p,p_x), \tag{11}$$

where $\bar{b} = (a_p p_x + a_u p + b_p)p_x + b_u p + (a_x + b_x)$, and the initial condition gives

$$p(0,x) = 0. \tag{12}$$

The second condition from (3) implies

$$p(t,-l) = p(t,l). \tag{13}$$

Further, letting $x = \pm l$ in (1) and taking (4) into account, from the first condition of (3) we obtain

$$p_x(t,-l) = p_x(t,l). \tag{14}$$

Now, by applying Theorem 1 (formula (7); the functions p, a and $\bar{b}$ are assumed to satisfy the conditions of the theorem) to problem (11)–(14) we have $|u_x(t,x)|_{1+\gamma}^{Q} \le C$. The estimate for u_t follows from (1). □

3. The Solution Existence and Uniqueness. Using the established a priori estimates, we shall prove a theorem on the unique solvability of problem (1)–(3).

Theorem 3. *Let the conditions of Theorem* 2 *and the compatibility condition* $b(0,-l,0,0) = b(0,l,0,0)$ *hold. Then for some* $\alpha \in (0,1)$ *there exists a unique solution* $u \in C^{2+\alpha}(Q)$ *of the problem* (1)–(3).

Proof. We shall use Shauder's fixed point principle. Let $\overline{C}^{1+\gamma}(Q)$ be the set of all functions from $C^{1+\gamma}(Q)$ satisfying conditions (2), (3). □

Consider the problem

$$v_t = a(t,x,u,u_x)v_{xx} + b(t,x,u,u_x), \tag{15}$$

$$v(0,x) = 0, \tag{16}$$

$$v(t,-l) = v(t,l), \quad v_x(t,-l) = v_x(t,l), \tag{17}$$

where $u(t,x) \in \overline{C}^{1+\gamma}(Q)$.

Equation (15) can be treated as a linear equation with Hölder continuous coefficients. Here we cannot refer to the well-known results, since we consider the problem with boundary conditions in nonlocal terms.

First we shall prove the following theorem for linear parabolic equations.

Theorem 4. *Let the function* $u(t,x)$ *be a solution of the parabolic equation*

$$Lu \equiv a(t,x)u_{xx} + b(t,x)u_x + e(t,x)u - u_t = f(t,x), \quad (t,x) \in Q, \tag{18}$$

satisfying conditions (2), (3). *If the coefficients* a, b, e *and* $f(t,x)$ *satisfy the Hölder condition, then there exists a solution of problem* (18), (2), (3) *such that*

$$|u|_{2+\alpha}^{Q} \le C|f|_{\alpha}^{Q}. \tag{19}$$

Proof. By virtue of Ciliberto's results [6] we have $|u|_{2+\alpha}^{Q_0^\delta} \le C_0|f|_{\alpha}^{Q}$. Therefore it will be enough to establish estimates (19) under $t > 0$. Let us reduce equation (18) to the homogeneous one by means of the solution

$$u_0(t,x) = \int_0^t d\eta \int_{-l}^{l} \Gamma(t,x,\eta,\xi) f(\eta,\xi) d\xi,$$

where $\Gamma(t,x;\eta,\xi)$ is the fundamental solution of (18) where $f\equiv 0$. Substituting $v=u+u_0$, and putting $Lu_0=-f$, we get the problem

$$Lv=0, \tag{20}$$
$$v(t,-l)=v(t,l)+r_1(t), \tag{21}$$
$$v_x(t,-l)=v_x(t,l)+r_2(t), \tag{22}$$
$$v(0,x)=0, \tag{23}$$

where $r_1(t)=u_0(t,-l)-u_0(t,l)$, $r_2(t)=u_{0x}(t,-l)-u_{0x}(t,l)$.

Now we shall improve the smoothness of the functions $r_i(t)$, $i=1,2$. As known from the theory of parabolic potentials [7, 8], if $f(t,x)$ satisfies the Hölder condition, then there exist continuous derivatives u_{0x}, u_{0xx} and u_{0t} in Q and the estimate $|u_0|^Q_{2+\alpha}\le C_0|f|^Q_\alpha$ is true. Here the definitions of functions $r_i(t)$ include the values $u_0(t,l)$ $(u_0(t,-l))$ and $u_x(t,l)(u_{0x}(t,-l))$. As is known, the function $\Gamma(t,x;\eta,\xi)$ is smooth enough if $x\ne\xi$. Therefore $r_i(t)$, $i=1,2$, are smooth enough functions. The solution of problem (20)–(23) is found as the sum of parabolic simple layer potentials

$$v(t,x)=\int_o^t \Gamma(t,x;\eta,-l)\psi_1(\eta)d\eta+\int_0^t \Gamma(t,x;\eta,l)\psi_2(\eta)d\eta,$$

where $\psi_i(t)$, $i=1,2$, are the unknown densities. It is known that for the simple layer potential $V(t,x)=\int_0^t \Gamma(t,x;l,\eta)\psi(\eta)d\eta$ we have the estimate $|V|^Q_{2+\alpha}\le C|\psi|_{1+\alpha}$. Therefore by showing that $\psi_i(t)\in C^{1+\alpha}[0,T]$ we shall prove the theorem.

By the boundary conditions (21), (22) we find

$$\begin{aligned}\int_0^t \Gamma(t,-l;\eta,-l)\psi_1(\eta)d\eta+\int_0^t \Gamma(t,-l;\eta,l)\psi_2(\eta)d\eta-\\ -\int_0^t \Gamma(t,-l;\eta,l)\psi_1(\eta)d\eta-\int_0^t \Gamma(t,l;\eta,l)\psi_2(\eta)d\eta=r_1(t),\end{aligned} \tag{24}$$

$$\begin{aligned}-\frac{\sqrt{\pi}}{\sqrt{a(t,-l)}}\psi_1(t)+\int_0^t \Gamma_x(t,-l;\eta,-l)\psi_1(\eta)d\eta+\\ +\int_0^t \Gamma_x(t,-l;\eta,l)\psi_2(\eta)d\eta-\frac{\sqrt{\pi}}{\sqrt{a(t,l)}}\psi_2(t)-\\ -\int_0^t \Gamma_x(t,l;\eta,-l)\psi_1(\eta)d\eta-\int_0^t \Gamma_x(t,l;\eta,l)\psi_2(\eta)d\eta=r_2(t).\end{aligned} \tag{25}$$

Next we reduce equation (24) to the second kind integral equation

$$\frac{d}{dt}\int_0^t \frac{r_1(\eta)}{\sqrt{t-\eta}}d\eta=\pi\psi_1(t)+\int_0^t \frac{\partial}{\partial t}\left[\int_\eta^t \frac{\Gamma(z,-l;\eta,-l)}{\sqrt{t-z}}dz\right]\psi_1(\eta)d\eta-\pi\psi_2(t)-$$

$$-\int_0^t \frac{\partial}{\partial t}\left[\int_\eta^t \frac{\Gamma(z,l;\eta,l)}{\sqrt{t-z}}dz\right]\psi_2(\eta)d\eta+\int_0^t \frac{\partial}{\partial t}\left[\int_\eta^t \frac{\Gamma(z,-l;\eta,l)}{\sqrt{t-z}}dz\right]\psi_2(\eta)d\eta-$$
$$-\int_0^t \frac{\partial}{\partial t}\left[\int_\eta^t \frac{\Gamma(z,l;\eta,-l)}{\sqrt{t-z}}dz\right]\psi_1(\eta)d\eta. \tag{26}$$

If $\sqrt{a(t,-l)}+\sqrt{a(t,l)} \neq 0$, then (25), (26) is the system of Volterra integral equations of second kind. Clearly, $\psi_i(0)=0$, $i=1,2$. By Lemmas 11 and 12 from [9] the kernels of this system have a weak singularity. Therefore the system has a solution in the class of functions to which the known function belongs.

Using the properties of parabolic potentials and applying integral inequalities one can obtain estimates for the solution of the system in the form $|\psi_1|_{1+\alpha} \leq C_1|f|_\alpha$, $\quad |\psi_2|_{1+\alpha} \leq C_2|f|_\alpha$. On combining the obtained estimates, we find $|u|^Q_{2+\alpha} \leq C|f|^Q_\alpha$. $\quad\square$

Thus we can state that problem (15)–(17) has the solution $v(t,x) \in C^{2+\alpha}(Q)$. On the other hand, for the solution we have the estimate

$$|v|_{1+\gamma} \leq N(M,a_0,a_1,K). \tag{27}$$

Let $D(N)$ be the set of functions from $\overline{C}^{1+\gamma}(Q)$ which satisfy estimate (27).

If the functions $u(t,x)$ from (15) belong to $D(N)$, then the considered problem is the mapping $v=Tu$ of the sphere $D(N)$ into itself. Now let us verify that the conditions of Shauder's principle are fulfilled.

We shall prove that the operator T is continuous.

Let v_1 and v_2 be the solutions of equation (15) corresponding to u_1 and u_2, respectively. The function $v=v_1-v_2$ satisfies the equation

$$v_t = a(t,x,u_1,u_{1x})v_{xx}+F(t,x),$$

and conditions (16), (17), where

$$F(t,x)=\Big[(u_1-u_2)\int_0^1 a_u(t,x,\tau u_1+(1-\tau)u_2,u_{1x})d\tau+$$
$$+(u_{1x}-u_{2x})\int_0^1 a_p(t,x,u_2,\tau u_{1x}+(1-\tau)u_{2x})d\tau\Big]v_{2xx}+$$
$$+(u_1-u_2)\int_0^1 b_u(t,x,\tau u_1+(1-\tau)u_2,u_{1x})d\tau+$$
$$+(u_{1x}-u_{2x})\int_0^1 b_p(t,x,u_2,\tau u_{1x}+(1-\tau)u_{2x})d\tau.$$

It is clear that $|F|^Q_\alpha \leq N_1|u_1-u_2|^Q_{1+\gamma}$. By the results for linear equations we have $|v|^Q_{2+\alpha} \leq N_2|F|^Q_\alpha$. Then $|v|^Q_{1+\gamma} \leq N_3|v|^Q_{2+\alpha} \leq N_4|F|^Q_\alpha \leq N_5|u_1-u_2|^Q_{1+\gamma}$. Let us now prove that the operator T is completely continuous. The set of solutions $v(t,x)$ is bounded in the space $C^{2+\alpha}(Q)$ and, if $\gamma \leq \alpha$, is compact in the space $C^{1+\gamma}(Q)$. Thus the mapping $v = Tu$ transfers the bounded set of $u \in D(N)$ to the compact set of $v \in D(N)$. Therefore the conditions of Shauder's principle are fulfilled. Thus there exists a solution $u(t,x)$ of problem (1)–(3).

The uniqueness of the problem follows from the extremum principle.

Remark. Since the solution uniqueness has been proved by assuming that the coefficients are differentiable, the same assumption has been made when obtaining a priori estimates for higher derivatives. Actually, it could be possible first to prove Theorem 4 and then, using it and Theorem 1, to obtain immediately the required estimate. In that case the esimates of higher derivatives could be obtained by imposing the Hölder condition on the coefficients.

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