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ON THE DIRICHLET PROBLEM FOR A SECOND ORDER ELLIPTIC SYSTEM WITH DEGENERATION ON THE ENTIRE DOMAIN BOUNDARY

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ABSTRACT. The solvability of the first boundary value problem is investigated for a second order elliptic system with degeneration on the entire domain boundary.

Consider a system of the form

$$L(u) \equiv Au_{xx} + 2Bu_{xy} + Cu_{yy} + au_x + bu_y + cu = 0 \tag{1}$$

in the bounded simply connected domain D of the plane of variables x, y, where $A, B, C, a, b \in H^\alpha(\overline{D}) \cap C^3(D)$ are given scalar functions and c is a given negatively definite $n \times n$-matrix from the class $H^\alpha(\overline{D}) \cap C^3(D)$, $n > 1$, i.e.,

$$(\xi, c(x,y)\xi) \leq c_0(\xi, \xi), \quad c_0 = const < 0 \quad \forall \xi \in R^n, \quad \forall (x,y) \in \overline{D}, \tag{2}$$

$u = (u_1, u_2, \dots, u_n)$ is the desired n-dimensional vector function, and $(\cdot, \cdot)$ is a scalar product.

It will be assumed below that system (1) is elliptic in the domain D and degenerates on the boundary $\Gamma = \partial D$, i.e.,

$$(B^2 - AC)|_D < 0, \tag{3}$$

$$(B^2 - AC)|_{\partial D} = 0. \tag{4}$$

Obviously, since system (1) is elliptic in D, it can be assumed without loss of generality that $A|_D > 0$.

Let us write the equation ∂D in the form $H(x, y) = 0$, where $H|_D > 0$, $H \in C^2(\overline{D})$, $H|_\Gamma = 0$, $\nabla H|_\Gamma \neq 0$.

A vector u of the class $C^2(D)$ satisfying system (1) in the domain D is called a regular solution of this system.

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Let us pose the *Dirichlet problem* for the system (1). Find, in the domain D, a regular solution of equation (1) which is continuous in the closed domain $\overline{D}$ and satisfies the boundary condition

$$u|_\Gamma = f, \tag{5}$$

where $f = (f_1, f_2, \dots, f_n)$ is a given continuous vector function on Γ.

Note that the Dirichlet problem was investigated in [1], [5] for some specific classes of second order elliptic systems.

When investigating the Dirichlet problem, the following two cases are to be distinguished:

$$(AH_x^2 + 2BH_xH_y + CH_y^2)|_\Gamma \neq 0, \tag{6}$$

$$(AH_x^2 + 2BH_xH_y + CH_y^2)|_\Gamma = 0. \tag{7}$$

Remark 1. Equality (4) together with condition (6) imply that equation (1) parabolically degenerates on Γ and, at every point of the boundary, the tangential direction does not coincide with the characteristic direction. Conditions (4) and (7) are equaivalent to the fact that at the points of the boundary either the equation order degenerates or parabolic degeneration takes place and the characteristic direction coincides with the tangential direction.

When condition (3) is fulfilled, the following extremum principle [1] takes place: the norm $R(x,y) = \sqrt{(u,u)}$ of a regular solution u of system (1) in the domain D cannot attain a nonzero relative maximum at any point $(x,y) \in D$.

The uniqueness of the solution of the Dirichlet problem for system (1) follows from the above-mentioned extremum principle.

To construct the solution of the Dirichlet problem we shall use the Wiener method. Using arbitrary vector functions f of the class $C(\overline{D}) \cap C^3(D)$ that are continuously extendable onto D, let us construct a sequence of domains D_h increasing as $h \to 0$ and having a smooth boundary of the class C^2 such that $\overline{D}_{h_1} \subset D_{h_2}$ for $h_1 > h_2$

Let $u_h(x,y)$ be the solution of the Dirichlet problem for system (1) in D_h that coincides with f on the boundary of D_h. As is known, the solution $u_h(x,y)$ exists and is unique, since in $\overline{D}_h$ system (1) does not degenerate [6]. By virtue of (2) in D_h we have the inequality $\|u_h\| \leq M$, where $M = \max\|f(x,y)\|$ in $\overline{D}$. Let us show that the set of functions $\{u_h(x,y)\}$ is compact within D. Indeed, let h_0 be an arbitrary fixed value of h. For $h \leq h_0$ the set $\{u_h(x,y)\}$ will be uniformy bounded in D_{h_0},

$$\|u_h(x,y)\| \leq M. \tag{8}$$

On the other hand, we have the representation [6]

$$u_h(x,y) = \int_{\partial D_{h_0}} K_{h_0}(x,y;s)u_h(s)ds, \quad (x,y) \in D_{h_0}, \tag{9}$$

where the kernel is itself a regular solution of system (1) with respect to the variables x, y.

This immediately implies the equicontinuity of the family $\{u_h(x,y)\}$. From (8) and (9) we obtain the equicontinuity of the set of function $\{u_h\}$ in D_{h_0}. According to Arcela's theorem [1] the latter set may contain a subsequence uniformly converging to some function $u(x,y)$ which by virtue of (9) is the solution of system (1) in the domain D. It remains to find out whether the constructed solution coincides with f on Γ.

We shall show that in the case of (6) for each point $Q \in \Gamma = \partial D$ there exists a function $v(x,y)$ called a barrier and possessing the following properties: (a) being continuous in some neighborhood of this point $\omega_Q = \{P \in \overline{D} : \ |P - Q| < \epsilon\}$; (b) being equal to zero at the point Q; (c) $v(x,y) > 0$ in $\omega_Q \backslash Q$; (d) everywhere in this neighborhood satisfying the condition

$$L_1^0(u) \equiv Au_{xx} + 2Bu_{xy} + Cu_{yy} + au_x + bu_y < 0.$$

In the considered case of (6) the function

$$v(x,y) = (x - x_0)^2 + (y - y_0)^2 + H^\beta(x,y), \quad 0 < \beta < 1,$$

can serve a barrier. Indeed, the function $v(x,y)$ evidently satisfies the conditions (a), (b) and (c). Let us verify the condition (d). Substituting the expression for $v(x,y)$ into $L_1^0(v)$, we obtain

$$\begin{aligned} L_1^0(v) = &\beta(\beta - 1)H^{\beta-2}(AH_x^2 + 2BH_xH_y + CH_y^2) + \\ &+ \beta H^{\beta-1}(AH_{xx} + 2BH_{xy} + CH_{yy} + aH_x + bH_y) + \\ &+ 2A + 2C + 2a(x - x_0) + 2b(y - y_0). \end{aligned} \tag{10}$$

Hence we immediately conclude that by virtue of $0 < \beta < 1$ there exists a neigborhood σ_Q of the point Q such that $L_1^0(v) < 0$.

Since the function $f(P)$ is continuous, for given positive ϵ we can find a semicircular neighborhood $\sigma'_Q \subset \sigma_Q$ of the point Q such that the inequality

$$\|f(P) - f(Q)\| < \epsilon, \ P \in \sigma'_Q, \ 0 < \epsilon < 1, \tag{11}$$

be fulfilled.

Consider two functions $v_1(P) = \epsilon + Kv(P)$, $K > 0$, $u_h^*(P) = u_h(P) - f(Q)$, where $P \in \sigma'_Q$. In the domain $\omega_h = \sigma'_Q \cap D_h$, where h is a sufficiently small positive number, we have

$$\sup_{(x,y)\in\omega_h} L_1^0(v) \le \alpha_0 = const < 0, \tag{12}$$

$$L_1^0(u_h^*) = L_1^0(u_h), \quad \|u_h^*\|_{C(\omega_h)} \le 2M, \; M = \max_{\overline{D}} \|f\|. \tag{13}$$

We set $g(P) = (u_h^*(P), u_h^*(P))$, $P \in \omega_h$. By virtue of (3), (10), (12), (13) and sufficiently large K we obtain

$$\begin{aligned} L_1^0(v_1 - g) &= KL_1^0(v) - 2[A(u_{hx}^*, u_{hx}^*) + 2B(u_{hx}^*, u_{hy}^*) + C(u_{hy}^*, u_{hy}^*)] - \\ &\quad -2(u_h^*, Au_{hxx}^* + 2Bu_{hxy}^* + Cu_{hyy}^* + au_{hx}^* + bu_{hy}^*) \le \\ &\le KL_1^0(v) - 2(u_h(P) - f(Q), L_1^0(u_h)) = KL_1^0(v) - 2(u_h(P) - f(Q), -cu_h) \le \\ &\le K\alpha_0 - 2(f(Q), cu_h) = KL_1^0(v) + 2(u_h, cu_h) - 2(f(Q), cu_h) \le \\ &\le K\alpha_0 + 2\|f(Q)\|\,\|c\|\,\|u_h\| \le K\alpha_0 + 2M^2\|c\| < 0. \end{aligned} \tag{14}$$

Let us now find out which sign of $v_1 - g$ takes place on the domain boundary ω_h; $\partial\omega_h = \gamma_h \cup \gamma_{1h}$, where $\gamma_{1h} = \partial\omega_h \cap \partial D_h$, $\gamma_h = \partial\omega_h \backslash \gamma_{1h}$. By virtue of (11) we have

$$\begin{aligned} g|_{\gamma_{1h}} &= (u_h(P) - f(Q), u_h(P) - f(Q))|_{\gamma_{1h}} = \|(f(P) - f(Q)\|^2 < \epsilon, \\ v_1|_{\gamma_{1h}} &= \epsilon + Kv(P)|_{\gamma_{1h}} > \epsilon + K \min_{\gamma_{1h}} v(P), \quad g|_{\gamma_h} \le 4M^2. \end{aligned} \tag{15}$$

By (15), for sufficiently large K we have

$$\begin{aligned} (v_1 - g)|_{\gamma_{1h}} &= (\epsilon + Kv - g)|_{\gamma_{1h}} > \epsilon + K \min_{(x,y)\in\gamma_{1h}} v - \epsilon = K \min_{(x,y)\in\gamma_{1h}} v > 0, \\ (v_1 - g)|_{\gamma_h} &= (\epsilon + Kv - g)|_{\gamma_h} \ge \epsilon + K \min_{(x,y)\in\gamma_h} v - 4M^2 > 0. \end{aligned}$$

By (14) and (15) and applying the extremum principle we have [1] $(v_1 - g)|_{\omega_h} \ge 0$, i.e., $g \le v_1$ throughout the domain ω_h, or

$$(u_h(P) - f(Q), u_h(P) - f(Q)) \le \epsilon + Kv(P). \tag{16}$$

Due to the properties of the barrier v there exists $\delta = \delta(\epsilon) > 0$ such that for $\|P - Q\| < \delta$ and $P \in \sigma'_Q$ we have

$$Kv(P) < \epsilon. \tag{17}$$

From (16) and (17) we find that $\|u(P) - f(Q)\| \le \sqrt{2\epsilon}$ for $\|P - Q\| < \delta$, which was to be proved.

Thus the following assertion is true.

Theorem 1. *The Dirichlet problem* (1), (5) *always has a unique solution.*

When considering the case of (7) it will be assumed that in some neighborhood of the boundary Γ there is a representation

$$AH_x^2 + 2BH_xH_y + CH_y^2 = H^pG, \tag{18}$$

where $p = const > 0$, G is a positive, continuous and bounded function in this neigborhood.

Lemma 1. *If there exists a function $W(P)$, which is positive in $\overline{D}$, uniformly converging to infinity as $\rho(P, \partial D) \to 0$ and satisfying the inequality $L_1^0(W) < 0$, then system* (1) *has only a trivial solution in the class of bounded vector functions in D.*

Proof. Indeed, let $u(x,y)$ be a bounded regular solution of system (1) in D. Consider the expression $L_1^0(\epsilon W - (u,u)) = \epsilon L_1^0(W) - L_1^0((u,u))$. By calculating $L_1^0((u,u))$ we obtain

$$\begin{aligned} L_1^0((u,u)) = 2[A(u_x,u_x) + 2B(u_x,u_y) + C(u_y,u_y)] + \\ +2(u, Au_{xx} + 2Bu_{xy} + Cu_{yy} + au_x + bu_y) \geq 0, \end{aligned} \tag{19}$$

since by virtue of (3), $A|_D > 0$, and the Cauchy–Bunyakovskii inequality we have

$$A(u_x,u_x) + 2B(u_x,u_y) + C(u_y,u_y) \geq A\|u_x\|^2 - 2|B||(u_x,u_y)| + C\|u_y\|^2 \geq$$

$$\geq A\|u_x\|^2 - 2\sqrt{A}\sqrt{C}\|u_x\| \cdot \|u_y\| + C\|u_y\|^2 = \left(\sqrt{A}\|u_x\| - \sqrt{C}\|u_y\|\right)^2 \geq 0.$$

The conditions of Lemma 1 and (19) lead to $L_1^0(\epsilon W - (u,u)) < 0$, which by virtue of the extremum principle gives that the function $\epsilon W - (u,u)$ cannot have a negative minimum in D, and since its limit value on the boundary are positive, we have $(u,u) \leq \epsilon W$ throughout D. Hence, since $\epsilon > 0$ is arbitrary, it folows that $\|u\| = 0$ in D. □

Taking into account (18), expression (10) can be rewritten as

$$\begin{aligned} L_1^0(v) = \beta(\beta-1)H^{\beta+p+2}G + \beta H^{\beta-1}L_1^0(H) + 2A + 2C + \\ +2a(x-x_0) + 2b(y-y_0). \end{aligned}$$

Theorem 2. *If* (2) *holds and one of the conditions* (i) $0 < p < 1$*;* (ii) $p = 1$, $(1 - IG^{-1})|_\Gamma > 0$, $I = L_1^0(H)$*;* (iii) $1 < p < 2$, $I|_\Gamma \leq 0$*;* (iv) $p \geq 2$, $I|_\Gamma < 0$ *is fulfilled, then the Dirichlet problem has a solution.*

Proof. The existence of a solution of the Dirichlet problem follows from the existence of a barrier function. For $0 < p < 1$ the sign of $L_1^0(v)$ coincides with the sign of $\beta(\beta-1)H^{p+\beta-2}$, i.e., $L_1^0(v) < 0$. If the condition (ii) is fulfilled, then the sign of $L_1^0(v)$ coincides with the sign of $\beta[(\beta-1)G + I]H^{\beta-1}$ and, assuming that $\beta < (1 - IG^{-1})|_\Gamma$, we shall have $L_1^0(v) < 0$. Under the condition (iv) the sign of $L_1^0(v)$ coincides with the sign of $\beta H^{\beta-1}I$, i.e., $L_1^0(v) < 0$. $\square$

Lemma 2. *Let any one of the conditions* (i) $p = 1$, $IG^{-1}|_\Gamma \geq 1$; (ii) $1 < p < 2$, $I|_\Gamma > 0$; (iii) $p \geq 2$, $I|_\Gamma \geq 0$ *be fulfilled and, at each point* $P_0 \in D$, *either the function* $\Phi(P_0) = (Ag_x^2 + 2Bg_xg_y + Cg_y^2)(P_0) \neq 0$, *where* $g \in C^2(\overline{D})$, $g > 1$, *or* $\Phi(P_0) = 0$ *and* $c(P_0) < 0$. *Then there exists a function* $W(x,y)$ *possessing the following properties:* (a) $W(P) > 0$, $P \in \overline{D}$, (b) $\lim\limits_{\rho(P,\partial D)\to 0} W(P) = +\infty$; (c) $L_1^0(W) < 0$, $P \in D$ [7].

We can take as W, for instance, an expression $W(x,y) = -\log H(x,y) - g^n(x,y) + K_0$ for some natural n and positive constant K_0.

Lemmas 1 and 2 give rise to the following assertion.

Theorem 3. *When the conditions of Lemma* 2 *are fulfilled, system* (1) *has only a trivial solution in the class of bounded vector functions in* D.

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