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ON THE APPROXIMATION OF FUNCTIONS ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. Questions of approximative nature are considered for a space of functions $L^p(G,\mu)$, $1 \le p \le \infty$, defined on a locally compact abelian Hausdorff group G with Haar measure μ. The approximating subspaces which are analogs of the space of exponential type entire functions are introduced.

Let us consider a locally compact abelian group G with Haar measure μ, assuming that the topology of the space G is Hausdorff. By $L^p(G,\mu)$ or, simply, by $L^p(G)$ we denote a space of real- or complex-valued functions defined on G and integrable on it with respect to the measure μ to the p-th power with the usual norm $\|f\|_p = \{\int_G |f|^p d\mu\}^{1/p}$ for $1 \le p < \infty$. $L^\infty(G,\mu)$ is a space of functions, essentially bounded on G with respect to μ and having norm $\|f\|_\infty = \operatorname*{vrai\,sup}_{g\in G} |f(g)|$. We shall briefly recall some definitions and facts from the theory of commutative harmonic analysis. A unitary irreducible representation of G, i.e., a complex-valued continuous function χ on G with the properties $|\chi(g)| = 1$, $\forall g \in G$ and $\chi(g_1g_2) = \chi(g_1)\cdot\chi(g_2)$, $\forall g_1, g_2 \in G$, is called a character of the group G. An Abelian group structure is naturally introduced into the set of all characters defined on G. The obtained group is denoted by $\widehat{G}$ and called a group dual to G. $\widehat{G}$ is usually topologized by the following two topologies: the first one is the weakest topology containing continuous functionals $\widehat{f}$ defined by the formula

$$\widehat{f}(\chi) = \int\limits_G f(g)\overline{\chi(g)}\,d\mu(g) \equiv \int\limits_G f(g)\overline{\chi(g)}\,dg, \quad \chi \in \widehat{G}, \quad f \in L^1(G,\mu), \tag{1}$$

and the second one is the topology of uniform convergence of characters on compact subsets of the group G. These topologies are equivalent and with their aid $\widehat{G}$ transforms to a locally compact Abelian group. A function $\widehat{f}$ of

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form (1) defined on $\widehat{G}$ for $f \in L^1(G)$ is called the Fourier transform of the function f. Similarly to (1), we define the inverse transform $\check{f}$ of f by the formula

$$\check{f}(\chi) = \int\limits_G f(g)\chi(g)\,dg.$$

Of much importance here is the Pontryagin duality principle by which the natural mapping of G into $\widehat{\widehat{G}}$, which to an element $g \in G$ assigns the character f_g on $\widehat{G}$, is an isomorphism of topological groups. The notions of direct and inverse Fourier transform can be extended by the well known technique to the case of spaces $L^p(G,\mu)$ for $1 < p \le 2$. By Plancherel's theorem the Fourier transform is a linear isometry of $L^2(G)$ on $L^2(\widehat{G})$ but the inverse Fourier transform is a linear isometry from $L^2(\widehat{G})$ on $L^2(G)$. These mappings are mutually inverse ([1], v. 2, §31). Usually, Haar measures on G and $\widehat{G}$ are normalized so that the inversion formula $f = (\widehat{f})\check{}$ holds for functions $f \in L^1(G)$, $\widehat{f} \in L^1(\widehat{G})$. These measures are called mutually dual and pairs of such measures will be considered below. For $f \in L^p(G,\mu)$, $1 \le p \le 2$, we have the Hausdorff–Young inequality by which

$$\|\widehat{f}\|_q \le \|f\|_p, \tag{2}$$

where q (here and in what follows) is the conjugate number of p ($\frac{1}{p} + \frac{1}{q} = 1$).

A great part of the approximation theory deals with questions of approximation of functions on an n-dimensional Euclidean space $\mathbb{R}^n$ and on an n-dimensional torus $\mathbb{T}^n$ which are Abelian groups. As an approximating subspace, a space of exponential type entire functions and a space of trigonometric polynomials [2] are usually taken as an approximating subspace in the first and the second case, respectively. We know that $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ and $\widehat{\mathbb{T}} = Z$ to within an isomorphism, where Z is a set of integer numbers. In both cases the Fourier transforms of elements from the approximating space have compact supports in dual spaces ([2], Ch. 3, §§3.1, 3.2). When investigating the problem of approximation of functions defined on compact or locally compact Abelian groups one should consider sets with properties similar to the properties of approximating sets of some well known classical groups. For example, by analogy with groups $\mathbb{R}^n$ or $\mathbb{T}^n$ one can try to consider as an approximating set the set of functions on G whose Fourier transform lies in some compact K of the dual space $\widehat{G}$. But such an approach cannot simultaneously be used for all spaces $L^p(G,\mu)$, $1 \le p \le \infty$, since, generally speaking, for $p > 2$ the Fourier transform of a function $f \in L^p$ does not exist. Therefore below we shall give a modified definition of approximating subspaces which is simultaneously applicable for all p, $1 \le p \le \infty$.

By U_G we shall denote a space of all symmetric compact sets from G which are the closures of neighborhoods of unity in G. Sets from U_G will be called compact neighborhoods of unity. $KT = \{g : g = g_1 g_2,\ g_1 \in K,\ g_2 \in T\}$ will stand for the product of the sets K and T, while $(1)_K$ will denote the characteristic function of the set K. For arbitrary K and T from $U_{\widehat{G}}$ we shall consider the function defined on G ([3], Ch. 5, §1)

$$V_{K,T}(g) = (\operatorname{mes} T)^{-1} \widehat{(1)}_T(g) \cdot \widehat{(1)}_{TK}(g), \tag{3}$$

which, for simplicity, will be denoted by V.

It is at once obvious that $V \in L^1(G)$. Indeed,

$$\|V_{K,T}\| \le \frac{1}{\operatorname{mes} T} \|\widehat{(1)}_T\|_2 \cdot \|\widehat{(1)}_{TK}\|_2 = \\ = \frac{1}{\operatorname{mes} T} \|(1)_T\|_2 \cdot \|(1)_{TK}\|_2 = \Big(\frac{\operatorname{mes} TK}{\operatorname{mes} T}\Big)^{1/2}. \tag{4}$$

Also note that V is a real-valued function. This follows from the fact that

$$\widehat{(1)}_K(g) = \int\limits_K \overline{\chi(g)}\, d\chi = \int\limits_K \chi(g^{-1})\, d\chi = \int\limits_K \chi(g)\, d\chi = \overline{\widehat{(1)}_K(g)}.$$

By Parceval's theorem ([1], v. 2, §31) we obtain

$$\int\limits_G V_{K,T}(g)\, dg = \frac{1}{\operatorname{mes} T} \int\limits_G \widehat{(1)}_T(g) \cdot \widehat{(1)}_{TK}(g)\, d\mu = \\ = \frac{1}{\operatorname{mes} T} \int\limits_{\widehat{G}} (1)_T(\chi) \cdot (1)_{TK}(\chi)\, d\chi = \frac{1}{\operatorname{mes} T} \int\limits_T d\chi = 1.$$

Using the function $f \in L^p(G)$ and kernel V, we introduce the function

$$P_{K,T}(f, g) = (f * V_{K,T})(g) = \int\limits_G f(h) \cdot V_{K,T}(h^{-1} g)\, dh. \tag{5}$$

Definition 1. Let $K \in U_{\widehat{G}}$ and $p \in [1, \infty]$. By $W^p(K)$ we shall denote the set of functions f from the space $L^p(G, \mu)$ for which we have the formula

$$f(g) = P_{K,T}(f, g) \quad \forall g \in G \quad \text{and} \quad T \in U_{\widehat{G}}. \tag{6}$$

Remark. If $G = \mathbb{R}^n$ and $K = \Delta_\nu = \{x(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \le \nu_i,\ \nu = (\nu_1, \dots, \nu_n)\}$, then the class defined by us coincides with the known class $W_{\nu p}(\mathbb{R}^n)$ of entire functions of exponential type ν. A function $f \in L^p(\mathbb{R}^n)$ belongs to $W_{\nu p}(\mathbb{R}^n)$ ([2], Ch. 3, §3.1) if it is analytically extendable onto the whole n-dimensional complex Euclidean space $\mathbb{C}^n$, and for any $\varepsilon > 0$ there exists a number A_ε such that the inequality

$$|g(z)| \leq A_\varepsilon \cdot \exp \sum_{j=1}^{n} (\nu_j + \varepsilon) \cdot |z_j|$$

is fulfilled for all $z(z_1, \dots, z_n) \in \mathbb{C}^n$.

But for $f \in W_{\nu p}(\mathbb{R}^n)$ we have representation (6) ([2], Ch. 8, §8.6) with $\widehat{(1)}_K = 2^n \prod_{i=1}^{n} (x_i)^{-1} \sin \nu_i x_i$. Conversely, let (6) hold for $f \in L^p(\mathbb{R}^n)$. We take an n-dimensional cube $T_\varepsilon = \{x \in \mathbb{R}^n : |x_i| \leq \varepsilon,\, i = 1, \dots, n\}$, $\varepsilon > 0$, as a neighborhood of T. The kernel V_{K,T_ε} will be a exponential type entire function for any vector ε with constant coordinates ε. The convolution $f * V_{K,T_\varepsilon}$ belongs to the same class too ([2], Ch. 3, §3.6), which by virtue of (6) means that f belongs to $W_{\nu p}(\mathbb{R}^n)$. Thus $W^p(K)$ coincides with $W_{\nu p}(\mathbb{R}^n)$ for $G = \mathbb{R}^n$ and $K = \Delta_\nu$. Such an equivalence can be proved by a similar reasoning for more general $K \subset \mathbb{R}^n$ too if one uses the results of [4].

Lemma 1. *$W^p(K)$ is the shift-invariant closed subspace of the space $L^p(G)$.*

Proof. Let $f_n \in W^p(K)$ be an arbitrary converging sequence and $\|f_n - f\|_p \to 0$ as $n \to \infty$. Using the well known estimate of a convolution norm ([1], v. 2, §31) for L^p, we obtain

$$\|P_{K,T}(f_n - f, g)\|_p \leq \|f_n - f\|_p \cdot \|V_{K,T}\|_1, \quad \forall T \in U_{\widehat{G}}.$$

But $P_{K,T}(f_n) = f_n$, $\forall n \in N$, and thus f_n tends in L^p simultaneously to f and $P_{K,T}(f)$. Therefore $f = P_{K,T}(f)$ and $f \in W^p(K)$. The operation of shift by the element h will be denoted by L_h. Let $f \in W^p(K)$ and $T \in U_{\widehat{G}}$. We obtain

$$(L_h f)(g) = f(hg) = \int_G f(g_1) V_{K,T}(g_1^{-1} gh)\, dg_1 = \int_G f(\xi h) V_{K,T}(\xi^{-1} h^{-1} gh)\, d\xi =$$
$$= \int_G (L_h f)(\xi) V_{K,T}(\xi^{-1} g)\, d\xi = (L_h f * V_{K,T})(g)$$

and Lemma 1 is proved. $\square$

Definition 2. When $p \in [1, 2]$, for an arbitrary fixed compact K from the dual space $\widehat{G}$ we shall denote by $\mathcal{F}^p(K)$ the set of functions from the space $L^p(G, \mu)$ whose Fourier transform supports belong to K. Using Parceval's theorem ([1], v. 2, §31) by which $\int_G f \widehat{\Phi}\, dg = \int_{\widehat{G}} \widehat{f} \Phi\, d\chi$, substituting the function $\Phi(\chi) = \frac{\chi(g)}{\operatorname{mes} T}((1)_T * (1)_{TK})(\chi)$ into it, and calculating $\widehat{\Phi}$ by the rule $(\chi f)\widehat{}(\cdot) = \widehat{f}(\chi^{-1}\cdot)$ ([1], v. 2, §31), for $1 \le p \le 2$ we obtain another representation of $P_{K,T}(f)$, namely:

$$P_{K,T}(f, g) = (\operatorname{mes} T)^{-1} \int\limits_{\widehat{G}} \widehat{f}(\chi) \cdot \chi(g) \cdot ((1)_T * (1)_{TK})(\chi)\, d\chi,$$

where integration actually occurs on the compact T^2K. If G is a compact, then this representation of $P_{K,T}$ certainly holds for all $1 \le p \le \infty$.

Note the following important property of functions from $\mathcal{F}^p(K)$. If $f \in \mathcal{F}^p(K)$, $1 \le p \le 2$, then $f \in L^{p_1}(G)$ for arbitrary $p_1 \in [p, \infty]$. Indeed, by virtue of (2), $\widehat{f} \in L^q(G)$, where q is the conjugate number of p. Since $\operatorname{supp} \widehat{f} = K$ and $q \ge 2$, we have $\widehat{f} \in L^2(\widehat{G})$ and $(\widehat{f})\check{} = f$ almost everywhere on G. Thus $\widehat{f} \in L^1(\widehat{G}) \cap L^2(\widehat{G})$ and the L^2 inverse Fourier transform of $\widehat{f}$ is f. But ([1], v. 2, §31) the L^1 inverse Fourier transform of $\widehat{f}$ is also f. Hence it is clear that f is a continuous and bounded function on G. Since $f \in L^p \cap L^\infty$, we have $f \in L^{p_1}(G)$ for any $p_1 \in [p, \infty]$. Moreover, for $p = \infty$, by (2) we obtain the inequality

$$\|f\|_\infty \le \|\widehat{f}\|_1 = \int\limits_K |\widehat{f}|\, d\chi \le \Big(\int\limits_K d\chi\Big)^{1/p} \|\widehat{f}\|_q \le (\operatorname{mes} K)^{1/p} \cdot \|f\|_p.$$

Lemma 2. *If $1 \le p \le 2$, then the sets $W^p(K)$ and $\mathcal{F}^p(K)$ coincide.*

Proof. Let us first assume that $f \in W^p(K)$. Then for $\forall T \in U_{\widehat{G}}$ we have

$$\widehat{f * V_{K,T}}(\chi) = \widehat{f}(\chi)$$

and therefore

$$\widehat{f}(\chi) \cdot \widehat{V}_{K,T}(\chi) = \widehat{f}(\chi) \tag{7}$$

for almost all $\chi \in \widehat{G}$.

Next, by (3) we obtain

$$0 \le \widehat{V}_{K,T}(\chi) = \frac{1}{\operatorname{mes} T}\, (1)_T * (1)_{TK}(\chi) = \frac{1}{\operatorname{mes} T} \int\limits_T (1)_{KT}(h^{-1}\chi) dh.$$

Let $\chi \notin K$. If $h^{-1}\chi \notin KT$ for any $h \in T$, then $\widehat{V}_{K,T}(\chi) = 0$. If however $\chi \notin K$ and $\chi \in KT$, then in T there exists a set U with $\operatorname{mes} U$ such that $\chi U \notin KT$ so that

$$\widehat{V}_{K,T}(\chi) = \frac{1}{\operatorname{mes} T}\Big(\int\limits_{U^{-1}} + \int\limits_{T\setminus U^{-1}}\Big) = \frac{1}{\operatorname{mes} T} \int\limits_{T\setminus U^{-1}} < 1.$$

Then from (7) it follows that for $\chi \notin K$ we have $\widehat{f}(\chi) = 0$ almost everywhere and therefore $f \in \mathcal{F}^p(K)$.

Let now $f \in \mathcal{F}^p(K)$ and T be an arbitrary set from $U_{\widehat{G}}$. We have

$$\widehat{P}_{K,T}(\chi) = \widehat{f}(\chi) \cdot \widehat{V}_{K,T}(\chi) = \frac{1}{\operatorname{mes} T} \int\limits_T (1)_{KT}(h^{-1}\chi) dh \cdot \widehat{f}(\chi).$$

If $\chi \in K$ and $h \in T$, then $h^{-1}\chi \in KT$ and we obtain $\widehat{f}(\chi)$ on the right-hand side. If however $\chi \notin K$, then we have the equality $\widehat{f}(\chi) = 0$ which by virtue of the fact that $f \in \mathcal{F}^p(K)$ implies $\widehat{P_{K,T}(f)}(\chi) = 0$. Therefore $\widehat{P_{K,T}(f)}(\chi) = \widehat{f}(\chi)$ for almost all $\chi \in \widehat{G}$. Keeping in mind that the functions f and $P_{K,T}(f)$ are continuous on G, by the uniqueness theorem on the Fourier transform of f we obtain representation (6) for any $T \in U_{\widehat{G}}$ and $g \in G$. $\square$

Remark. If $G = \mathbb{R}^n$, by applying the well known Peley–Wiener theorem ([5], Ch. 6, §4) we can prove that the classes $W^p(K)$ and $\mathcal{F}^p(K)$ coincide for $p > 2$ too provided that by $\mathcal{F}^p(K)$ we shall understand the class of functions $f \in L^p(\mathbb{R}^n)$ whose generalized Fourier transform supports (in the sense of generalized functions) belong to $K \subset \mathbb{R}^n$.

For our further discussion it is important to note that the above-stated property that $f \in \mathcal{F}^p(K)$, $1 \le p \le 2$, gives that $f \in L^{p_1}(G)$ for any $p_1 \in [p, \infty]$ holds for functions from $W^p(K)$ for $p \in [1, \infty]$. Indeed, if $f \in W^p(K)$, then $f = f * V_{K,T}$ for any $T \in U_{\widehat{G}}$. But $V_{K,T} \in \mathcal{F}^1(KT^2)$ and, as we already know, this implies that $V_{K,T} \in L^r(G)$ for all $r \in [1, \infty]$. By the Young inequality ([1], v. 1, §20)

$$\|f\|_{p_1} \le \|f\|_p \cdot \|V_{K,T}\|_r, \quad \text{where } \frac{1}{p_1} = \frac{1}{p} + \frac{1}{r} - 1. \tag{8}$$

If $1 \le r \le \frac{p}{p-1}$, then p_1 takes all values from the interval $[p, \infty]$. Let us prove that the functions from $W^p(K)$ are continuous for any $1 \le p \le \infty$. Indeed, if $f \in W^p(K)$, then $f \in W^\infty(K)$

$$|f(g_1) - f(g)| = \left| \int\limits_G f(h) \big[V_{K,T}(h^{-1}g_1) - V_{K,T}(h^{-1}g) \big] dh \right| \le$$
$$\le \|f\|_\infty \cdot \big\| V_{K,T}(h^{-1}g_1) - V_{K,T}(h^{-1}g) \big\|_1,$$

and the continuity of functions follows from the continuity of the shift operator in L^1 ([3], Ch. 3, §5).

Let $C^0(G)$ be the Banach space of continuous functions on a locally compact but not compact group G, which vanish at infinity ([1], v. 1, §11, [3], Ch. 2, §3). $f \in C^0(G)$ if for any $\varepsilon > 0$ there exists a compact G_ε depending on f and such that the inequality $|f(g)| < \varepsilon$ holds everywhere outside G_ε. We shall show that for $1 \le p < \infty$ the functions from $W^p(K)$ belong to $C^0(G)$. Indeed, in view of the fact the Haar measure is regular and taking into account the integral representation (6), for any $\varepsilon > 0$ we can find a compact G_ε such that

$$\left| f(g) - \int\limits_{G_\varepsilon} f(h)V(h^{-1}g)dh \right| < \varepsilon.$$

Next, using (4), we obtain

$$\left| \int\limits_{G_\varepsilon} f(h)V(h^{-1}g)dh \right| \le \|f\|_p \cdot \left\{ \int\limits_{G_\varepsilon} |V(hg)|^q dh \right\}^{\frac{1}{q}} \le$$
$$\le \left(\frac{\operatorname{mes} TK}{\operatorname{mes} T} \right)^{\frac{1}{2q}} \cdot \|f\|_p \cdot \sup_{h \in G_\varepsilon} |V(hg)|^{\frac{q-1}{q}}$$

(for $p = 1$, $q = \infty$ we have $\|f\|_\infty \cdot \sup\limits_{h \in G_\varepsilon} |V(hg)|$ on the right-hand side).

It is the well known fact that the Fourier transform of a function from $L^1(\widehat{G})$ belongs to $C^0(G)$ ([1], v. 2, §31). Hence by (3) it is clear that $V(hg) \in C^0(G)$, $\forall h \in G_\varepsilon$. Since $p \ne \infty$, $q > 1$ and the fact that $V(hg)$ belongs to $C^0(G)$, together with the last two estimates, enable us to conclude that $f \in C^0(G)$. A constant function $f(g) \equiv C \ne 0$ on G may serve as an example showing that for $p = \infty$ the inclusion $W^\infty(K) \subset C^0(G)$ is not valid.

If $G = \mathbb{R}^n$ and $1 \le p < \infty$, then the obtained inclusion $W_{\nu p}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$ is equivalent to the well known fact that $\lim\limits_{|x| \to \infty} f(x) = 0$ if $f \in W_{\nu p}(\mathbb{R}^n)$ which was proved by a different method in [2] (Ch. 3, §3.2).

Lemma 3. *Let $f \in L^p(G)$, $1 \le p \le \infty$, $K \in U_{\widehat{G}}$, $T \in U_{\widehat{G}}$. Then*

$$P_{K,T}(f) \subseteq W^p(KT^2).$$

Proof. From the equality

$$\widehat{V}_{K,T}(\chi) = (\operatorname{mes} T)^{-1} \int\limits_T (1)_{KT}(h^{-1}\chi)dh$$

it follows that $V_{K,T} \in \mathcal{F}^1(KT^2)$ and, by Lemma 2 we have $V_{K,T} \in W^1(KT^2)$. After multiplying both sides of equality (5) by $V_{KT^2,T_1}(g^{-1}t)$, where $T_1 \in$

$U_{\widehat{G}}$, and integrating them over G, we obtain

$$\int\limits_G P_{K,T}(f,g)V_{KT^2,T_1}(g^{-1}t)dg = \int\limits_G\int\limits_G f(h)V_{K,T}(h^{-1}g)dh\cdot V_{KT^2,T_1}(g^{-1}t)dg =$$
$$= \int\limits_G f(h)dg \int\limits_G V_{K,T}(h^{-1}g)\cdot V_{KT^2,T_1}(g^{-1}t)dg.$$

Here changing the integration order is righful by virtue of the Fubini theorem ([1], v. 1, §13) and in view of the fact that

$$\int\limits_G dg \int\limits_G \big|f(h)V_{K,T}(h^{-1}g)\cdot V_{KT^2,T_1}(g^{-1}t)\big|dh \le$$
$$\le \bigg\|\int\limits_G \big|f(h)V_{K,T}(h^{-1}g)\big|dh\bigg\|_p \cdot \|V_{KT^2,T_1}\|_{\frac{p}{p-1}} < \infty.$$

Next, by the change of the variable we obtain

$$\int\limits_G P_{K,T}(f,g)V_{KT^2,T_1}(g^{-1}t)dg = \int\limits_G f(h)dh \int\limits_G V_{K,T}(\xi)\cdot V_{KT^2,T_1}(\xi^{-1}h^{-1}t)d\xi.$$

But $V_{K,T}\in W^1(KT^2)$ and, according to the definition of the class W^1, the internal integral can be replaced by $V_{K,T}(h^{-1}t)$. As a result,

$$\int\limits_G P_{K,T}(f,g)V_{KT^2,T_1}(g^{-1}t)dg = \int\limits_G f(h)V_{K,T}(h^{-1}t)dh = P_{K,T}(f,t).$$

This implies that $P_{K,T}(f)\in W^p(KT^2)$. $\square$

Remark. While proving Lemma 3, concomittantly we have actually proved the following property of functions from $W^p(K)$: if $f\in W^p(K)$, $p\in[1,\infty]$, and K_1 is an arbitrary compact from $U_{\widehat{G}}$ containing K, then $f\in W^p(K_1)$. This can be shown by repeating the proof of Lemma 3, where V_{KT^2,T_1} is replaced by V_{K_1,T_1} for any $T_1\in U_{\widehat{G}}$, and taking into account that $V_{K,T}\in W^1(K_1T_1)$. The latter inclusion is valid because $V_{K,T}\in\mathcal{F}^1(KT^2)$. But T can be chosen so that $KT^2\subset K_1T_1$ ([3], Ch. 3, §1). Then $V_{K,T}\in\mathcal{F}^1(K_1T_1)$ and, by virtue of Lemma 2, $V_{K,T}\in W^1(K_1T_1)$. Finally, we obtain $f=P_{K,T}(f)\in W^p(K_1T_1)$.

Theorem 1. *Let $p\in[1,\infty]$. The set of functions from $W^p(K)$ for all possible compacts from $U_{\widehat{G}}$ is dense everywhere in $L^p(G,\mu)$.*

Proof. First we shall prove that if $f \in W^p(K)$ and χ is an arbitrary character of the group G, then $f \cdot \chi \in W^p(K_1)$ for some compact $K_1 \in U_{\widehat{G}}$. Indeed, since $f \in W^p(K)$, we have

$$\chi(g)f(g) = \int_G f(h)\chi(g)V_{K,T}(h^{-1}g)dh =$$

$$= \int_G f(h)\chi(h)\chi(h^{-1}g)V_{K,T}(h^{-1}g)dh = (f\chi * \chi V_{K,T})(g). \tag{9}$$

Next we obtain

$$(\widehat{V_{K,T} \cdot \chi})(\chi_0) = \int_G V_{K,T}(g) \cdot \chi(g) \cdot \overline{\chi_0(g)}\, dg =$$

$$= \int_G V_{K,T}(g) \cdot \overline{(\overline{\chi}\chi_0)}(g)\, dg = \widehat{V}_{K,T}(\chi^{-1}\chi_0).$$

Since $V_{K,T} \in \mathcal{F}^1(KT^2)$, this implies that $\chi \cdot V_{K,T} \in \mathcal{F}^1(K_1)$ for some compact $K_1 \in U_{\widehat{G}}$, i.e., $\chi \cdot V_{K,T} \in W^1(K_1)$ and $\chi \cdot V_{K,T} = \chi V_{K,T} * V_{K_1,T_1}$ for any $T, T_1 \in U_{\widehat{G}}$. Next, by (9) we have

$$\chi(g)f(g) = (f\chi * (\chi V_{K,T} * V_{K_1,T_1}))(g) =$$
$$= ((f\chi * \chi V_{K,T}) * V_{K_1,T_1})(g) = (\chi f * V_{K_1,T_1})(g),$$

which implies that $\chi f \in W^p(K_1)$.

We shall show that for all $K \in U_{\widehat{G}}$ the functions from $W^p(K)$ are dense everywhere in $L^p(G,\mu)$. Let us assume that the opposite is true. Let $\mathcal{A}^p$ be the closure of the union $\bigcup_K W^p(K)$ and $\mathcal{A}^p$ not coinciding with $L^p(G,\mu)$.

There exists a well defined nontrivial linear functional on $L^p(G)$ which is equal to zero on $\mathcal{A}^p$. This functional is defined by a nontrivial function $\varphi \in L^q(G)$ ([1], v. 1, §12). Since $f \cdot \chi \in \mathcal{A}^p$ for arbitrary $\chi \in \widehat{G}$, then

$$\int_G f\chi\varphi\, d\mu = 0, \quad f \in L^p(G), \quad \varphi \in L^q(G).$$

The latter equality implies

$$\widehat{f \cdot \varphi}(\chi) = 0, \quad f \cdot \varphi \in L^1(G)$$

for any $\chi \in \widehat{G}$. Hence it follows that ([1], v. 2, §31) $f\varphi = 0$ almost everywhere on G for any function $f \in W^p(K)$, where K is an arbitrary compact from $U_{\widehat{G}}$. By the continuity of f we readily conclude that $\varphi = 0$ almost everywhere on G, which is impossible. $\square$

For $p = 1$, Theorem 1 actually establishes the density of functions from $\mathcal{F}^1(K)$, for all possible compacta K, in $L^1(G)$ and this density is known ([3], Ch. 5, §4, [6], Ch. 2, §8.8).

Remark. Theorem 1 does not hold for $p = \infty$ but remains valid for its subspace $C^0(G) \subset L^\infty(G)$.

The dual space to $C^0(G)$ is the space $M(G)$ of complex-valued measures defined on some σ-algebra containing all Borel sets in G ([1], v. 1, §14). To verify this, i. e., the density of $W^\infty(K) \cap C^0(G)$ in $C^0(G)$, we must repeat the proof of Theorem 1, which will lead us to the existence of a nontrivial measure $\mu \in M(G)$ such that $\int_G f\chi d\mu = 0$, $\forall f \in C^0(G) \cap W^\infty(K)$, $\chi \in \widehat{G}$. Hence the Fourier transform of the measure $fd\mu$ is equal to zero on the entire $\widehat{G}$. Therefore $fd\mu$ ([1], v. 2, §28) and thus $\mu = 0$, which is impossible.

Let us consider the best approximation $E_K(f)_{p,G}$ of the function $f \in L^p(G)$ by the subspace $W^p(K)$, i. e., the value

$$E_K(f)_{p,G} = \inf_{g \in W^p(K)} \|f - g\|_{L^p(G)}. \tag{10}$$

If we obtain inf on some element $g_0 \in W^p(K)$, then it will be called the best approximation element of f in $W^p(K)$. We shall prove that such an element necessarily exists if $1 \le p \le \infty$. Indeed, let $f \in L^p(G)$, and $t_n \in W^p(K)$ be the minimizing sequence, i.e.,

$$\|f - t_n\|_p \le d + \varepsilon_n,$$

where $d = E_K(f)_{p,G}$ and $\varepsilon \downarrow 0$. Clearly, the sequence t_n is bounded with respect to the norm of the space L^p, and by applying inequality (8) to $p_1 = \infty$ we have

$$\|t_n\|_\infty \le C \cdot \|t_n\|_p \le A,$$

where A does not depend on n. Next we obtain

$$\|t_n(h^{-1}g) - t_n(g)\|_p = \left\| \int\limits_G t_n(\xi)\big[V_{K,T}(\xi^{-1}h^{-1}g) - V_{K,T}(\xi^{-1}g)\big] d\xi \right\|_p \le$$
$$\le \|t_n\|_p \cdot \|V_{K,T}(h^{-1}g) - V_{K,T}(g)\|_1.$$

The continuity of the shift operator in L^1 ([3], Ch. 3, §5) implies equicontinuity of the family $\{t_n\}$ in $L^p(G)$ and therefore in $L^\infty(G)$ as well. Hence by virtue of the well known theorem ([7], §5) it follows that from t_n we can extract a subsequence t_{n_ν} converging uniformly on arbitrary compacts from G to a certain function t. Let us prove that $t \in W^p(K)$. The sequence t_{n_ν} will again be denoted by t_n. In the first place, for any compact $G_1 \subset G$ we have

$$\|t\|_{L^p(G_1)} \le \|t - t_n\|_{L^p(G_1)} + \|t_n\|_{L^p(G_1)} \le C(G_1)\|t - t_n\|_{L^\infty(G_1)} + \|t_n\|_{L^p(G)}.$$

But since $\lim\limits_{n\to\infty} \|t - t_n\|_{L^\infty(G_1)} = 0$ and $\|t_n\|_{L^p(G)} \le C$, for any compact $G_1 \subset G$ we have $\|t\|_{L^p(G_1)} \le C$, where C is a constant not depending on G_1. Therefore $t \in L^p(G)$. Let consider an arbitrary compact $G_1 \subset G$. Since $t_n \in W^p(K)$, we have

$$t_n(g) = \int\limits_G t_n(h) V_{K,T}(h^{-1}g)dh =$$
$$= \int\limits_{G_1} t_n(h) V_{K,T}(h^{-1}g)dh + \int\limits_{G\setminus G_1} t_n(h) V_{K,T}(h^{-1}g)dh.$$

Assuming that $g \in G_1$ and passing to the limit as $n \to \infty$, we obtain

$$t(g) = \int\limits_{G_1} t(h) V_{K,T}(h^{-1}g)dh + A(G \setminus G_1),$$

where

$$|A(G \setminus G_1)| \le \sup_n \|t_n\|_p \cdot \int\limits_{G\setminus G_1} |V_{K,T}(h^{-1}g)|dh.$$

Now taking into account the fact that the Haar measure is regular and making G_1 tend to G, we find that $t(g) = \int_G t(h) V_{K,T}(h^{-1}g)dh$ and therefore $t \in W^p(K)$. Further, for any compact $G_1 \subset G$ we have

$$\|f - t\|_{L^p(G_1)} = \lim_{n\to\infty} \|f - t_n\|_{L^p(G_1)} \le \lim_{n\to\infty} \|f - t_n\|_{L^p(G)} = d$$

and thus t is the best approximation element of $f \in L^p(G)$ in $W^p(K)$, $1 \le p \le \infty$.

For $p = 2$ the best approximation element of $f \in L^2(G)$ in the subspace $W^2(K)$ can be constructed explicitly in the form

$$f_K(g) = \int\limits_K \widehat{f}(\chi)\chi(g)d\chi. \tag{11}$$

Indeed, let S be an arbitrary element from $W^2(K)$, i.e., from $\mathcal{F}^2(K)$. Then almost everywhere $(\widehat{S})\check{} = S$. Since $\widehat{S} \in L^1(\widehat{G})$ and its Fourier transforms in the sense of the spaces L^1 and L^2 coincide, we have

$$S(g) = \int\limits_K \widehat{S}(\chi)\chi(g)\, d\chi \quad \text{almost everywhere on } G.$$

Applying Parceval's theorem we obtain

$$\|f-S\|_2^2=\|f\|_2^2-\int\limits_K\widehat{S}(\chi)\overline{\widehat{f}(\chi)}\,d\chi-\int\limits_K\overline{\widehat{S}(\chi)}\widehat{f}(\chi)\,d\chi+\|S\|_2^2=$$
$$=\|f\|_2^2+\int\limits_K[\widehat{S}(\chi)-\widehat{f}(\chi)]^2d\chi-\int\limits_K[\widehat{f}(\chi)]^2d\chi.$$

The right-hand side of this equality will is minimal if $\widehat{S}=\widehat{f}$ almost everywhere on K, i.e., by virtue of (11), if

$$S(g)=\int\limits_K\widehat{S}(\chi)\chi(g)\,d\chi=\int\limits_K\widehat{f}(\chi)\chi(g)\,d\chi=f_K(g).$$

For $p\neq 2$, constructing a best approximation element is a difficult task even for simple cases. Moreover, we know of the cases for which it has been proved that one cannot construct a sequence of linear continuous projectors used to realize the best order approximation ([8], Ch. 9, §5.4, [9], Ch. 9, §5). In this connection, a problem is posed to construct a sequence of linear continuous operators, close in a certain sense to projectors, by means of which the best order approximation is realized. Such a problem, in which G is the unit circumference T, was posed and solved by de la Vallée-Poussin ([10]). The space of trigonometric polynomials of order n is considered as an approximating set and it is proved that for $1\le p\le\infty$ and $f\in L^p(T)$

$$\|f-\sigma_{n,m}(f)\|_p\le 2\,\frac{n+1}{m+1}\,E_n(f)_p,$$

where

$$(\sigma_{n,m}f)(x)=\frac{1}{m+1}\sum_{k=n}^{n+m}(S_kf)(x)=\int\limits_0^{2\pi}f(u)V_{n,m}(x-u)du,$$

$S_k(f)$ is the Fourier sum of the function f of order k, and

$$V_{n,m}(x)=\frac{\sin\frac{(m+1)x}{2}\cdot\sin\frac{(2n+m+1)x}{2}}{2\pi(m+1)\sin^2\frac{x}{2}}. \tag{12}$$

In [10], this result was obtained for the space of $C(T)$-continuous functions on T. If $m=[\frac{n}{2}]$, then the deviation $\|f-\sigma_{n,[n/2]}\|_p$ has a the best approximation order $E_n(f)_p$.

Theorem 2. *If for a locally compact Abelian group G the function f belongs to the space $L^p(G,\mu)$, $1\le p\le\infty$, and K and T are the compact* *symmetric neighborhoods of the unit of the dual group $\widehat{G}$, then*

$$\|f(g) - P_{K,T}(f,g)\|_p \leq \Big(1 + \Big(\frac{\operatorname{mes}(TK)}{\operatorname{mes} T}\Big)^{1/2}\Big) E_K(f)_{p,G}, \tag{13}$$

where $P_{K,T}(f,g)$ and $E_K(f)_{p,G}$ are defined by formulas (5) *and* (10).

Proof. Let $f_n \in W^p(K)$ be a minimizing sequence for the function $f \in L^p(G)$, $1 \leq p \leq \infty$. Then

$$\|f(g) - P_{K,T}(f,g)\|_p \leq \|f - t_n\|_p + \|P_{K,T}(f - t_n)\|_p \leq$$
$$\leq \|f - t_n\|_p + \|f - t_n\|_p \cdot \|V_{K,T}\|_1.$$

Hence, taking into account inequality (4) and the relation $\lim\limits_{n\to\infty} \|f - t_n\|_p = E_K(f)_{p,G}$, we obtain estimate (13). □

Remark. To have a best order approximation in (13), we must choose K and T from $U_{\widehat{G}}$ such that for E_K the multiplier be bounded from above by a number not depending on K and T. For example, if $T = K$ and $P_{K,K}$ is denoted by P_K, then (13) gives the estimate

$$\|f - P_K(f)\|_p \leq \Big(1 + \Big(\frac{\operatorname{mes} K^2}{\operatorname{mes} K}\Big)\Big)^{1/2} \cdot E_K(f)_{p,G},$$

and $\frac{\operatorname{mes} K^2}{\operatorname{mes} K}$ is bounded from above by the number not depending on K for a sufficiently wide class of groups G. For example, for $\widehat{G} = \mathbb{R}^n$ the Haar measure μ coincides with the Lebesgue measure, and if K is compact convex symmetric neighborhood of zero, then $\frac{\operatorname{mes} K^2}{\operatorname{mes} K} = 2^n$. If $\widehat{G} = Z^n$, and $K = \{x \in Z^n,\ -N \leq x_i \leq N,\ x_i \in Z,\ i = 1, 2, \dots, n\}$, then $\frac{\operatorname{mes} K^2}{\operatorname{mes} K} = (\frac{4N+1}{2N+1})^n < 2^n$. If however in the latter case $K = \{x \in Z^n,\ |x| \leq N\}$ is an integer-valued lattice from Z^n contained within a ball of radius N, then $\frac{\operatorname{mes} K^2}{\operatorname{mes} K}$ behaves as 2^n for large N . This follows from the known relation $\sum\limits_{|k|\leq N} 1 \sim \sum\limits_{|k|\leq N} \pi^{n/2}[\Gamma(\frac{n}{2} + 1)]^{-1} N^n$, where Γ is a Euler function of second kind.

Examples.

1. Let $G = \mathbb{R}^n$. Then $\widehat{G} = \mathbb{R}^n$ and the group G has a character $\chi(x) = e^{itx}$, $t \in \mathbb{R}^n$. We respectively take, as sets K and T, n-dimensional parallelepipeds with ribs of length $2N$ and $2s$: $K = \{-N \leq x_i \leq N\}$, $T = \{-s \leq x_i \leq s\}$, $i = 1, 2, \dots, n$. Since the measure is normalized, we readily obtain

$$V_{K,T}(x) = (s\pi)^{-n} \prod_{j=1}^{n} x_j^{-2} \cdot \sin s x_j \cdot \sin(N + s) x_j.$$

For $s = \frac{N}{2}$ this is the well known Vallée–Poussin kernel ([2], Ch. 8, §8.6).

If K and T are n-dimensional balls of radii N and s, then, after performing some calculations, we find that

$$V_{K,T}(x)=\frac{n}{\Omega_n}\Big(\frac{s}{N+s}\Big)^{\frac{n}{2}}|x|^{-n}\mathcal{I}_{\frac{n}{2}}(s|x|)\cdot\mathcal{I}_{\frac{n}{2}}((N+s)|x|),$$

where Ω_n is the surface area of the n-dimensional unit sphere and $\mathcal{I}_{n/2}$ is a first kind Bessel function of order $n/2$. In both cases the expression $\frac{\operatorname{mes}(TK)}{\operatorname{mes}T}$ is bounded from above by the number $(\frac{N+s}{N})^n$.

2. If $G=E$ is the unit circumference from $\mathbb{R}^2$, then $\widehat{G}=Z$ and the group G has a character of the form $\chi_n(t)=e^{-int}$, $t\in E$, $n\in Z$. Take $K=[-n,n]=\{-n,-n+1,\dots,n-1,n\}$, $T=[-s,s]$. Then for the kernel $V_{K,T}$ we obtain the above-mentioned de la Vallée-Poussin kernel $V_{n,m}$ defined by formula (12) with $m=2s$. If m is an odd number, then $V_{K,T}$ does not exactly coincide with $V_{n,m}$, but to obtain $V_{n,m}$ one should modify the definition of the kernel $V_{K,T}$ by means of averaging the kernels $\widehat{(1)}_{K_i}$, where $T\subset K_i\subset K$.

If G is an n-dimensional torus, then one can take, as $V_{K,T}$, the product of one-dimensional kernels (12), i.e.,

$$V_{K,T}(x)=(2\pi)^{-n}\prod_{j=1}^{n}(2s_j+1)^{-n}\cdot\sin^{-2}\frac{x_j}{2}\cdot\sin\Big(s_j+\frac{1}{2}\Big)x_j\times$$
$$\times\sin\Big(N_j+s_j+\frac{1}{2}\Big)x_j,\quad t_j,N_j\in Z^{+}.$$

3. If $G=Z$, then $\widehat{G}=E$. Characters of the group G have the form $\chi_t(n)=t^n$, $n\in Z$, $t\in E$. Let $K=\{e^{i\theta}:-\varphi\le\theta\le\varphi,\ 0<\varphi\le\pi\}$, $T=\{e^{i\theta}:-t\le\theta\le t,\ 0<t\le\pi\}$. Take, as a dual measure on E, the arc length divided into $i\sqrt{2\pi}$. We have

$$\widehat{(1)}_T(n)=\frac{1}{i\sqrt{2\pi}}\int_T\xi^n d\xi=\frac{1}{\sqrt{2\pi}}\int_{-t}^{t}e^{i\theta(n+1)}d\theta=$$
$$=\begin{cases}\frac{\sqrt{2}}{\sqrt{\pi}\cdot(n+1)}\sin t(n+1), & n\neq-1,\\ \frac{\sqrt{2}}{\sqrt{\pi}}\,t, & n=-1,\end{cases}$$

and $P_{K,T}(f,n)$ in this example takes the form

$$P_{K,T}(f,n)=$$
$$=\frac{1}{t\pi}\Big\{\sum_{\substack{k=-\infty\\k\neq n+1}}^{\infty}f(k)\,\frac{\sin t(n-k+1)\cdot\sin(\varphi+t)(n-k+1)}{(n-k+1)^2}+t(\varphi+t)\Big\}$$

and $\frac{\operatorname{mes}(TK)}{\operatorname{mes} T} = \frac{\varphi+t}{t}$.

4. Let $G = \mathbb{R}^+$ be a multiplicative group of positive integers with the unit $e = 1$. The group G has a character $\chi(\xi) = \xi^{ix}$, where $x \in \mathbb{R}$, $\widehat{G} = \mathbb{R}$ to within an isomorphism. Take, as K and T, the intervals $K = [-N, N]$, $T = [-t, t]$, $N, t \in \mathbb{R}^+$. Then

$$\widehat{(1)}_K(\xi) = \int_{-N}^{N} \xi^{ix}\, dx = 2\,\frac{\sin(N \ln \xi)}{\ln \xi},$$

$$P_{K,T}(f, x) = \frac{1}{t\pi} \int_0^\infty f(h)\, \frac{\sin(t \ln \frac{x}{h}) \cdot \sin((N+t) \ln \frac{x}{h})}{h \ln^2 \frac{x}{h}}\, dh.$$

5. Let $a = (a_0, a_1, \ldots, a_n, \ldots)$ be a given sequence, where all a_n are integer numbers greater than 1, and consider the Cartesian product $\Delta_a = \mathbf{P}_{n \in \mathbb{N} \cup \{0\}} \{0, 1, \ldots, a_n - 1\}$. If one applies the summation operation to Δ_a ([1], v. 1, §10), then Δ_a together with Tikhonov topology of the product becomes a compact Abelian group. This group $G = \Delta_a$ is called a group of integer a-adic numbers. Its dual group is the discrete group $\widehat{G} = Z(a^\infty)$ consisting of all numbers of the form

$$\exp\Big(2\pi i \frac{l}{a_0 a_1 \cdots a_r}\Big), \quad l, r \in Z, \quad r > 1$$

([1], v. 1, §10). For the fixed natural number N we take, as a compact symmetric neighborhood K of the unit, the set of numbers from $Z(a^\infty)$ for which $r \le N$. It is proved ([1], v. 1, §25) that the character corresponding to the number $\xi = \exp(2\pi i \frac{l}{a_0 a_1 \cdot a_r})$ is written as

$$\chi_\xi(g) = \exp\Big[\frac{2\pi i l}{a_0 a_1 \cdots a_r} (g_0 + a_0 g_1 + \cdots + a_0 a_1 \cdots a_{r-1} g_r)\Big],$$

where $g_0 + a_0 g_1 + \cdots + a_0 a_1 \cdots a_{r-1} g_r$ is the sum of the first $r+1$ terms of the a-adic expansion of the element $g \in \Delta_a$. Assuming that the Haar measure of each point in $Z(a^\infty)$ is 1, we have

$$\widehat{(1)}_K(g) = \int_K \chi_\xi(g)\, d\xi =$$

$$= \sum_{r=0}^{N} \sum_{l=1-a_0 a_1 \cdots a_r}^{a_0 a_1 \cdots a_r - 1} \exp\Big[\frac{2\pi i l}{a_0 a_1 \cdots a_r} \Big(g_0 + \sum_{k=1}^{r} a_0 \cdots a_{k-1} g_k\Big)\Big] =$$

$$= \sum_{r=0}^{N} \frac{\sin(a_0 a_1 \cdots a_r - \frac{1}{2}) x_r}{\sin \frac{x_r}{2}}, \quad x_r = \frac{2\pi}{a_0 a_1 \cdots a_r} \Big(g_0 + \sum_{k=1}^{r} a_0 \cdots a_{k-1} g_k\Big).$$

Clearly, $K^2 = K$ and, taking $T = K$, we obtain

$$P_K(f, g_1) = \frac{1}{(N+1)^2} \int\limits_{\Delta_a} f(gg_1) \Bigg[\sum_{r=0}^{N} \frac{\sin(a_0 a_1 \cdots a_r - \frac{1}{2})x_r}{\sin \frac{x_r}{2}} \Bigg]^2 dg,$$

where integration is performed with respect to the measure dual to the taken discrete measure on $Z(a^\infty)$.

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