GEORGIAN MATHEMATICAL JOURNAL: Vol. 6, No. 4, 1999, 363-378

MARKOV DILATION OF DIFFUSION TYPE PROCESSES AND ITS APPLICATION TO THE FINANCIAL MATHEMATICS

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Abstract. The Markov dilation of diffusion type processes is defined. Infinitesimal operators and stochastic differential equations for the obtained Markov processes are described. Some applications to the integral representation for functionals of diffusion type processes and to the construction of a replicating portfolio for a non-terminal contingent claim are considered.

1. Introduction

Let $\xi = (\xi_t)_{t\in[0,1]}$ be a stochastic process in the metric space X with the sample paths from the space $D([0,1],X)$ of functions which are right continuous with left limit (r.c.l.l.). It is easy to see that the $D([0,1],X)$-valued process defined for each $t \in [0,1]$ by

$$\xi^t = (\xi_{t\wedge s},\ s \in [0,1])$$

has a Markov property, i.e., for any Borel set B in $D[0,1]$

$$P[\xi^t \in B|\xi^{t_1}, \xi^{t_2}, \dots, \xi^{t_n}] = P[\xi^t \in B|\xi^{t_n}],\ \ 0 \le t_1 \le \dots \le t_n \le 1, \quad (1.1)$$

since σ-fields $\sigma(\xi^t) = \sigma(\xi_s, s \le t)$ increase as t increases.

Consider the case $X = R$ and suppose that ξ is a diffusion type process,i.e., it satisfies a stochastic differential equation (S.D.E.)

$$d\xi_t = a(t,\xi)dw_t + b(t,\xi)dt, \quad (1.2)$$

where $a(t,x)$, $b(t,x)$ are nonanticipative functionals and $w = (w_t)_{t\in[0,1]}$ is the Wiener process. We want to find a S.D.E. and infinitesimal operators for a random process ξ^t. This will allow us to write a parabolic equation for functionals of diffusion type processes and to derive Itô's formula for

1991 *Mathematics Subject Classification.* 60H05, 60H07.
Key words and phrases. Markov dilation, infinitesimal operator, Itô's formula, replicating portfolio.

1072-947X/99/0700-0363$15.00/0

nonanticipative functionals. In Section 2 we represent ξ^t as a solution of a S.D.E. in the space of square integrable paths. For this case the infinitesimal operator can be defined using the results of infinite dimensional stochastic analysis but one needs strong conditions on the coefficients a and b. In Section 3 more general and complicated case of the continuous path space is studied. Finally, we shall obtain a new proof of Clark's formula for Itô's processes [1] and we bring some applications to financial mathematics. When ξ is a homogeneous Markov process, our results have some intersection with the recent results of [2].

Here we use the following notation: $W[0,1]$ is the space of continuous functions, $D[0,1]$ the space of r.c.l.l. functions, $L^2_m[0,1]$ a space of square integrable functions w.r.t. measure m, $i_t = 1_{[0,t)}, j_t = 1_{[t,1]}$. $CB([0,1] \times W[0,1])$ denotes class of bounded continuous functions and $CB^{i,k}([0,1] \times W[0,1])$ denote the subclasses of functions from $CB([0,1] \times W[0,1])$ with continuous and bounded derivatives w.r.t. the first variable up to order i and continuous bounded Frechet derivatives w.r.t. the second variable up to order k.

For any $t \in [0,1]$ we shall consider the operators:

$$C_t x = x i_t + x(t) j_t, \quad C_t : D[0,1] \to D[0,1] \subset L^2_m[0,1],$$
$$R_t x = x(t) j_t, \quad R_t : D[0,1] \to D[0,1] \subset L^2_m[0,1],$$
$$L^\psi_t x(s) \equiv \psi \circ_t x(s) = \begin{cases} \psi(s) - \psi(t) + x(t) & \text{if } s > t, \\ x(s) & \text{if } s < t, \end{cases}$$
$$\psi \in D[0,1], \quad L^\psi_t : D[0,1] \to D[0,1] \subset L^2_m[0,1],$$
$$Q_t x(s) = \begin{cases} x(0) & \text{if } s > t, \\ x(t) - x(s) + x(0) & \text{if } s \le t, \end{cases}$$
$$Q_t : D[0,1] \to D[0,1] \subset L^2_m[0,1].$$

Their restrictions on the space $W[0,1]$ will be denoted by the same symbols. We shall also use the space $W[0,\infty)$ with metric $\|x - y\| = \sum\limits_{k=1}^{\infty} 2^{-k} \sup\limits_{s \le k} \{|x(s) - y(s)|, 1\}$ and spaces $CB(R_+ \times W)$, $CB^{i,k}(R_+ \times W)$ defined in the same way.

2. Repesentation of a Diffusion Type Process in the Space $L^2_m[0,1]$

Let $(\Omega, \mathcal{F}, P)$ be a probabilist space with filtration $(\mathcal{F}_t)_{t \in [0,1]}$. Let $(w_t, \mathcal{F}_t)_{t \in [0,1]}$ be a Wiener process and $(\beta_t, \mathcal{F}_t)_{t \in [0,1]}$ a random process with paths from the space $D[0,1] \cap V[0,1]$. Denote by $I_\beta(g)_t$ and $I_w(g)_t$ the integral $\int_0^t g_s d\beta_s$ and the stochastic integral $\int_0^t g_s dw_s$ respectively, for a process

$(g_t, \mathcal{F}_t)$ with values in the Hilbert space satisfy standart condition, which guarantees the existence of those integrales. If $\beta_t = t$, then $I_\beta(g)$ is shortly denoted by $I(g)$. Then the following theorem is valid.

Theorem 2.1. *Let $(a_t, \mathcal{F}_t)$, $(b_t, \mathcal{F}_t)$ be real processes such that*

$$P\biggl(\int_0^1 |b_t| d\beta_t < \infty\biggr) = 1, \quad E\int_0^1 |a_t|^2 dt < \infty.$$

Then the identities take place

$$C(I_\beta(B)) = I_\beta(R(b)), \ \ C(I_w(a)) = I_w(R(a)).$$

In other words, if $\xi_t = \int_0^t a_s dw_s + \int_0^t b_s d\beta_s$, then $C_t\xi = \int_0^t R_s(a) dw_s + \int_0^t R_s(b) d\beta_s$.

Proof. Let $\psi \in L_m^2[0,1]$. Then

$$\begin{aligned}
(\psi, C_t(I_\beta(b))) &= \int_0^t \psi_s \int_0^s b_u d\beta_u dm_s + \int_t^1 \psi_s \int_0^t b_u d\beta_u dm_s = \\
&= \int_0^t \int_0^t \psi_s b_u i_s(u) d\beta_u dm_s + \int_t^1 \int_0^t \psi_s b_u d\beta_u dm_s = \\
&= \int_0^t \int_0^t \psi_s b_u 1_{[u,1]}(s) d\beta_u dm_s + \int_t^1 \int_0^t \psi_s b_u d\beta_u dm_s = \\
&= \int_0^t b_u \int_0^t \psi_s j_u(s) dm_s d\beta_u + \int_0^t b_u \int_t^1 \psi_s dm_s d\beta_u = \\
&= \int_0^t b_u \int_u^1 \psi_s dm_s d\beta_u = \int_0^t (\psi, R_u(b)) d\beta_u = \\
&= \biggl(\psi, \int_0^t R_u(b) d\beta_u\biggr).
\end{aligned}$$

By the arbitrariness of ψ the first relation is established. Identitities for stochastic integrals may be derived in similar way. For instance, the transition

$$\int_0^t dm_s \int_0^t \psi_s a_u j_s(u) dw_u = \int_0^t \int_0^t \psi_s a_u j_s(u) dm_s dw_u$$

is true by Fubini's theorem for stochastic integrals [3, p. 217]. In particular, we must take

$$\tilde{\Omega} = [0,1], \tilde{\mathcal{F}}_t = \mathcal{B}[0,1], \tilde{p} = m, g_s(w, \tilde{w}) = \psi_u a_s(w) 1_{[0,s]}(\tilde{w})$$

in the equality

$$\int_{\tilde{\Omega}} \int_0^1 g_s(w, \tilde{w}) dw_s d\tilde{P} = \int_0^1 \int_{\tilde{\Omega}} g_s(w, \tilde{w}) d\tilde{P} dw_s. \qquad \square$$

Corollary. *Let $f : [0,1] \times L_m^2[0,1] \to R$ be a bounded function with bounded and continuous Frechet derivative f_t, $\nabla_x f$, $\nabla_x^2 f$ and let $\xi_t = \int_0^t a_s dw_s + \int_0^t b_s d\beta_s$ be the Itô process. Then*

$$f(t, C_t\xi) - f(t_0, C_{t_0}\xi) = \int_{t_0}^t \Big[\frac{\partial}{\partial s} f(s, C_s\xi) + b_s \frac{\partial}{\partial j_s} f(s, C_s\xi) + \\ + \frac{1}{2} a_s^2 \frac{\partial^2}{\partial j_s^2} f(s, C_s\xi)\Big] ds + \int_{t_0}^t a_s \frac{\partial}{\partial j_s} f(s, C_s\xi) dw_s, \tag{2.1}$$

where $\frac{\partial}{\partial j_t}$ denotes the Gateux derivative in the direction j_t.

Proof. Using Itô's formula for the $L_m^2[0,1]$-valued Itô process $\eta_t = \int_0^t a_s j_s dw_s + \int_0^t b_s j_s ds$ [4] and taking into account

$$\begin{aligned} (\nabla_x f(t,x), b_t j_t) &= b_t \frac{\partial}{\partial j_t} f(t,x), \\ \nabla_x^2 f(t,x)(a_t j_t, a_t j_t) &= a_t^2 \frac{\partial^2}{\partial j_s^2} f(t,x), \end{aligned} \tag{2.2}$$

we obtain (2.1). □

About $C_t\xi, Q_t\xi$ or $L_t^\psi \xi$ we shall say that each of them is the *Markov dilation* of ξ, since each of them has the Markov property.

Theorem 2.2. *Let $A, B : [0,1] \times L_m^2[0,1] \to R$ be Lipschitz function, satisfying linearly growth condition. Suppose that functions $a^\psi, b^\psi : [0,1] \times W[0,1] \to R$ defined by*

$$a^\psi(t,x) = A(t, L_t^\psi x), \ \ b^\psi(t,x) = B(t, L_t^\psi x)$$

and $\xi_s^{t,\psi}, \theta_s^{t,\psi}$ denote solutions of the S.D.E.

$$\xi_s = \psi_s + j_t(s) \int_t^s a^\psi(u,\xi) dw_u + j_t(s) \int_t^s b^\psi(u,\xi) du, \tag{2.3}$$

$$\theta_s = \psi + \int_t^s A(u, \theta_u) j_u dw_u + \int_t^s B(u,\theta_u) j_u du, \tag{2.4}$$

respectively. Then $\theta_s^{t,\psi} = L_s^\psi(\xi^{t,\psi})$.

Proof. Equation (2.3) has a unique strong solution ([3], [4]) and equation (2.4) has a unique strong solution only in the space $L_m^2[0,1]$ [5]. By Theorem 2.1 equation (2.3) gives

$$C_s\xi = \psi_s + \int_t^s a^\psi(u,\xi) j_u dw_u + \int_t^s b^\psi(u,\xi) j_u du.$$

Since $a_u^\psi(\xi) = A_u(L_u^\psi \xi)$, we have

$$L_s^\psi \xi = \psi + \int_t^s A(u, L_u^\psi \xi) j_u dw_u + \int_t^s B(u, L_u^\psi \xi) j_u du,$$

$\theta_s^{t,\psi} = L_s^\psi(\xi^{t,\psi})$ because of the solution is unique. □

Corollary 1. *Suppose that in addition to the conditions of Theorem* 2.2 *the functions* A, B *belong to* $C^{1,2}([0,1] \times L_m^2)$ *and* $\eta \in C^2(L_m^2)$. *Then the Cauchy problem*

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}(t)\Big) u(t,\varphi) \equiv \Big(\frac{\partial}{\partial t} + b(t,\varphi)\frac{\partial}{\partial j_t} + \frac{1}{2} a(t,\varphi)^2 \frac{\partial^2}{\partial j_t^2}\Big) u(t,\varphi) = 0, \quad (2.5)$$

$$u(1,\varphi) = \eta(\varphi) \quad (2.6)$$

has a solution which can be represented as

$$u(t,\varphi) = E\eta(\xi^{t,\varphi}), \quad (2.7)$$

where $\xi^{t,\varphi}$ *is a solution of* (2.3).

Proof. It follows from the results of [5, pp. 322, 325] taking into account (2.2). □

Corollary 2. $\xi_s = \xi_s^{0,\psi(0)j_0}$ *satisfies the S.D.E.*

$$\xi_s = \psi(0) + \int_0^s a(u,\xi) dw_u + \int_0^s b(u,\xi) du,$$

where $a(u,\xi) = A(u, C_u \xi), b(u,\xi) = B(u, C_u \xi)$, *and we have*

$$E[\eta(\xi)|\mathcal{F}_t^\xi] = E\eta(\xi^{t,\psi})|_{\psi = C_t \xi}.$$

Remark 1. If we suppose

$$\psi = \psi_0 j_0, \quad m = \delta_1, \quad A(t,x) = \tilde{A}(t, x(t)), \quad B(t,x) = \tilde{B}(t, x(t)),$$

then $a(t,x) = A(t, C_t x) = \tilde{A}(t, x(t)), b(t,x) = B(t, C_t x) = \tilde{B}(t, x(t))$.

Theorem 2.3. *Suppose that a continuous bounded function*

$$f : [0,1] \times L_m^2[0,1] \to R$$

possesses the continuous bounded derivatives $\frac{\partial}{\partial t} f(t,\psi)$, $\frac{\partial}{\partial j_t} f(s,\psi)$, $\frac{\partial^2}{\partial j_t^2} f(s,\psi)$ *for each* $t, s \in [0,1]$. *Then* (2.1) *holds.*

Proof. By a standard way the proof of formula (2.1) reduces to the case $\xi_t = x_{t_0} + (t-t_0)R_{t_0}(b) + (w(t)-w(t_0))R_{t_0}(a)$. Evidently, $\xi_t - x_{t_0} = \eta_t j_t$, $g(t,\eta_t) = f(t,\xi_t)$, where $g(t,q) = f(t, qx_{t_0}j_{t_0})$, $\eta_t = (t-t_0)b_{t_0} + (w(t)-w(t_0))a_{t_0}$. By the Itô formula for scalar case we obtain

$$g(t,\eta_t) - g(t_0,0) = \int_{t_0}^t \Big[\frac{\partial}{\partial s} g(s,\eta_s) + b_{t_0}\frac{\partial}{\partial q} g,\eta_s) + \\ + \frac{1}{2}a_{t_0}^2 \frac{\partial^2}{\partial q^2} g(s,\eta_s)\Big] ds + \int_{t_0}^t a_{t_0}\frac{\partial}{\partial q} g(s,\eta_s) dw_s.$$

Then the Itô formula is obtained for ξ_t, since $\frac{\partial^k}{\partial q^k} g(t,q) = \frac{\partial^k}{\partial j_{t_0}^k} f(t, x_{t_0} + qj_{t_0})$, $k=1,2$. □

The first and second Frechet derivatives ∇F, $\nabla^2 F$ for the functions $F : W[0,1] \to R$ by Riesz' theorem are represented as a Borel measure on $[0,1]$ and a symmetric (w.r.t. the inversion $(u,v) \to (v,u)$) Borel measure on $[0,1]^2$, respectively [6, p. 68]. These measures are denoted by $\nabla F(x,du)$ and $\nabla^2 F(x,dudv)$.

Remark 2. If $f : W[0,1] \to R$ is a twice continuous Frechet differentiable function, then by the results of [2] both $\frac{\partial}{\partial j_t} f(t, C_t x)$, $\frac{\partial^2}{\partial j_t^2} f(t, C_t x)$ are r.c.l.l. in t.

Theorem 2.4. *Suppose that* $f \in CB^{1,2}([0,1] \times W[0,1])$, $f^0(t,x) = \frac{\partial}{\partial t} f(t,x)$, $f^1(t,x) = \nabla f(t,x,[t,1])$, $f^2(t,x,[t,1]^2) = \nabla^2 f(t,x,[t,1]^2)$ *and* $\eta_t = \psi + \int_{t_0}^t a_s j_s dw_s + \int_{t_0}^t b_s j_s ds$, *where* $(a_t, \mathcal{F}_t)$, $(b_t, \mathcal{F}_t)$ *satisfy the conditions of Theorem* 2.1. *Then*

$$f(t,\eta_t) - f(t_0,\eta_{t_0}) = \int_{t_0}^t \Big[f(s,\eta_s) + b_s f^1(s,\eta_s) + \frac{1}{2}a_s^2 f^2(s,\eta_s)\Big] ds + \\ + \int_{t_0}^t a_s f^1(s,\eta_s) dw_s. \tag{2.8}$$

Proof. For each $h \in L^2[0,1]$ we denote

$$h_n(t) = n\int_0^t e^{-n(t-s)} h(s) ds. \tag{2.9}$$

Evidently, the mapping $L^2[0,1] \in h \to h_m \in W[0,1]$ is a lineary continuous function and $h_m \to h, m \to \infty$ in $L^2[0,1]$. The mapping $f^{(n)}(t,x) = f(t,x)$

is a differentiable w.r.t. t and twice Frechet differetiable w.r.t. x. For $h \in L^2[0,1]$ we have

$$\frac{\partial}{\partial h} f^{(n)}(t,x) = \int_0^1 f_x(t, x^n, ds) h_n(s),$$
$$\frac{\partial^2}{\partial h^2} F^{(n)}(t,x) = \int_0^1 \int_0^1 f_{xx}(t, x^n, dudv) h_n(u) h_n(v)$$

for $h = j_s,\ h_n(t) = e^{-nt}(e^{nt} - e^{ns})$, i.e.,

$$\frac{\partial}{\partial j_t} f^{(n)}(t,x) \to f^1(t,x), \quad n \to \infty.$$

Similarly,

$$\frac{\partial^2}{\partial j_t^2} f^{(n)}(t,x) \to f^2(t,x), \quad n \to \infty.$$

By Lebesgue's theorem we can pass to the limit in formula (2.1). $\square$

Remark 3. In [7] it was proposed to define derivatives of anticipative functions as follows. $f(t,x)$ called differentiable if there exist f^0 and f^1 such that $f(t,x) = f^0(t,x) + \int_0^t f^1(s,x) dx_s$ for $x \in V[0,1]$. If $f(t,x) = F(t, C_t x)$ then the derivatives $f^0(t,x), f^1(t,x)$ can be calculated as

$$\int_0^t \frac{\partial F}{\partial s}(s, C_s x) ds, \quad \frac{\partial F}{\partial j_t}(t, C_t x),$$

respectively.

Remark 4. It is possible to define the Markov dilation by the operator Q_t. Then the infinitesimal operator will have the form $\frac{\partial}{\partial t} + b(t,x)\frac{\partial}{\partial i_t} + \frac{1}{2} a(t,x)^2 \frac{\partial^2}{\partial i_t^2}$.

3. Markov dilation of a Diffusion Type Process in the Space $W[0,\infty)$

Let $a, b : R \times W[0,\infty) \to R$ be nonanticipative continuous functions such that the S.D.E.

$$\xi_s = \psi_s + \int_t^s a(u,\xi) dw_u + \int_t^s b(u,\xi) du, \quad t \le s, \tag{3.1}$$

being defined as $\psi(s)$ for $s < t$ has a unique strong solution for any t. For this it must satisfies the following condition [5]: There exists $K > 0$ such that

$$\begin{gathered} |a(t,x) - a(t,y)|^2 + |b(t,x) - b(t,y)|^2 \le K \|x - y\|_t^2, \\ |a(t,x)|^2 + |b(t,x)|^2 \le K(1 + \|x\|^2). \end{gathered} \tag{3.2}$$

Lemma 3.1. *Let* $\xi_s^{t,\psi}, s \geq t$ *be a solution of* (3.1). *Then*

$$\xi_\tau^{t,\psi} = \xi_\tau^{\psi \circ_s \xi^{t,\psi}}, \quad t \leq s \leq \tau.$$

Proof. By inserting $\varphi = \xi^{t,\psi}$ in the equality

$$\xi_\tau^{\psi \circ_s \varphi} = \psi \circ_s \varphi(\tau) + \int_t^\tau a(u, \xi^{\psi \circ_s \varphi}) dw_u + \int_t^\tau b(u, \xi^{\psi \circ_s \varphi}) du, \ \ \psi \in W,$$

and taking into account that

$$\int_s^\tau g(u,r) dw_u|_{r=\eta} = \int_s^\tau g(u,\eta) dw_u, \quad \eta \in \mathcal{F}_s,$$

we have

$$\begin{aligned}
\xi_\tau^{s,\psi \circ_s \xi^{t,\psi}} &= \psi(\tau) - \psi(s) + \xi_s^{t,\psi} + \\
&\quad + \int_s^\tau a(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) dw_u + \int_s^\tau b(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) du = \\
&= \xi_s^{t,\psi} - \psi(s) - \int_t^s a(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) dw_u - \int_t^s b(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) du + \\
&\quad + \psi(\tau) + \int_t^\tau a(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) dw_u + \int_t^\tau b(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) du = \\
&= \psi(\tau) + \int_t^\tau a(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) dw_u + \int_t^\tau b(u, \xi^{s,\psi \circ_s \xi^{t,\psi}}) du.
\end{aligned}$$

This equality is valid, since $\xi_u^{t,\psi} = \xi_u^{s,\psi \circ_s \xi^{t,\psi}}$ for $t \leq u \leq s$. By the uniqueness of a solution it follows that $\xi_\tau^{t,\psi} = \xi_\tau^{\psi \circ_s \xi^{t,\psi}}$. □

Theorem 3.1. $\theta_s^{t,\psi} \equiv \psi \circ_s \xi^{t,\psi} = L_s^\psi(\xi^{t,\psi})$ *is a* W*-valued Markov family of processes.*

Proof. Suppose $f \in CB(W)$. By Lemma 3.1 $\psi \circ_s \xi^{t,\psi} = \psi \circ_s \xi^{s,\psi \circ_s \xi_s^{t,\psi}}$, i.e., $\theta_\tau^{t,\psi} = \psi \circ_s \xi^{s,\psi \circ_s \xi_s^{t,\psi}}$, $t \leq s \leq \tau$, and

$$E[f(\theta_\tau^{t,\psi}) | \theta_s^{t,\psi}] = E[f(\theta_\tau^{s,\theta_s^{t,\psi}}) | \theta_s^{t,\psi}] = E[f(\theta_\tau^{s,\psi})]_{\psi = \theta_s^{s,\psi}}.$$

On the other hand, (1.1) is fulfilled, i.e., $E[f(\theta_\tau^{t,\psi}) | \theta_s^{t,\psi}] = Ef(\theta_\tau^{s,\psi})_{\psi = \theta_s^{s,\psi}}$, since

$$\sigma(\theta_s^{t,\psi}) = \sigma(\xi_u^{t,\psi}, \quad t \leq u \leq s) = \sigma(\theta_u^{t,\psi}, \ t \leq u \leq s). \qquad \square$$

Now we shall describe infinitesimal operators of Markov family of processes. Here j_t will denote $1_{[t,\infty)}$.

Theorem 3.2. *Suppose that the bounded function $f : R_+ \times L^2(R_+) \to R$ has the continuous bounded derivatives $\frac{\partial}{\partial t} f(t,\psi)$, $\frac{\partial}{\partial j_s} f(t,\psi)$, $\frac{\partial^2}{\partial j_s^2} f(t,\psi)$ for each $t, s \in R_+$. Then*

$$f(s,\theta_s^{t,\psi}) - f(t,\psi) = \int_t^s \Big[\frac{\partial}{\partial u} f(u,\theta_u^{t,\psi}) + \mathcal{A}(u)F(u,\theta_u^{t,\psi})\Big] du + \\ + \int_t^s a(u,\theta_u^{t,\psi}) \frac{\partial}{\partial j_u} f(u,\theta_u^{t,\psi}) dw_u.$$

Proof. It can be obtained immediately by the theorem 2.3 and the representation

$$\theta_s^{t,\psi} = \psi_s + \int_t^s j_u a(u,\psi) dw_u + \int_t^s j_u b(u,\psi) du. \quad \square$$

Let us denote by $CB_J^{1,2}(R_+ \times W)$ a class of function satisfying the conditions of Theorem 3.2.

Lemma 3.2. *Suppose $f \in CB(R_+ \times W)$, $f^{(n)}(t,x) = f(t,x^n)$, where x^n is defined by* (2.9)*. Then $f^{(n)}(t,x) \to f(t,x)$, $(t,x) \in R_+ \times W$, and $\{f^n\}$ is the uniformly bounded family.*

Proof. The results of [8] imply that $x_n(t) \to x(t)$ uniformly on each segment $[a,b]$, i.e., $x_n \to x$ in W. By the continuity of f we have $f^{(n)}(t,x) \to f(t,x)$, $f(t,x) \leq \|f\|_\infty$. $\square$

Corollary. *$CB_J^{1,2}(R_+ \times W)$ dense in $CB_J(R_+ \times W)$ in the topology of bounded pointwise convergence.*

Proof. It is sufficient to recall that Gateux differentiable functions in the Hilbert space H dense in $CB(H)$ [5]. $\square$

Now we introduce the $R_+ \times W$-valued homogeneous Markov family

$$\eta_s^{t,\psi} = (t+s, \theta_{t+s}^{t,\psi}), \quad s \geq 0.$$

Evidently, by Theorem 2.3 for each $f \in CB_J^{1,2}(R_+ \times W)$, we have the following decomposition

$$f(\eta_s^{t,\psi}) = f(t,\psi) + \int_0^s \mathcal{L}f(u+t, \theta_{u+t}^{t,\psi}) du + \\ + \int_0^s \frac{\partial f}{\partial j_{u+t}}(u+t, \theta_{u+t}^{t,\psi}) a(u+t, \theta_{u+t}^{t,\psi}) dw_u.$$

Therefore $CB_J^{1,2}$ belongs to the domain of the generator of the Markov family $\{\eta_s^{t,\psi}\}$ and the equation $\mathcal{L}f = (\frac{\partial}{\partial t} + \mathcal{A}(t))f$ holds for any $f \in CB_J^{1,2}(R_+ \times W)$. We will show the solvability of this equatiion.

Lemma 3.3. *Let* $a_n(t,u), b_n(t,u), c_n(t)$, $(t,u)\in R_+\times R$, $n=0,1,\dots$, *be the processes adapted with filtration* $(\mathcal{F}_t)$ *for each fixed* u. *Furtermore,* $a_n(t,\cdot)$, $b_n(t,\cdot)$ *are functions of finite variation. Assume that there exist* K *such that*

$$\|a_n(t,\cdot)\|\le K,\quad \|b_n(t,\cdot)\|\le K,\quad E\sup_{s\le t}|c_n(s)|\le K,$$

and

$$\|a_n(t,\cdot)-a_0(t,\cdot)\|_{V[0,t]}\to 0,\quad \|b_n(t,\cdot)-b_0(t,\cdot)\|_{V[0,t]}\to 0,$$
$$E\sup_{s\le t}|c_n(s)-c_0(s)|\to 0$$

in probability. Let ζ_n, $n=0,1,\dots$, *be solutions of*

$$\zeta_n(t)=c_n(t)+\int_0^t\int_0^s a_n(s,du)\zeta_n(u)dw_s+\int_0^t\int_0^s b_n(s,du)\zeta_n(u)ds.$$

Then $E\|\zeta_n-\zeta_0\|_t^2\to 0$.

Proof. We shall use the proof from [9]. Using the inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$ we obtain

$$\begin{aligned}E\sup_{s\le t}&|\zeta_n(\tau)-\zeta_0(\tau)|_t^2\le\\ &\le 3E\sup_{\tau\le t}\int_0^\tau\bigg|\int_0^s(a_n(s,du)\zeta_n(du)-a_0(s,du)\zeta_0(u))dw_s\bigg|^2+\\ &\quad+3E\sup_{\tau\le t}\int_0^\tau\bigg|\int_0^s(b_n(s,du)\zeta_n(du)-b_0(s,du)\zeta_0(u))ds\bigg|^2+\\ &\quad+3E\sup_{\tau\le t}|c_n(t)-c_0(t)|^2.\end{aligned}$$

By Doob's inequality we have

$$\begin{aligned}E\sup_{\tau\le t}&\int_0^\tau\bigg|\int_0^s(a_n(s,du)\zeta_n(du)-a_0(s,du)\zeta_0(u))dw_s\bigg|^2\le\\ &\le 4E\int_0^t\bigg|\int_0^s(a_n(s,du)\zeta_n(du)-a_0(s,du)\zeta_0(u))\bigg|^2ds,\end{aligned}$$

i.e.,

$$\begin{aligned}E\sup_{\tau\le t}|\zeta_n(\tau)-\zeta_0(\tau)|^2&\le 3E\sup_{\tau\le t}|c_n(t)-c_0(t)|^2+\\ &+12E\int_0^t\bigg|\int_0^s(a_n(s,du)\zeta_n(u)-a_0(s,du)\zeta_0(u))\bigg|^2ds+\end{aligned}$$

$$+ 3tE\int_0^t \left|\int_0^s (b_n(s,du)\zeta_n(u) - b_0(s,du)\zeta_0(u))\right|^2 ds \leq$$

$$\leq 3E\sup_{\tau\leq t}|c_n(\tau) - c_0(\tau)|^2 +$$

$$+ 24E\int_0^t \left|\int_0^s [(a_n(s,du) - a_0(s,du)]\zeta_0(u))\right|^2 ds +$$

$$+ 6tE\int_0^t \left|\int_0^s [b_n(s,du) - b_0(s,du)]\zeta_0(u))\right|^2 ds +$$

$$+ 24E\int_0^t \left|\int_0^s a_n(s,du)[\zeta_n(u) - \zeta_0(u)]\right|^2 ds +$$

$$+ 6tE\int_0^t \left|\int_0^s b_n(s,du)(\zeta_n(u) - \zeta_0(u))\right|^2 ds \leq$$

$$\leq 3E\sup_{\tau\leq t}|c_n(\tau) - c_0(\tau)|^2 +$$

$$+ 24E\int_0^t \left|\int_0^s a_n(s,du)[\zeta_n(u) - \zeta_0(u)]\right|^2 ds +$$

$$+ 6tE\int_0^t \left|\int_0^s b_n(s,du)(\zeta_n(u) - \zeta_0(u))\right|^2 ds \leq$$

$$\leq \delta_n(t) + 6K^2t\int_0^t E\sup_{\tau\leq s}|\zeta_n(u) - \zeta_0(u)|^2 ds +$$

$$+ 24K^2\int_0^t E\sup_{\tau\leq s}|\zeta_n(u) - \zeta_0(u)|^2 ds,$$

where

$$\delta_n(t) = 3E\sup_{s\leq t}|c_n(s) - c_0(s)| +$$

$$+ 24E\sup_{s\leq t}|\zeta_0(s)|^2 E\int_0^t \|a_n(s,\cdot) - a_0(s,\cdot)\|^2 ds +$$

$$+ 6tE\sup_{s\leq t}|\zeta_0(s)|^2 E\int_0^t \|b_n(s,\cdot) - b_0(s,\cdot)\|^2 ds.$$

Evidently, $\delta_n(t) \to 0, t \geq 0$. By Gronwall's lemma

$$E|\zeta_n(s) - \zeta_0(s)|^2 \leq \sup_{s\leq t}|\delta_n(s)|^2 e^{Ht},$$

where $H = 24K^2 + 6K^2t$. □

Theorem 3.3. *Let $a, b \in CB^{0,2}(R_+ \times W)$. Then the solution of equation (3.1) is Frechet differeniable w.r.t. ψ for each t, s. Moreover, $Y(s,u) =$*

$\frac{\partial}{\partial j_u}\xi_s^{t,\psi}$ *satisfies the equation*

$$d_sY(s,u) = (\nabla a(s,\xi), Y(\cdot,u))dw_s + (\nabla b(s,\xi), Y(\cdot,u))ds, \quad s \geq u,$$
$$Y(u,u) = 1, \tag{3.3}$$
$$Y(s,u) = 0, \quad s < u.$$

Proof. Let t, ψ be fixed. Denote $\zeta_s^{\varphi,\epsilon} = \frac{1}{\epsilon}[\zeta_s^{t,\psi+\epsilon\varphi} - \zeta_s^{t,\psi}]$ for any direction φ. Then

$$\zeta_s^{\varphi,\epsilon} = \psi_s + j_t \int_t^s \frac{1}{\epsilon}[a(u,\xi^{t,\psi+\epsilon\varphi}) - a(u,\xi^{t,\psi})]dw_u + \\ + j_t \int_t^s \frac{1}{\epsilon}[b(u,\xi^{t,\psi+\epsilon\varphi}) - b(u,\xi^{t,\psi})]du.$$

Using the mean value theorem we obtain

$$\zeta_s^{\psi,\epsilon} = \psi_s + j_t \int_t^s (\nabla a(u,\xi^{t,\psi} + \epsilon\theta\zeta^{\varphi,\epsilon}), \zeta^{\varphi,\epsilon})dw_u + \\ + j_t \int_t^s (\nabla b(u,\xi^{t,\psi} + \epsilon\theta\zeta^{\varphi,\epsilon}), \zeta^{\varphi,\epsilon})du$$

for some θ, $0 \leq \theta \leq 1$. Introduce the notation $a_n = \nabla a(u,\xi^{t,\psi} + \epsilon_n\zeta^{\varphi,\epsilon_n})$, $b_n = \nabla b(u,\xi^{t,\psi} + \epsilon_n\zeta^{\varphi,\epsilon_n})$, where $\epsilon_n \to 0$. By Lemma 3.3 it follows that $\zeta^{\varphi,\epsilon_n} \to \frac{\partial}{\partial\varphi}\xi^{t,\psi}$ and consequently $\sup_{s\leq t}[\epsilon_n\zeta_s^{\varphi,\epsilon_n}] \to 0$. Thus $a_n \to \nabla a(u,\xi^{t,\psi})$, $b_n \to \nabla b(u,\xi^{t,\psi})$ as $n \to \infty$. Evidently $(\frac{\partial}{\partial\varphi}\xi^{t,\psi})(s) = \varphi(s)$, $s < t$. It remains to take $\varphi = j_u$. The existence of second derivatives at ψ is proved in a similar way [9]. □

Theorem 3.4. *Let the conditions of Theorem* 3.3 *hold and* $\eta \in C^2(W)$. *Then* $u(t,\psi) = E\eta(\xi^{t,\psi})$ *belongs to* $C^{1,2}(R_+ \times W)$ *and satisfies*

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}(t)\Big)u(t,\psi) = 0, \quad \lim_{t\to\infty} u(t,\psi) = \eta(\psi).$$

Proof. The differentiability of the functions $u(t,\psi)$ w.r.t. ψ follows from Theorem 3.2. We have

$$u(t,\psi) = E\eta(\xi^{t,\psi}) = E\eta(\xi^{s,\psi\circ_s\xi^{t,\psi}}) = E[\eta(\xi^{s,\psi\circ_s y})|_{y=\xi^{t,\psi}}] = Eu(s,\psi\circ_s\xi^{t,\psi}),$$

$y \in W$, i.e., $u(t,\psi) = Eu(s,\theta_s^{t,\psi})$. Let $s = t + h$. By Itô's formula

$$u(t+h,\theta_{t+h}^{t,\psi}) - u(t+h,\psi) = \\ = \int_t^{t+h} \Big[b(s,\theta_s^{t,\psi})\frac{\partial}{\partial j_s} + \frac{1}{2}a(s,\theta_s^{t,\psi})^2\frac{\partial^2}{\partial j_s^2}\Big]u(t+h,\theta_s^{t,\psi})ds + \\ + \int_t^{t+h} a(s,\theta_s^{t,\psi})\frac{\partial}{\partial j_s}u(t+h,\theta_s^{t,\psi})dw_s.$$

Consequently,

$$u(t,\psi)-u(t+h,\psi)=Eu(t+h,\theta^{t,\psi}_{t+h})-u(t+h,\psi)=$$
$$+E[b(s',\theta^{t,\psi}_{s'})\frac{\partial}{\partial j_s}u(t+h,\theta^{t,\psi}_{s'})+\frac{1}{2}a(s',\theta^{t,\psi}_{s'})^2\frac{\partial^2}{\partial j_s^2}u(t+h,\theta^{t,\psi}_{s'})]h,$$

for some s', $t\le s'\le t+h$, i.e.,

$$\frac{1}{h}(u(t,\psi)-u(t+h,\psi))\to\mathcal{A}(t)u(t,\psi),\quad h\to 0,$$

and $(\frac{\partial}{\partial t}+\mathcal{A}(t))u(t,\psi)=0$. Evidently, $\xi^{t,\psi}\to\psi$ when t tends to infinity. □

Remark 5. Applying a similar reasoning, one can prove the solvability of the Cauchy problem

$$\Big(\frac{\partial}{\partial t}+\mathcal{A}(t)-r(t,\varphi)\Big)u(t,\varphi)=0,\quad \lim_{t\to\infty}u(t,\varphi)=\eta(\varphi),$$

where $r\in C^{0,2}(R_+\times W)$.

4. Applications

The obtained results allow us to derive a representation formula for functionals of a diffusion type process (Clark's formula) [1]. However our conditions will be stronger than those in [1] and more general than those in the recent work [2].

Theorem 4.1. *Let $a(t,\psi)$, $b(t,\psi)$, $\eta(\psi)$ be the functions with bounded continuous Frechet derivatives of first and second order w.r.t. $\psi\in W[0,1]$. Suppose $(\xi_t)_{t\in[0,1]}$ is a solution of* (1.2). *Then*

$$E[\eta(\xi)|\mathcal{F}_t]=E[\eta(\xi)]+\int_0^t E\left[\int_u^1\nabla\eta(\xi,ds)Y(s,u)|\mathcal{F}_u\right]a(u,\xi)dw_u,\quad (4.1)$$

where Y satisfies (3.3).

Proof. By the Markov property we have $E[\eta(\xi)|\mathcal{F}_t]=V(t,C_t\xi)$, where $V(t,\psi)=E\eta(\xi^{t,\psi})$. By Theorem 3.4 and Itô's formula for $V(t,C_t\xi)$ we have

$$V(t,C_t\xi)=V(0,\xi_0)+\int_0^t\frac{\partial}{\partial j_u}V(u,C_u\xi)a(u,\xi)dw_u.$$

By Theorem 3.3

$$\frac{\partial}{\partial j_u}V(u,\psi)=E(\nabla\eta(\xi),Y(\cdot,u))=E\int_u^1\nabla\eta(\xi,ds)Y(s,u).$$

Using Corollary 2 of Theorem 2.2 we can write

$$\frac{\partial}{\partial j_u} V(u, C_u \xi) = E\bigg[\int_u^1 \nabla \eta(\xi, ds) Y(s, u) | \mathcal{F}_u\bigg].$$

Thus relation (4.1) is valid. □

The other application refers to financial mathematics. Suppose that the stock price process S_t satisfies

$$dS_t = \mu(t, S)dt + \sigma(t, S)dw_t$$

and the bond price process satisfies

$$dB_t = r(t, S)B_t dt.$$

Also suppose that $g(S)$ is a contingent claim under the stock S_t with delivery time 1.

The portfolio process $(\alpha(t, S), \beta(t, S))$, where α denotes the number of stocks and β the number of bonds, is called a self-financing if the wealth process $h(t, S) = \alpha(t, S)S_t + \beta(t, S)B_t$ can be represented as

$$h(t, S) = h(0, S_0) + \int_0^t \alpha(s, S)dS_s + \int_0^t \beta(s, S)dB_s.$$

The process $(\alpha(t, S), \beta(t, S))$, is called a replicating portfolio for the contingent claim $g(S)$, if, additionaly, $h(1, S) = g(S)$ [10].

Theorem 4.2. *Suppose that* $\sigma(t,\psi)$, $r(t,\psi)$ *belongs to* $C^{0,2}([0,1]\times W[0,1])$ *and* $g(\psi)$ *belongs to* $BC^2(W[0,1])$. *Then there exists a replicating portfolio process* (α_t, β_t) *whose wealth process is a solution of the Cauchy problem*

$$\begin{gathered} \frac{\partial}{\partial t} h(t, \psi) + \psi_t r(t, \psi) \frac{\partial}{\partial j_t} h(t, \psi) + \\ + \frac{1}{2} \sigma(t, \psi)^2 \frac{\partial^2}{\partial j_t^2} h(t, \psi) - r(t, \psi) h(t, \psi) = 0, \\ h(1, \psi) = g(\psi). \end{gathered} \tag{4.2}$$

Moreover, $\alpha(t, \psi) = \frac{\partial}{\partial j_t} h(t, \psi)$.

Proof. Let $h(t, \psi)$ be a solution of problem (4.2) existing by virtue of Theorem 3.4. Define the portfolio process by

$$\alpha(t, S) = \frac{\partial}{\partial j_t} h(t, S), \;\; \beta(t, S) = \frac{1}{B_t}\Big(h(t, S) - S_t \frac{\partial}{\partial j_t} h(t, S)\Big).$$

This strategy replicates the contingent claim, since $h(1,S) = g(S)$. To verify the self-financing property we perform the transformation

$$\int_0^t \alpha_u dS_u + \int_0^t \beta_u dB_u =$$

$$= \int_0^t \frac{\partial}{\partial j_u} h(u,\psi)\sigma(u,S)dw_u + \int_0^t \mu(u,S)\frac{\partial}{\partial j_u} h(u,\psi)du +$$

$$+ \int_0^t (h(u,S) - S_u \frac{\partial}{\partial j_u} h(u,\psi))r(u,S)du =$$

$$= \int_0^t \sigma(u,S)\frac{\partial}{\partial j_u} h(u,\psi)dw_u,$$

$$\int_0^t [h(u,S)r(u,S) + (\mu(u,S) - S_u r(u,S))\frac{\partial}{\partial j_u} h(u,\psi)]du =$$

$$= \int_0^t \sigma(u,S)\frac{\partial}{\partial j_u} h(u,\psi)dw_u +$$

$$+ \int_0^t [h(u,S)r(u,S) + (\mathcal{A}^\mu(u) - \mathcal{A}^r(u))h(u,S)]du =$$

$$= \int_0^t \sigma(u,S)\frac{\partial}{\partial j_u} h(u,\psi)dw_u + \int_0^t \mathcal{A}^\mu(u)h(u,S)du =$$

$$= h(t,S) - h(0,S_0),$$

where we use the notation

$$\mathcal{A}^\mu(s) = \mu(u,S)\frac{\partial}{\partial j_u} + \frac{1}{2}\sigma(u,S)^2\frac{\partial^2}{\partial j_u^2},$$

$$\mathcal{A}^r(s) = r(u,S)S_u\frac{\partial}{\partial j_u} + \frac{1}{2}\sigma(u,S)^2\frac{\partial^2}{\partial j_u^2}.$$

The equality is valid by Itô's formula. □

Acknowledgement

This research was supported by the Grant 1.26 of the Georgian Academy of Sciences.

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(Received 21.08.1997)

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