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OSCILLATION AND NONOSCILLATION CRITERIA FOR TWO-DIMENSIONAL SYSTEMS OF FIRST ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

A. LOMTATIDZE AND N. PARTSVANIA

Abstract. Sufficient conditions are established for the oscillation and nonoscillation of the system

$$\begin{aligned} u' &= p(t)v\,, \\ v' &= -q(t)u\,, \end{aligned}$$

where $p, q : [0, +\infty[\to [0, +\infty[$ are locally summable functions.

§ 1. Statement of the Problem and Formulation of the Main Results

Consider the system

$$\begin{aligned} u' &= p(t)v\,, \\ v' &= -q(t)u\,, \end{aligned} \tag{1}$$

where $p, q : [0, +\infty[\to [0, +\infty[$ are locally summable functions. Under a solution of system (1) is understood a vector-function $(u, v) : [0, +\infty[\to]-\infty, +\infty[$ with locally absolutely continuous components satisfying (1) almost everywhere.

A nontrivial solution (u, v) of system (1) is said to be *oscillatory* if the function u has at least one zero in any neighbourhood of $+\infty$; otherwise it is said to be *nonoscillatory*.

It is known (cf., for example, [1]) that if system (1) has an oscillatory solution, then everyone of its solutions is oscillatory.

Definition. System (1) is said to be *oscillatory* if it has at least one oscillatory solution; otherwise it is said to be *nonoscillatory*.

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It is known (cf., for example, [2]) that if

$$\int^{+\infty} p(s)\,ds = +\infty \quad \text{and} \quad \int^{+\infty} q(s)\,ds = +\infty,$$

then system (1) is oscillatory, and if

$$\int^{+\infty} p(s)\,ds < +\infty \quad \text{and} \quad \int^{+\infty} q(s)\,ds < +\infty,$$

then system (1) is nonoscillatory (see also Remark 5).

Therefore we will assume that either

$$\int^{+\infty} p(s)\,ds = +\infty \quad \text{and} \quad \int^{+\infty} q(s)\,ds < +\infty \tag{2}$$

or

$$\int^{+\infty} p(s)\,ds < +\infty \quad \text{and} \quad \int^{+\infty} q(s)\,ds = +\infty.$$

It is easily seen that if (u,v) is an oscillatory solution of (1), then the function v also has zero in any neighbourhood of $+\infty$, and the vector-function $(\overline{u},\overline{v}) \equiv (v,-u)$ is an oscillatory solution of the system

$$\begin{aligned} \overline{u}\,' &= q(t)\overline{v}\,, \\ \overline{v}\,' &= -p(t)\overline{u}\,. \end{aligned}$$

In view of this fact, it is sufficient to consider the case where conditions (2) are fulfilled.

It results from [3] that if conditions (2) are fulfilled and for some $\lambda < 1$

$$\int^{+\infty} f^{\lambda}(s)q(s)\,ds = +\infty,$$

where

$$f(t) = \int_0^t p(s)\,ds \quad \text{for} \quad t \geq 0, \tag{3}$$

then system (1) is oscillatory (for the second order linear equation, i.e., when $p(t) \equiv 1$, this assertion goes back to W. B. Fite [4] and E. Hille [5]).

Therefore, unless the contrary is specified, throughout the paper we will assume that

$$\int^{+\infty} p(s)\,ds = +\infty \quad \text{and} \quad \int^{+\infty} f^{\lambda}(s)q(s)\,ds < +\infty \quad \text{for} \quad \lambda < 1, \tag{4}$$

where f is defined by (3).

Introduce the notation

$$g_*(\lambda) = \liminf_{t\to+\infty} f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s)q(s)\,ds,$$

$$g^*(\lambda) = \limsup_{t\to+\infty} f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s)q(s)\,ds \quad \text{for} \quad \lambda < 1,$$

$$g_*(\lambda) = \liminf_{t\to+\infty} f^{1-\lambda}(t) \int_0^{t} f^\lambda(s)q(s)\,ds,$$

$$g^*(\lambda) = \limsup_{t\to+\infty} f^{1-\lambda}(t) \int_0^{t} f^\lambda(s)q(s)\,ds \quad \text{for} \quad \lambda > 1.$$

Below the new criteria for the oscillation and nonoscillation of system (1) are given in terms of the numbers $g_*(\lambda)$ and $g^*(\lambda)$. Analogous results for second order linear equations, second order nonlinear equations of the Emden–Fowler type and third order linear equations are contained in [6], [7] and [8], respectively.

Proposition 1. *If either $g_*(0) > \frac{1}{4}$ or $g_*(2) > \frac{1}{4}$, then system* (1) *is oscillatory.*

In the case of the second order equation, i.e., when $p(t) \equiv 1$, this result slightly generalizes E. Hille's theorem [5].

According to Proposition 1, it is natural to restrict our investigation to the case where

$$g_*(0) \le \frac{1}{4} \quad \text{and} \quad g_*(2) \le \frac{1}{4}\,. \tag{5}$$

Theorem 1. *Let* (5) *be fulfilled and*

$$g^*(0) > g_*(0) + \frac{1}{2}\Big(\sqrt{1-4g_*(0)} + \sqrt{1-4g_*(2)}\Big). \tag{6}$$

Then system (1) *is oscillatory.*

From this theorem we obtain in particular that if $g^*(0) > 1$, then (1) is oscillatory (for the second order equation this assertion goes back to E. Hille [5]).

Theorem 2. *Let* (5) *be fulfilled and either*

$$g^*(\lambda) > \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2}\Big(1 + \sqrt{1-4g_*(2)}\Big) \tag{7}$$

for some $\lambda < 1$ or

$$g^*(\lambda) > \frac{\lambda^2}{4(\lambda-1)} - \frac{1}{2}\Big(1 - \sqrt{1-4g_*(0)}\Big) \tag{8}$$

for some $\lambda > 1$. *Then system* (1) *is oscillatory.*

For the second order linear equation, this theorem generalizes Z. Nehari's theorem [9].

Remark 1. Below we will show (see Lemma 5 and Lemma 6) that the mapping $\lambda \longmapsto |1-\lambda| g^*(\lambda)$ does not increase for $\lambda < 1$ and does not decrease for $\lambda > 1$, while the mapping $\lambda \longmapsto |1-\lambda| g_*(\lambda)$ does not decrease for $\lambda < 1$ and does not increase for $\lambda > 1$. Moreover,

$$\lim_{\lambda \to 1-} |1-\lambda| g_*(\lambda) = \lim_{\lambda \to 1+} |1-\lambda| g_*(\lambda)$$

and

$$\lim_{\lambda \to 1-} |1-\lambda| g^*(\lambda) = \lim_{\lambda \to 1+} |1-\lambda| g^*(\lambda).$$

Corollary 1. *Let* (5) *be fulfilled and*

$$\lim_{\lambda \to 1} |1-\lambda| g^*(\lambda) > \frac{1}{4}\,. \tag{9}$$

Then system (1) *is oscillatory.*

Corollary 2. *Let* (5) *be fulfilled and for some* $\lambda \neq 1$

$$|1-\lambda| g_*(\lambda) > \frac{1}{4}\,. \tag{10}$$

Then system (1) *is oscillatory.*

Corollary 3. *Let* (5) *be fulfilled and*

$$\limsup_{t \to +\infty} \frac{1}{\ln f(t)} \int_1^t f(s) q(s)\, ds > \frac{1}{4}\,. \tag{11}$$

Then system (1) *is oscillatory.*

Corollary 4. *Let* (5) *be fulfilled and*

$$\limsup_{\lambda \to 1-} (1-\lambda) \int_1^{+\infty} f^{\lambda}(s) q(s)\, ds > \frac{1}{4}\,. \tag{12}$$

Then system (1) *is oscillatory.*

Remark 2. Inequalities (9)–(12) are exact and cannot be weakened. Indeed, let $p(t) > 0$ for $t > 0$ and $\int^{+\infty} p(s)\, ds = +\infty$. Suppose that $q(t) =$

$\frac{f'(t)}{4f^2(t)}$ for $t > 1$, where the function f is defined by (3). It is easy to check that

$$|1-\lambda|g^*(\lambda) = \frac{1}{4}\,, \quad |1-\lambda|g_*(\lambda) = \frac{1}{4}\,,$$
$$\lim_{t\to+\infty} \frac{1}{\ln f(t)} \int_1^t f(s)q(s)\,ds = \frac{1}{4}\,,$$
$$\lim_{\lambda\to 1-} (1-\lambda) \int_1^{+\infty} f^\lambda(s)q(s)\,ds = \frac{1}{4}\,.$$

However system (1) has the nonoscillatory solution $(\sqrt{f}, \frac{1}{2\sqrt{f}})$.

Theorem 3. *Let for some* $\lambda \neq 1$

$$|1-\lambda|g_*(\lambda) > \frac{(2\lambda-1)(3-2\lambda)}{4} \quad \textit{and} \quad |1-\lambda|g^*(\lambda) < \frac{1}{4}\,. \tag{13}$$

Then system (1) *is nonoscillatory.*

Remark 3. As it will be seen from the proof, this theorem is also valid for the case where q is not, in general, of constant sign.

Corollary 5. *Let conditions* (5) *hold and for some* $\lambda \in]-\infty, 1-\sqrt{\frac{1}{4}-g_*(0)}[\cup]1+\sqrt{\frac{1}{4}-g_*(2)}, +\infty[$ *the inequality*

$$|1-\lambda|g^*(\lambda) < \frac{1}{4}$$

be fulfilled. Then system (1) *is nonoscillatory.*

Remark 4. Consider the system

$$\begin{aligned} u' &= p_1(t)u + p_2(t)v\,, \\ v' &= q_1(t)u + q_2(t)v\,, \end{aligned} \tag{14}$$

where $p_i, q_i : [0,+\infty[\,\to\,]-\infty,+\infty[$ $(i=1,2)$ are locally summable functions such that $p_2(t) \geq 0$ and $q_1(t) \leq 0$ for $t > 0$. It is easy to see that system (14) is equivalent to system (1) with

$$p(t) = p_2(t)\exp\left[\int_0^t \big(q_2(s)-p_1(s)\big)\,ds\right],$$
$$q(t) = -q_1(t)\exp\left[-\int_0^t \big(q_2(s)-p_1(s)\big)\,ds\right] \quad \text{for} \quad t > 0.$$

Therefore from Theorems 1–3 the oscillation and nonoscillation criteria for system (14) can be obtained.

§ 2. Some Auxiliary Statements

Throughout this section, we will assume that $q \not\equiv 0$ in any neighbourhood of $+\infty$ and $p(t) > 0$ for $0 < t < 1$.

Lemma 1. *Let* (u, v) *be a nonoscillatory solution of* (1). *Then there exists* $t_0 > 0$ *such that*

$$u(t)v(t) > 0 \quad \textit{for} \quad t > t_0.$$

Proof. For the sake of definiteness, we assume that $u(t) > 0$ for $t > t_0$. Suppose that $v(t_1) < 0$ for some $t_1 > t_0$. Then from (1) we find that $v(t) \le v(t_1)$ for $t > t_1$ and

$$u(t) = u(t_1) + \int_{t_1}^{t} p(s)v(s)\,ds \le u(t_1) + v(t_1)\int_{t_1}^{t} p(s)\,ds \quad \text{for} \quad t > t_1.$$

According to (4), from the latter inequality we obtain the contradiction $u(t_2) < 0$ for some $t_2 > t_1$. □

Lemma 2. *Let* (5) *be fulfilled and system* (1) *be nonoscillatory. Then there exists* $t_0 > 0$ *such that the equation*

$$\rho' = -q(t) - p(t)\rho^2 \tag{15}$$

has the solution $\rho : [t_0, +\infty[\, \to\,]0, +\infty[$. *Moreover,*

$$\begin{aligned} \liminf_{t\to+\infty} f(t)\rho(t) &\ge \frac{1 - \sqrt{1 - 4g_*(0)}}{2}\,, \\ \limsup_{t\to+\infty} f(t)\rho(t) &\le \frac{1 + \sqrt{1 - 4g_*(2)}}{2}\,, \end{aligned} \tag{16}$$

where f *is defined by* (3).

Proof. Let (u, v) be a nonoscillatory solution of system (1). Choose $a > 0$ so that $u(t) \ne 0$ for $t > a$. It is easy to see that the function $\rho(t) = \frac{v(t)}{u(t)}$ satisfies equation (15) for $t > a$. By Lemma 1, there exists $t_0 > a$ such that $\rho(t) > 0$ for $t > t_0$.

Introduce the notation

$$r = \liminf_{t\to+\infty} f(t)\rho(t), \quad R = \limsup_{t\to+\infty} f(t)\rho(t). \tag{17}$$

From (15), we have

$$-\frac{\rho'(t)}{\rho^2(t)} = \frac{q(t)}{\rho^2(t)} + p(t) \quad \text{for} \quad t > t_0.$$

Integrating this equality from t_0 to t, we obtain

$$\rho(t)\int_{t_0}^{t} p(s)\,ds < 1 \quad \text{for} \quad t > t_0.$$

Hence, by (4), we have

$$\lim_{t\to+\infty} \rho(t) = 0. \tag{18}$$

Taking into account (18) and integrating (15) from t to $+\infty$, we obtain

$$f(t)\rho(t) = f(t)\int_{t}^{+\infty} q(s)\,ds + f(t)\int_{t}^{+\infty} p(s)\rho^2(s)\,ds \quad \text{for} \quad t > t_0. \tag{19}$$

Hence we easily find that

$$r \geq g_*(0). \tag{20}$$

Multiplying (15) by f^2 and integrating from t_0 to t, we obtain

$$\begin{aligned} f(t)\rho(t) = \frac{1}{f(t)}\,f^2(t_0)\rho(t_0) - \frac{1}{f(t)}\int_{t_0}^{t} f^2(s)q(s)\,ds\, + \\ + \frac{1}{f(t)}\int_{t_0}^{t} p(s)f(s)\rho(s)\big(2 - f(s)\rho(s)\big)\,ds \quad \text{for} \quad t > t_0, \end{aligned} \tag{21}$$

whence we get

$$R \leq 1 - g_*(2). \tag{22}$$

Now suppose that $g_*(0) \neq 0$ and $g_*(2) \neq 0$ (otherwise, estimates (16) follow from (20) and (22)). Let $0 < \varepsilon < \min\{g_*(0), g_*(2)\}$. Choose $t_\varepsilon > t_0$ so that

$$r - \varepsilon < f(t)\rho(t) < R + \varepsilon \quad \text{for} \quad t > t_\varepsilon,$$

$$f(t)\int_{t}^{+\infty} q(s)ds > g_*(0) - \varepsilon, \quad \frac{1}{f(t)}\int_{t_0}^{t} f^2(s)q(s)\,ds > g_*(2) - \varepsilon \quad \text{for} \quad t > t_\varepsilon.$$

From (19) and (21) we have

$$f(t)\rho(t) \geq g_*(0) - \varepsilon + (r - \varepsilon)^2 \quad \text{for} \quad t > t_\varepsilon,$$

$$f(t)\rho(t) \leq \frac{1}{f(t)}\,f^2(t_0)\rho(t_0) - g_*(2) + \varepsilon + (R + \varepsilon)(2 - R - \varepsilon) \quad \text{for} \quad t > t_\varepsilon.$$

These inequalities readily imply

$$r \geq g_*(0) + r^2, \quad R \leq R(2 - R) - g_*(2). \tag{23}$$

Therefore

$$r \geq \frac{1}{2}\left(1-\sqrt{1-4g_*(0)}\right), \quad R \leq \frac{1}{2}\left(1+\sqrt{1-4g_*(2)}\right).$$

Thus the lemma is proved. □

Lemma 3. *For the nonoscillation of system* (1) *it is necessary and sufficient that for some* $\lambda \neq 1$ *the system*

$$\begin{aligned} u' &= p(t)v\,, \\ v' &= l_\lambda(t)u + h_\lambda(t)v \end{aligned} \tag{24}$$

be nonoscillatory, where

$$\begin{aligned} l_\lambda(t) &= \frac{p(t)}{f^2(t)}\left[\lambda F_\lambda(t)\operatorname{sgn}(1-\lambda) - F_\lambda^2(t)\right], \\ h_\lambda(t) &= -\frac{2p(t)}{f^{2-\lambda}(t)}\,F_\lambda(t)\operatorname{sgn}(1-\lambda) \quad \textit{for} \quad t>0, \end{aligned} \tag{25}$$

$$\begin{aligned} F_\lambda(t) &= f^{1-\lambda}(t)\int_t^{+\infty} f^\lambda(s)q(s)\,ds \quad \textit{for} \quad t \geq 0 \quad \textit{and} \quad \lambda<1, \\ F_\lambda(t) &= f^{1-\lambda}(t)\int_0^{t} f^\lambda(s)q(s)\,ds \quad \textit{for} \quad t \geq 0 \quad \textit{and} \quad \lambda>1. \end{aligned} \tag{26}$$

Proof. The equality

$$\rho(t) = f^\lambda(t)\,\frac{v(t)}{u(t)} - \frac{F_\lambda(t)}{f^{1-\lambda}(t)}\,\operatorname{sgn}(1-\lambda)$$

establishes a correlation between a nonoscillatory solution (u,v) of system (1) and a solution ρ (defined in some neighbourhood of $+\infty$) of the equation

$$\begin{aligned} \rho' = -\frac{p(t)}{f^\lambda(t)}\,\rho^2 &+ \Big(\frac{\lambda p(t)}{f(t)} - \frac{2p(t)F_\lambda(t)}{f^{2-\lambda}(t)}\,\operatorname{sgn}(1-\lambda)\Big)\rho + \\ &+ \frac{p(t)}{f^{2-\lambda}(t)}\,\big(\lambda F_\lambda(t)\operatorname{sgn}(1-\lambda) - F_\lambda^2(t)\big). \end{aligned} \tag{27}$$

On the other hand, the equality

$$\rho(t) = f^\lambda(t)\,\frac{v(t)}{u(t)}$$

establishes a correlation between a nonoscillatory solution of system (24) and a solution ρ (defined in some neighbourhood of $+\infty$) of equation (27). Consequently the nonoscillation of each of systems (1) and (24) implies the nonoscillation of the other. □

In the next lemma a sufficient condition is established for the nonoscillation of the system

$$\begin{aligned} u' &= p(t)v\,, \\ v' &= l(t)u + h(t)v\,, \end{aligned} \tag{28}$$

where $l, h : [0, +\infty[\to] - \infty, +\infty[$ are locally summable functions.

Lemma 4. *Let the function $\rho : [t_0, +\infty[\to] - \infty, 0[\cup]0, +\infty[$ be locally absolutely continuous and*

$$\rho'(t) \le l(t) + h(t)\rho(t) - p(t)\rho^2(t) \quad \textit{for} \quad t > t_0.$$

Then system (28) *is nonoscillatory.*

Proof. Assume the contrary. Let (u, v) be an oscillatory solution of system (28). Let $t_2 > t_1 > t_0$ be chosen so that

$$u(t) > 0 \quad \text{for} \quad t_1 < t < t_2, \quad u(t_1) = 0, \quad u(t_2) = 0.$$

It is clear that

$$v(t_1) > 0 \quad \text{and} \quad v(t_2) < 0.$$

Introduce the notation

$$\varphi(t) = \exp\left[-\int_{t_1}^t h(s)\,ds\right], \quad \sigma(t) = \frac{v(t)}{u(t)}\,\varphi(t), \quad \rho_0(t) = \rho(t)\varphi(t),$$

$$l_0(t) = l(t)\varphi(t), \quad p_0(t) = \frac{p(t)}{\varphi(t)} \quad \text{for} \quad t_1 \le t \le t_2.$$

It is easily seen that

$$\begin{aligned} &\sigma'(t) = l_0(t) - p_0(t)\sigma^2(t) \quad \text{for} \quad t_1 < t < t_2, \\ &\rho_0'(t) \le l_0(t) - p_0(t)\rho_0^2(t) \quad \text{for} \quad t_1 < t < t_2. \end{aligned} \tag{29}$$

For the sake of definiteness, we assume that $\rho(t) > 0$ for $t > t_0$. Since $\sigma(t_1+) = +\infty$ and $\sigma(t_2-) = -\infty$, there exist $t_3 \in]t_1, t_2[$ and $\varepsilon \in]0, t_2 - t_3[$ such that $\sigma(t_3) = \rho_0(t_3)$ and

$$0 < \sigma(t) < \rho_0(t) \quad \text{for} \quad t_3 < t < t_3 + \varepsilon. \tag{30}$$

Due to this fact, from (29) we have

$$\sigma(t) = \sigma(t_3) + \int_{t_3}^t l_0(s)\,ds - \int_{t_3}^t p_0(s)\sigma^2(s)\,ds \ge$$

$$\ge \rho_0(t_3) + \int_{t_3}^t l_0(s)\,ds - \int_{t_3}^t p_0(s)\rho_0^2(s)\,ds \ge \rho_0(t) \quad \text{for} \quad t_3 < t < t_3 + \varepsilon.$$

But the latter inequality contradicts (30). The obtained contradiction proves the validity of the lemma. □

Remark 5. Let

$$\int^{+\infty} q(s)\,ds < +\infty, \qquad \int^{+\infty} p(s)\,ds < +\infty$$

and

$$\rho(t) = \int_t^{+\infty} q(s)\,ds + \int_t^{+\infty} p(s)\,ds \quad \text{for} \quad t > 0.$$

Choose $t_0 > 0$ so that $\rho(t) < 1$ for $t > t_0$. Then it is obvious that

$$\rho'(t) \le -q(t) - p(t)\rho^2(t) \quad \text{for} \quad t > t_0.$$

Consequently, according to Lemma 4, system (1) is nonoscillatory.

Lemma 5. *Let* $g^*(\lambda) < +\infty$ *for* $\lambda \ne 1$. *Then the mapping* $\lambda \longmapsto |1-\lambda| g^*(\lambda)$ $(\lambda \longmapsto |1-\lambda| g_*(\lambda))$ *does not increase (does not decrease) for* $\lambda < 1$ *and does not decrease (does not increase) for* $\lambda > 1$.

Proof. We prove this lemma only for the case where $\lambda < 1$. For $\lambda > 1$ the lemma is proved in a similar way. Let $\varepsilon > 0$. Choose $t_\varepsilon > 0$ such that

$$g_*(\lambda) - \varepsilon < f^{1-\lambda}(t) \int_t^{+\infty} f^{\lambda}(s) q(s)\,ds < g^*(\lambda) + \varepsilon \quad \text{for} \quad t > t_\varepsilon. \tag{31}$$

It is easy to see that

$$f^{1-\mu}(t) \int_t^{+\infty} f^{\mu}(s) q(s)\,ds = f^{1-\lambda}(t) \int_t^{+\infty} f^{\lambda}(s) q(s)\,ds +$$
$$+(\mu-\lambda) f^{1-\mu}(t) \int_t^{+\infty} f^{\lambda-2}(s) p(s) \Big[f^{1-\lambda}(s) \int_s^{+\infty} f^{\lambda}(\xi) q(\xi)\,d\xi \Big] ds$$
$$\text{for} \quad \mu < 1 \quad \text{and} \quad t > 0.$$

Hence we find

$$\big(g_*(\lambda) - \varepsilon\big)\Big(1 + \frac{\mu-\lambda}{1-\mu}\Big) < f^{1-\mu}(t) \int_t^{+\infty} f^{\mu}(s) q(s)\,ds <$$
$$< \big(g^*(\lambda) + \varepsilon\big)\Big(1 + \frac{\mu-\lambda}{1-\mu}\Big) \quad \text{for} \quad \lambda < \mu \quad \text{and} \quad t > t_\varepsilon.$$

Consequently

$$(1-\lambda) g_*(\lambda) \le (1-\mu) g_*(\mu), \quad (1-\mu) g^*(\mu) \le (1-\lambda) g^*(\lambda) \quad \text{for} \quad \lambda < \mu.$$

Thus the lemma is proved. □

Lemma 6. *Let* $g^*(\lambda) < +\infty$ *for* $\lambda \ne 1$. *Then*

$$\begin{aligned} \lim_{\lambda \to 1-} (1-\lambda) g_*(\lambda) &= \lim_{\lambda \to 1+} (\lambda - 1) g_*(\lambda), \\ \lim_{\lambda \to 1-} (1-\lambda) g^*(\lambda) &= \lim_{\lambda \to 1+} (\lambda - 1) g^*(\lambda). \end{aligned} \tag{32}$$

Proof. Let $\lambda < 1$, $\mu > 1$ and $\varepsilon > 0$. Choose $t_\varepsilon > 0$ such that inequalities (31) be fulfilled and

$$g_*(\mu) - \varepsilon < f^{1-\mu}(t) \int_0^t f^\mu(s) q(s)\, ds < g^*(\mu) + \varepsilon \quad \text{for} \quad t > t_\varepsilon.$$

It is easy to see that

$$f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s) q(s)\, ds = -f^{1-\mu}(t) \int_0^t f^\mu(s) q(s)\, ds +$$
$$+(\mu-\lambda) f^{1-\lambda}(t) \int_t^{+\infty} p(s) f^{\lambda-2}(s) \left[f^{1-\mu}(s) \int_s^{+\infty} f^\mu(\xi) q(\xi)\, d\xi \right] ds \ \text{ for } \ t>0,$$
$$f^{1-\mu}(t) \int_0^t f^\mu(s) q(s)\, ds = -f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s) q(s)\, ds +$$
$$+(\mu-\lambda) f^{1-\mu}(t) \int_1^t p(s) f^{\mu-2}(s) \left[f^{1-\lambda}(s) \int_s^{+\infty} f^\lambda(\xi) q(\xi)\, d\xi \right] ds \ \text{ for } \ t>0.$$

From these equalities we have

$$\frac{\mu-\lambda}{1-\lambda} \big(g_*(\mu) - \varepsilon \big) - f^{1-\mu}(t) \int_0^t f^\mu(s) q(s)\, ds < f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s) q(s)\, ds <$$
$$< \frac{\mu-\lambda}{1-\lambda} \big(g^*(\mu) + \varepsilon \big) \ \text{ for } \ t>t_\varepsilon,$$
$$\frac{\mu-\lambda}{\mu-1} \big(g_*(\lambda) - \varepsilon \big) - f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s) q(s)\, ds < f^{1-\mu}(t) \int_0^t f^\mu(s) q(s)\, ds <$$
$$< \frac{\mu-\lambda}{\mu-1} \big(g^*(\lambda) + \varepsilon \big) \ \text{ for } \ t > t_\varepsilon.$$

Hence we obtain

$$(\mu-\lambda) g_*(\mu) - (1-\lambda) g^*(\mu) \le (1-\lambda) g_*(\lambda), \ \ (1-\lambda) g^*(\lambda) \le (\mu-\lambda) g^*(\mu),$$
$$(\mu-\lambda) g_*(\lambda) - (\mu-1) g^*(\lambda) \le (\mu-1) g_*(\mu), \ \ (\mu-1) g^*(\mu) \le (\mu-\lambda) g^*(\lambda).$$

Now by Lemma 5 we can conclude that equalities (32) are valid. $\square$

The next lemma can be proved similarly by using the equalities

$$\frac{1}{\ln f(t)} \int_1^t f(s) q(s)\, ds = -\frac{f^{1-\lambda}(t)}{\ln f(t)} \int_t^{+\infty} f^\lambda(s) q(s)\, ds +$$
$$+ \frac{1}{\ln f(t)} f^{1-\lambda}(t) \int_1^{+\infty} f^\lambda(s) q(s)\, ds +$$
$$+ \frac{1-\lambda}{\ln f(t)} \int_1^{+\infty} \frac{f'(s)}{f(s)} \left[f^{1-\lambda}(s) \int_s^{+\infty} f^\lambda(\xi) q(\xi)\, d\xi \right] ds \ \text{ for } \ t>1 \ \text{ and } \ \lambda<1,$$

$$\int_1^{+\infty} f^\lambda(s)q(s)\,ds = (\mu-\lambda)\int_1^{+\infty} f^{\lambda-2}(s)f'(s)\left[f^{1-\mu}(s)\int_1^s f^\mu(\xi)q(\xi)\,d\xi\right]ds$$
$$\text{for } \lambda<1 \text{ and } \mu>1.$$

Lemma 7. *Let $g^*(\lambda)<+\infty$ for some $\lambda<1$ and $g^*(\mu)<+\infty$ for some $\mu>1$. Then*

$$\limsup_{t\to+\infty}\frac{1}{\ln f(t)}\int_1^t f(s)q(s)\,ds \le (1-\lambda)g^*(\lambda),$$
$$\limsup_{\lambda\to1-}(1-\lambda)\int_1^{+\infty} f^\lambda(s)q(s)\,ds \le (\mu-1)g^*(\mu).$$

§ 3. Proof of the Main Results

Proof of Proposition 1. Assume the contrary. Let (u,v) be a nonoscillatory solution of system (1). Then according to Lemma 1, there exists $t_0>0$ such that $u(t)v(t)>0$ for $t_0>0$. It is easy to see that the function $\rho(t)=\frac{v(t)}{u(t)}$ for $t>t_0$ satisfies equation (15). Similarly to the proof Lemma 2, we can see that inequalities (23) are fulfilled, where r and R are defined by (17). However from (23) we have

$$g_*(0)\le\frac14 \quad\text{and}\quad g_*(2)\le\frac14.$$

But this contradicts the conditions of the proposition. □

Proof of Theorem 1. Assume the contrary. Let system (1) be nonoscillatory. Then according to Lemma 2, equation (15) has the solution $\rho:[t_0,+\infty[\to]0,+\infty[$ satisfying (16). Integrating (15) from t to $+\infty$ and taking into account (18), we can conclude that equality (19) is valid. Suppose that $g_*(0)\ne0$. Let $0<\varepsilon<g_*(0)$. Choose $t_\varepsilon>t_0$ so that

$$\frac12\left(1-\sqrt{1-4g_*(0)}\right)-\varepsilon<f(t)\rho(t)<\frac12\left(1+\sqrt{1-4g_*(2)}\right)+\varepsilon \quad\text{for } t>t_\varepsilon.$$

From (19) we easily find that

$$f(t)\int_t^{+\infty} q(s)\,ds<\frac12\left(1+\sqrt{1-4g_*(2)}\right)+\varepsilon-$$
$$-\left[\frac12\left(1-\sqrt{1-4g_*(0)}\right)-\varepsilon\right]^2 \quad\text{for}\quad t>t_\varepsilon.$$

Hence we have

$$g^*(0)\le g_*(0)+\frac12\left(\sqrt{1-4g_*(0)}+\sqrt{1-4g_*(2)}\right).$$

For $g_*(0) = 0$ the validity of the latter inequality is proved in a similar way. On the other hand, this inequality contradicts condition (6). $\square$

Proof of Theorem 2. Suppose that system (1) is nonoscillatory. Then according to Lemma 2, equation (15) has the solution $\rho : [t_0, +\infty[\to]0, +\infty[$ satisfying estimates (16). Multiplying (15) by f^λ and integrating from t to $+\infty$ if $\lambda < 1$ or from t_0 to t if $\lambda > 1$, we get

$$f^{1-\lambda}(t) \int_t^{+\infty} f^\lambda(s) q(s)\, ds = f(t)\rho(t) +$$

$$+ f^{1-\lambda}(t) \int_t^{+\infty} f'(s) f^{\lambda-1}(s)\rho(s)\big(\lambda - f(s)\rho(s)\big)\, ds \le f(t)\rho(t) + \frac{\lambda^2}{4(1-\lambda)}$$

$$\text{for} \quad t > t_0 \quad \text{and} \quad \lambda < 1,$$

$$f^{1-\lambda}(t) \int_{t_0}^t f^\lambda(s) q(s)\, ds = -f(t)\rho(t) + f^\lambda(t_0)\rho(t_0) f^{1-\lambda}(t) +$$

$$+ f^{1-\lambda}(t) \int_{t_0}^t f'(s) f^{\lambda-1}(s)\rho(s)\big(\lambda - f(s)\rho(s)\big)\, ds \le$$

$$\le -f(t)\rho(t) + f^\lambda(t_0)\rho(t_0) f^{1-\lambda}(t) + \frac{\lambda^2}{4(\lambda-1)}\Big(1 - f^{1-\lambda}(t) f^{\lambda-1}(t_0)\Big)$$

$$\text{for} \quad t > t_0 \quad \text{and} \quad \lambda > 1.$$

Taking into account (16), from these inequalities we find

$$g^*(\lambda) \le \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2}\Big(1 + \sqrt{1 - 4g_*(2)}\Big) \quad \text{for} \quad \lambda < 1,$$

$$g^*(\lambda) \le \frac{\lambda^2}{4(\lambda-1)} - \frac{1}{2}\Big(1 - \sqrt{1 - 4g_*(0)}\Big) \quad \text{for} \quad \lambda > 1.$$

But this contradicts inequalities (7) and (8). $\square$

Proof of Corollary 1. Suppose that $g^*(\lambda) < +\infty$ for $\lambda \neq 1$ (otherwise, by Theorem 2 system (1) is oscillatory). By Lemma 6, the limit in the left-hand side of (9) exists. Obviously,

$$\lim_{\lambda \to 1-} \left[(1-\lambda) g^*(\lambda) - \frac{\lambda^2}{4} - \frac{1-\lambda}{2}\Big(1 + \sqrt{1 - 4g_*(2)}\Big)\right] > 0.$$

This implies that (7) is fulfilled for some $\lambda < 1$. Therefore by Theorem 2 system (1) is oscillatory. $\square$

Taking into account Corollary 1, Lemma 5 and Lemma 7, we can easily make sure that Corollaries 2–4 are valid.

Proof of Theorem 3. Define the functions l_λ, h_λ and F_λ by (25) and (26). According to (13), there exists $t_0 > 0$ such that

$$F_\lambda^2(t) + \frac{2\lambda^2 - 4\lambda + 1}{2|1-\lambda|} F_\lambda(t) + \frac{(2\lambda-1)(3-2\lambda)}{16(1-\lambda)^2} < 0 \quad \text{for} \quad t > t_0.$$

Consequently

$$\rho'(t) \le l_\lambda(t) + h_\lambda(t)\rho(t) - p(t)\rho^2(t) \quad \text{for} \quad t > t_0,$$

where $\rho(t) = \frac{1-2\lambda}{4(1-\lambda)f(t)}$ for $t > t_0$. In view of this fact, by Lemma 3 and Lemma 4 system (1) is nonoscillatory. □

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References

1. J. D. Mirzov, On some analogues of Sturm's and Kneser's theorems for nonlinear systems. *J. Math. Anal. Appl.* **53**(1976), No. 2, 418–425.
2. J. D. Mirzov, Asymptotic behaviour of solutions of systems of nonlinear non-autonomous ordinary differential equations. (Russian) *Maikop*, 1993.
3. J. D. Mirzov, On oscillation of solutions of a certain system of differential equations. (Russian) *Mat. Zametki* **23**(1978), No. 3, 401–404.
4. W. B. Fite, Concerning the zeros of the solutions of certain differential equations. *Trans. Amer. Math. Soc.* **19**(1917), 341–352.
5. E. Hille, Non-oscillation theorems. *Trans. Amer. Math. Soc.* **64** (1948), 234–252.
6. A. Lomtatidze, Oscillation and nonoscillation criteria for second order linear differential equation. *Georgian Math. J.* **4**(1997), No. 2, 129–138.
7. A. Lomtatidze, Oscillation and nonoscillation of Emden-Fowler type equation of second order. *Arch. Math.* **32**(1996), No. 3, 181–193.
8. A. Lomtatidze, On oscillation of the third order linear equation. (Russian) *Differentsial'nye Uravneniya* **32**(1996), No. 10.
9. Z. Nehari, Oscillation criteria for second-order linear differential equations. *Trans. Amer. Math. Soc.* **85**(1957), 428–445.

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Authors' addresses:

A. Lomtatidze
Masaryk University
Department of Mathematics
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic

N. Partsvania
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia