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OSCILLATION AND NONOSCILLATION IN DELAY OR ADVANCED DIFFERENTIAL EQUATIONS AND IN INTEGRODIFFERENTIAL EQUATIONS

I.-G. E. KORDONIS AND CH. G. PHILOS

ABSTRACT. Some new oscillation and nonoscillation criteria are given for linear delay or advanced differential equations with variable coefficients and not (necessarily) constant delays or advanced arguments. Moreover, some new results on the existence and the nonexistence of positive solutions for linear integrodifferential equations are obtained.

1. Introduction and Preliminaries

With the past two decades, the oscillatory behavior of solutions of differential equations with deviating arguments has been studied by many authors. The problem of the oscillations caused by the deviating arguments (delays or advanced arguments) has been the subject of intensive investigations. Among numerous papers dealing with the study of this problem we choose to refer to the papers by Arino, Györi and Jawhari [1], Györi [2], Hunt and Yorke [3], Jaroš and Stavroulakis [4], Koplatadze and Chanturija [5], Kwong [6], Ladas [7], Ladas, Sficas and Stavroulakis [8, 9], Ladas and Stavroulakis [10], Li [11, 12], Nadareishvili [13], Philos [14, 15, 16], Philos and Sficas [17], Tramov [18], and Yan [19] and to the references cited therein; see also the monographs by Erbe, Kong and Zhang [20], Györi and Ladas [21], and Ladde, Lakshmikantham and Zhang [22] and the references therein. In particular, we mention the sharp oscillation results by Ladas [7] and Koplatadze and Chanturija [5] (see also Kwong [6]); for some very recent related results we refer to Jaroš and Stavroulakis [4], Li [11, 12], and Philos and Sficas [17] (see also the references cited therein). In the special case of an autonomous delay or advanced differential equation it is known that a necessary and sufficient condition for the oscillation of all solutions is that its characteristic equation have no real roots; such a result was proved

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by Arino, Györi and Jawhari [1], Ladas, Sficas and Stavroulakis [8, 9], and Tramov [18] (see also Arino and Györi [23] for the general case of neutral differential systems and Philos, Purnaras and Sficas [24] and Philos and Sficas [25] for some general forms of neutral differential equations). Also, for a class of delay differential equations with periodic coefficients, a necessary and sufficient condition for the oscillation of all solutions is given by Philos [15] (in this case a characteristic equation is also considered). For the existence of positive solutions of delay differential equations we refer to the paper by Philos [26]. The reader is referred to the books by Driver [27], Hale [28], and Hale and Vertuyn Lunel [29] for the basic theory of delay differential equations.

The literature is scarce concerning the oscillation and nonoscillation of solutions of integrodifferential equations. We mention the papers by Gopalsamy [30, 31, 32], Györi and Ladas [33], Kiventidis [34], Ladas, Philos and Sficas [35], Philos [36, 37, 38], and Philos and Sficas [39] dealing with the problem of the existence and the nonexistence of positive solutions of integrodifferential equations or of systems of such equations. Integrodifferential equations belong to the class of differential equations with unbounded delays; for a survey on equations with unbounded delays see the paper by Corduneanu and Lakshmikantham [40]. For the basic theory of integrodifferential equations (and, more generally, of integral equations) we refer to the books by Burton [41] and Corduneanu [42].

In this paper we deal with the oscillation and nonoscillation problem for first order linear delay or advanced differential equations as well as for first order linear integrodifferential equations. The discrete analogs of the results of this paper have recently been obtained by the authors [43] and the second author [44].

Consider the delay differential equation

$$x'(t) + \sum_{j\in J} p_j(t)x(t-\tau_j(t)) = 0 \tag{E$_1$}$$

and the advanced differential equation

$$x'(t) - \sum_{j\in J} p_j(t)x(t+\tau_j(t)) = 0, \tag{E$_2$}$$

where J *is an* (*nonempty*) *initial segment of natural numbers and for* $j \in J$ p_j *and* τ_j *are nonnegative continuous real-valued functions on the interval* $[0,\infty)$. *For the delay equation* (E_1) *it will be supposed that the set* J *is necessarily finite and that the delays* τ_j *for* $j \in J$ *satisfy*

$$\lim_{t\to\infty}[t-\tau_j(t)] = \infty \quad \text{for} \quad j \in J;$$

with respect to the advanced equation (E_2) *the set* J *may be infinite.*

Let $t_0 \geq 0$. By a *solution on* $[t_0, \infty)$ of the delay differential equation (E_1) we mean a continuous real-valued function x defined on the interval $[t_{-1}, \infty)$, where

$$t_{-1} = \min_{j \in J} \min_{t \geq t_0} [t - \tau_j(t)],$$

which is continuously differentiable on $[t_0, \infty)$ and satisfies (E_1) for all $t \geq t_0$. (Note that $t_{-1} \leq t_0$ and that t_{-1} depends on the delays τ_j for $j \in J$ and the initial point t_0.) A *solution on* $[t_0, \infty)$ of the advanced differential equation (E_2) is a continuously differentiable function x on the interval $[t_0, \infty)$, which satisfies (E_2) for all $t \geq t_0$.

As usual, a solution of (E_1) or (E_2) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise the solution is called *nonoscillatory.*

Consider also the integrodifferential equations

$$x'(t) + q(t) \int_0^t K(t-s)x(s)ds = 0 \tag{E_3}$$

and

$$x'(t) + r(t) \int_{-\infty}^t K(t-s)x(s)ds = 0 \tag{E_4}$$

as well as the integrodifferential inequalities

$$y'(t) + q(t) \int_0^t K(t-s)y(s)ds \leq 0 \tag{I_1}$$

and

$$y'(t) + r(t) \int_{-\infty}^t K(t-s)y(s)ds \leq 0, \tag{I_2}$$

where *the kernel K is a nonnegative continuous real-valued function on the interval* $[0, \infty)$, *and the coefficients q and r are nonnegative continuous real-valued functions on the interval* $[0, \infty)$ *and the real line* $\mathbb{R}$, *respectively.*

If $t_0 \geq 0$, by a *solution on* $[t_0, \infty)$ of the integrodifferential equation (E_3) (resp. of the integrodifferential inequality (I_1)) we mean a continuous real-valued function x [resp. y] defined on the interval $[0, \infty)$, which is continuously differentiable on $[t_0, \infty)$ and satisfies (E_3) [resp. (I_1)] for all $t \geq t_0$. In particular, a *solution on* $[0, \infty)$ of (I_1) is a continuously differentiable real-valued function y on the interval $[0, \infty)$ satisfying (I_1) for every $t \geq 0$. Moreover, if $t_0 \in \mathbb{R}$, then a *solution on* $[t_0, \infty)$ of the integrodifferential equation (E_4) [resp. of the integrodifferential inequality (I_2)] is a continuous real-valued function x [resp. y] defined on the real line $\mathbb{R}$, which is continuously differentiable on $[t_0, \infty)$ and satisfies (E_4) [resp. (I_2)] for all $t \geq t_0$. Also, a continuously differentiable real-valued function y on the real line $\mathbb{R}$, which satisfies (I_2) for every $t \in \mathbb{R}$, is called a *solution on* $\mathbb{R}$ of (I_2).

The results of the paper will be presented in Sections 2, 3, 4 and 5. Section 2 contains some results which provide sufficient conditions for the oscillation of all solutions of the delay differential equation (E_1) or of the advanced differential equation (E_2). Conditions which guarantee the existence of a positive solution of the delay equation (E_1) or of the advanced equation (E_2) will be given in Section 3. Section 4 deals with the nonexistence of positive solutions of the integrodifferential inequalities (I_1) and (I_2) (and, in particular, of the integrodifferential equations (E_3) and (E_4)). More precisely, in Section 4 necessary conditions are given for (E_3) or, more generally, for (I_1) to have solutions on $[t_0, \infty)$, where $t_0 \geq 0$, which are positive on $[0, \infty)$; analogously, necessary conditions are derived for (E_4) or, more generally, for (I_2) to have solutions on $[t_0, \infty)$, where $t_0 \in \mathbb{R}$, which are positive on $\mathbb{R}$. In Section 5, sufficient conditions are obtained for the equation (E_3) to have a solution on $[t_0, \infty)$, where $t_0 > 0$, which is positive on $[0, \infty)$ and tends to zero at ∞; similarly, sufficient conditions are given for the existence of a solution on $[t_0, \infty)$, where $t_0 \in \mathbb{R}$, of the equation (E_4) which is positive on $\mathbb{R}$ and tends to zero at ∞.

2. Sufficient Conditions for the Oscillation of Delay or Advanced Differential Equations

In this section, we will give conditions which guarantee the oscillation of all solutions of the delay differential equation (E_1) (Theorem 2.1) or of the advanced differential equation (E_2) (Theorem 2.2).

To state Theorem 2.1, it is needed to consider *the points* T_i $(i = 0, 1, \dots)$ *defined as*

$$T_0 = 0$$

and for $i = 1, 2, \dots$

$$T_i = \min\left\{ s \geq 0 : \min_{j \in J} \min_{t \geq s} [t - \tau_j(t)] \geq T_{i-1} \right\}.$$

(It is clear that $0 \equiv T_0 \leq T_1 \leq T_2 \leq \dots$.)

Theorem 2.1. *Assume that*

$$p \equiv \inf_{t \geq 0} \sum_{j \in J_0} p_j(t) > 0 \quad \text{and} \quad \tau \equiv \min_{j \in J_0} \inf_{t \geq 0} \tau_j(t) > 0$$

for a nonempty set $J_0 \subseteq J$. Moreover, suppose that there exists a nonnegative integer m such that

$$\int_{t^\star - \tau}^{t^\star} P_m(s) ds > \log \frac{4}{(p\tau)^2} \quad \text{for a sufficiently large } t^\star \geq T_m + \tau,$$

where

$$P_0(t) = \sum_{j \in J} p_j(t) \quad \text{for } t \geq 0 \equiv T_0$$

and, when $m > 0$, for $i = 0, 1, \dots, m-1$

$$P_{i+1}(t) = \sum_{j\in J} p_j(t) \exp\left[\int_{t-\tau_j(t)}^{t} P_i(s)ds\right] \quad \textit{for } t \geq T_{i+1}.$$

Then all solutions of the delay differential equation (E_1) *are oscillatory.*

Proof. Let x be a nonoscillatory solution on an interval $[t_0, \infty)$, $t_0 \geq 0$, of the delay differential equation (E_1). Without restriction of generality one can assume that $x(t) > 0$, $t \in [0, \infty)$. Furthermore, there is no loss of generallity to suppose that x is positive on the whole interval $[t_{-1}, \infty)$, where

$$t_{-1} = \min_{j\in J} \min_{t\geq t_0} [t - \tau_j(t)].$$

(Clearly, $-\infty < t_{-1} \leq t_0$.) Then it follows from (E_1) that $x'(t) \leq 0$ for all $t \geq t_0$ and so x is decreasing on the interval $[t_0, \infty)$.

Now we define

$$S_0 = \min\left\{s \geq 0 : \min_{j\in J} \min_{t\geq s} [t - \tau_j(t)] \geq t_0\right\}$$

and, provided that $m > 0$,

$$S_i = \min\left\{s \geq 0 : \min_{j\in J} \min_{t\geq s} [t - \tau_j(t)] \geq S_{i-1}\right\} \quad (i = 0, 1, \dots, m).$$

It is obvious that $t_0 \leq S_0 \leq S_1 \leq \dots \leq S_m$. Moreover, we can immediately see that $T_i \leq S_i$ $(i = 0, 1, \dots, m)$.

We will show that

$$x'(t) + P_m(t)x(t) \leq 0 \quad \text{for every } t \geq S_m. \tag{2.1}$$

By the decreasing nature of x on $[t_0, \infty)$ it follows from (E_1) that for $t \geq S_0$

$$0 = x'(t) + \sum_{j\in J} p_j(t)x(t - \tau_j(t)) \geq x'(t) + \left[\sum_{j\in J} p_j(t)\right] x(t),$$

i.e.,

$$x'(t) + P_0(t)x(t) \leq 0 \quad \text{for every } t \geq S_0. \tag{2.2}$$

Hence (2.1) is satisfied if $m = 0$. Let us assume that $m > 0$. Then by (2.2) we obtain for $j \in J$ and $t \geq S_1$

$$\log \frac{x(t - \tau_j(t))}{x(t)} = -\int_{t-\tau_j(t)}^{t} \frac{x'(s)}{x(s)} ds \geq \int_{t-\tau_j(t)}^{t} P_0(s)ds.$$

So we have

$$x(t - \tau_j(t)) \geq x(t) \exp\left[\int_{t-\tau_j(t)}^{t} P_0(s)ds\right] \quad \text{for } j \in J \text{ and } t \geq S_1.$$

Thus (E_1) gives for $t \geq S_1$

$$0 = x'(t) + \sum_{j \in J} p_j(t) x(t - \tau_j(t)) \geq x'(t) + \left\{ \sum_{j \in J} p_j(t) \exp\left[\int_{t-\tau_j(t)}^{t} P_0(s)ds \right] \right\} x(t),$$

i.e.,

$$x'(t) + P_1(t)x(t) \leq 0 \quad \text{for every } t \geq S_1. \tag{2.3}$$

This means that (2.1) is fulfilled when $m = 1$. Let us consider the case where $m > 1$. Then it follows from (2.3) that

$$x(t - \tau_j(t)) \geq x(t) \exp\left[\int_{t-\tau_j(t)}^{t} P_1(s)ds \right] \quad \text{for } j \in J \text{ and } t \geq S_2$$

and so (E_1) yields

$$x'(t) + P_2(t)x(t) \leq 0 \quad \text{for every } t \geq S_2. \tag{2.4}$$

Thus (2.1) holds if $m = 2$. If $m > 2$, we can use (2.4) and (E_1) to obtain an inequality similar to (2.4) with P_3 in place of P_2 and S_3 in place of S_2. Following the same procedure in the case where $m > 3$, we can finally arrive at (2.1).

Next, it follows from (2.1) that for $t \geq S_m + \tau$

$$\log \frac{x(t-\tau)}{x(t)} = -\int_{t-\tau}^{t} \frac{x'(s)}{x(s)} ds \geq \int_{t-\tau}^{t} P_m(s)ds$$

and so we have

$$x(t - \tau) \geq x(t) \exp\left[\int_{t-\tau}^{t} P_m(s)ds \right] \quad \text{for all } t \geq S_m + \tau. \tag{2.5}$$

On the other hand, by the decreasing character of x on $[t_0, \infty)$, from (E_1) we obtain for $t \geq S_0$

$$\begin{aligned} 0 = x'(t) + \sum_{j \in J} p_j(t) x(t - \tau_j(t)) &\geq x'(t) + \sum_{j \in J_0} p_j(t) x(t - \tau_j(t)) \geq \\ &\geq x'(t) + \left[\sum_{j \in J_0} p_j(t) \right] x(t - \tau) \geq x'(t) + p x(t - \tau), \end{aligned}$$

i.e.,

$$x'(t) + p x(t - \tau) \leq 0 \quad \text{for every } t \geq S_0. \tag{2.6}$$

Following the same arguments used in the proof of Lemma in [8] (see also Lemma 1.6.1 in [21]), from (2.6) it follows that

$$x(t - \tau) \leq \frac{4}{(p\tau)^2} x(t) \quad \text{for all } t \geq S_0 + \tau/2. \tag{2.7}$$

Combining (2.5) and (2.7), we get

$$\exp\left[\int_{t-\tau}^{t} P_m(s)ds\right] \leq \frac{4}{(p\tau)^2} \quad \text{for all } t \geq S_m + \tau$$

or, equivalently,

$$\int_{t-\tau}^{t} P_m(s)ds \leq \log\frac{4}{(p\tau)^2} \quad \text{for every } t \geq S_m + \tau.$$

This is a contradiction, since $t^\star$ is sufficiently large and so it can be supposed that $t^\star \geq S_m + \tau$. □

Theorem 2.2. *Let J_0 be a nonempty subset of J and assume that $p > 0$ and $\tau > 0$, where p and τ are defined as in Theorem 2.1. Moreover, suppose that there exists a nonnegative integer m such that*

$$\int_{t^\star}^{t^\star+\tau} P_m(s)ds > \log\frac{4}{(p\tau)^2} \quad \textit{for a sufficiently large} \quad t^\star \geq 0,$$

where

$$P_0(t) = \sum_{j\in J} p_j(t) \quad \textit{for } t \geq 0$$

and, when $m > 0$, for $i = 0, 1, \ldots, m-1$

$$P_{i+1}(t) = \sum_{j\in J} p_j(t)\exp\left[\int_{t}^{t+\tau_j(t)} P_i(s)ds\right] \quad \textit{for } t \geq 0.$$

Then all solutions of the advanced differential equation (E_2) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that the advanced differential equation (E_2) has a nonoscillatory solution x on an interval $[t_0, \infty)$, where $t_0 \geq 0$. Without loss of generality, we can suppose that x is eventually positive. Furthermore, we may (and do) assume that x is positive on the whole interval $[t_0, \infty)$. Then (E_2) gives $x'(t) \geq 0$ for every $t \geq t_0$ and so the solution x is increasing on the interval $[t_0, \infty)$.

We will prove that

$$x'(t) - P_m(t)x(t) \geq 0 \quad \text{for every } t \geq t_0. \tag{2.8}$$

By taking into account the fact that x is increasing on $[t_0, \infty)$, from (E_2) we obtain for $t \geq t_0$

$$0 = x'(t) - \sum_{j\in J} p_j(t)x(t+\tau_j(t)) \leq x'(t) - \left[\sum_{j\in J} p_j(t)\right]x(t)$$

and consequently

$$x'(t) - P_0(t)x(t) \geq 0 \quad \text{for every } t \geq t_0. \tag{2.9}$$

Thus, (2.8) holds when $m = 0$. Let us consider the case where $m > 0$. Then we can use (2.9) to derive for $j \in J$ and $t \geq t_0$

$$\log \frac{x(t+\tau_j(t))}{x(t)} = \int_t^{t+\tau_j(t)} \frac{x'(s)}{x(s)} ds \geq \int_t^{t+\tau_j(t)} P_0(s) ds.$$

This gives

$$x(t+\tau_j(t)) \geq x(t) \exp\left[\int_t^{t+\tau_j(t)} P_0(s) ds\right] \quad \text{for } j \in J \text{ and } t \geq t_0.$$

Hence from (E_2) it follows that for $t \geq t_0$

$$0 = x'(t) - \sum_{j \in J} p_j(t) x(t+\tau_j(t)) \leq x'(t) - \left\{\sum_{j \in J} p_j(t) \exp\left[\int_t^{t+\tau_j(t)} P_0(s) ds\right]\right\} x(t)$$

i.e.,

$$x'(t) - P_1(t)x(t) \geq 0 \quad \text{for every } t \geq t_0. \tag{2.10}$$

So (2.8) is satisfied if $m = 1$. Let us suppose that $m > 1$. Then, using the same arguments as above with (2.10) in place of (2.9), we can obtain

$$x'(t) - P_2(t)x(t) \geq 0 \quad \text{for every } t \geq t_0.$$

Thus (2.8) is fulfilled when $m = 2$. Repeating the above procedure if $m > 2$, we can finally arrive at (2.8).

Now from (2.8) we get for $t \geq t_0$

$$\log \frac{x(t+\tau)}{x(t)} = \int_t^{t+\tau} \frac{x'(s)}{x(s)} ds \geq \int_t^{t+\tau} P_m(s) ds$$

and consequently

$$x(t+\tau) \geq x(t) \exp\left[\int_t^{t+\tau} P_m(s) ds\right] \quad \text{for all } t \geq t_0. \tag{2.11}$$

Next, taking into account the fact that x is increasing on $[t_0, \infty)$, from (E_2) we derive for $t \geq t_0$

$$\begin{aligned} 0 = x'(t) - \sum_{j \in J} p_j(t) x(t+\tau_j(t)) &\leq x'(t) - \sum_{j \in J_0} p_j(t) x(t+\tau_j(t)) \leq \\ &\leq x'(t) - \left[\sum_{j \in J_0} p_j(t)\right] x(t+\tau) \leq x'(t) - p x(t+\tau) \end{aligned}$$

and so

$$x'(t) - p x(t+\tau) \geq 0 \quad \text{for all } t \geq t_0. \tag{2.12}$$

As in the proof of Lemma 1.6.1 in [21], (2.12) gives

$$x(t+\tau) \le \frac{4}{(p\tau)^2}x(t) \quad \text{for every } t \ge t_0. \tag{2.13}$$

A combination of (2.11) and (2.13) yields

$$\int_t^{t+\tau} P_m(s)ds \le \log\frac{4}{(p\tau)^2} \quad \text{for all } t \ge t_0.$$

The point $t^\star$ is sufficiently large and so we can assume that $t^\star \ge t_0$. We have thus arrived at a contradiction. This contradiction completes the proof of the theorem. □

3. Existence of Positive Solutions of Delay or Advanced Differential Equations

Our results in this section are Theorems 3.1 and 3.2 below. Theorem 3.1 provides conditions under which the delay differential equation (E_1) has a positive solution; analogously, the conditions which ensure the existence of a positive solution of the advanced differential equation (E_2) are established by Theorem 3.2.

Let us consider the delay differential inequality

$$y'(t) + \sum_{j\in J} p_j(t)y(t-\tau_j(t)) \le 0 \tag{H_1}$$

and the advanced differential inequality

$$y'(t) - \sum_{j\in J} p_j(t)y(t+\tau_j(t)) \ge 0, \tag{H_2}$$

which are associated with the delay differential equation (E_1) and the advanced differential equation (E_2), respectively. For the delay inequality (H_1) it will be assumed that J is finite and that $\lim_{t\to\infty}[t-\tau_j(t)] = \infty$ for $j \in J$, while for the advanced inequality (H_2) the set J may be infinite.

Let $t_0 \ge 0$ and define $t_{-1} = \min_{j\in J}\min_{t\ge t_0}[t-\tau_j(t)]$. (Clearly, $-\infty < t_{-1} \le t_0$.) By a *solution on* $[t_0,\infty)$ of the delay differential inequality (H_1) we mean a continuous real valued function y defined on the interval $[t_{-1},\infty)$, which is continuously differentiable on $[t_0,\infty)$ and satisfies (H_1) for all $t \ge t_0$. A solution on $[t_0,\infty)$ of the delay inequality (H_1) or, in particular, of the delay equation (E_1) will be called *positive* if it is positive on the whole interval $[t_{-1},\infty)$.

Let again $t_0 \ge 0$. A *solution on* $[t_0,\infty)$ of the advanced differential inequality (H_2) is a continuously differentiable function y on the interval $[t_0,\infty)$, which satisfies (H_2) for all $t \ge t_0$. A solution on $[t_0,\infty)$ of the advanced inequality (H_2) or, in particular, of the advanced equation (E_2) is said to be *positive* if all its values for $t \ge t_0$ are positive numbers.

In order to prove Theorems 3.1 and 3.2 we need Lemmas 3.1 and 3.2 below, respectively. These lemmas guarantee that if there exists a positive solution of the delay inequality (H_1) or of the advanced inequality (H_2), then the delay equation (E_1) or the advanced equation (E_2), respectively, also has a positive solution.

Lemma 3.1 below is similar to Lemma in [16] concerning the particular case of constant delays. The method of proving Lemma 3.1 is similar to that of Lemma in [16] (see also the proof of the Lemma in [14] and the proof of Theorem 1 in [26].

Lemma 3.1. *Let $t_0 \geq 0$ and let y be a positive solution on $[t_0, \infty)$ of the delay differential inequality (H_1). Set*

$$t_1 = \min\left\{s \geq 0 : \min_{j \in J} \min_{t \geq s} [t - \tau_j(t)] \geq t_0\right\}$$

and assume that $t_1 > t_0$. (Clearly, we have $\min_{j\in J} \min_{t \geq t_1} [t - \tau_j(t)] = t_0$.) Moreover, suppose that there exists a nonempty subset J_0 of J such that the functions τ_j for $j \in J_0$, and $\sum_{j\in J_0} p_j$ are positive on $[t_1, \infty)$.

Then there exists a positive solution x on $[t_1, \infty)$ of the delay differential equation (E_1) with $\lim_{t\to\infty} x(t) = 0$ and such that $x(t) \leq y(t)$ for all $t \geq t_0$.

Proof. It follows from the inequality (H_1) that for $\widetilde{t} \geq t \geq t_0$

$$y(t) \geq y(\widetilde{t}) + \int_t^{\widetilde{t}} \sum_{j\in J} p_j(s) y(s - \tau_j(s))ds > \int_t^{\widetilde{t}} \sum_{j\in J} p_j(s) y(s - \tau_j(s))ds.$$

Thus, as $\widetilde{t} \to \infty$, we obtain

$$y(t) \geq \int_t^{\infty} \sum_{j\in J} p_j(s) y(s - \tau_j(s))ds \quad \text{for every} \quad t \geq t_0. \tag{3.1}$$

Let $\mathcal{X}$ be the space of all nonnegative continuous real-valued functions x on the interval $[t_0, \infty)$ with $x(t) \leq y(t)$ for every $t \geq t_0$. Then using (3.1) we can easily show that the formulae

$$(Lx)(t) = \int_t^{\infty} \sum_{j\in J} p_j(s) x(s - \tau_j(s))ds, \quad \text{if } t \geq t_1$$

and

$$(Lx)(t) = \int_{t_1}^{\infty} \sum_{j\in J} p_j(s) x(s - \tau_j(s))ds + \\ + \int_t^{t_1} \sum_{j\in J} p_j(s) y(s - \tau_j(s))ds, \quad \text{if } t_0 \leq t < t_1$$

are meaningful for any function $x \in \mathcal{X}$ and that, by these formulae, an operator $L : \mathcal{X} \to \mathcal{X}$ is defined. Furthermore, we see that, for any pair of functions x_1 and x_2 in $\mathcal{X}$ such that $x_1(t) \leq x_2(t)$ for $t \geq t_0$, we have $(Lx_1)(t) \leq (Lx_2)(t)$ for $t \geq t_0$. This means that the operator L is monotone. Next, we set

$$x_0 = y|[t_0, \infty) \quad \text{and} \quad x_\nu = Lx_{\nu-1} \quad (\nu = 1, 2, \dots).$$

Clearly, $(x_\nu)_{\nu \geq 0}$ is a decreasing sequence of functions in $\mathcal{X}$. (Note that the decreasing character of this sequence is considered with the usual pointwise ordering in $\mathcal{X}$.) Define

$$x = \lim_{\nu \to \infty} x_\nu \quad \text{pointwise on } [t_0, \infty).$$

By the Lebesgue dominated convergence theorem, we obtain $x = Lx$, i.e.,

$$x(t) = \int_t^\infty \sum_{j \in J} p_j(s) x(s - \tau_j(s)) ds, \quad \text{if } t \geq t_1 \tag{3.2}$$

and

$$x(t) = \int_{t_1}^\infty \sum_{j \in J} p_j(s) x(s - \tau_j(s)) ds + \\ + \int_t^{t_1} \sum_{j \in J} p_j(s) y(s - \tau_j(s)) ds, \quad \text{if } t_0 \leq t < t_1. \tag{3.3}$$

Equation (3.2) gives

$$x'(t) = -\sum_{j \in J} p_j(t) x(t - \tau_j(t)) \quad \text{for all } t \geq t_1,$$

which means that the function x is a solution on $[t_1, \infty)$ of the delay equation (E_1). Clearly, we have $0 \leq x(t) \leq y(t)$ for every $t \geq t_0$. Moreover, from (3.2) it follows that x tends to zero at ∞. Hence it remains to show that x is positive on the whole interval $[t_0, \infty)$. From (3.3) we obtain for any $t \in [t_0, t_1)$

$$x(t) \geq \int_t^{t_1} \sum_{j \in J} p_j(s) y(s - \tau_j(s)) ds \geq \\ \geq \left[\min_{j \in J} \min_{t_0 \leq s \leq t_1} y(s - \tau_j(s)) \right] \int_t^{t_1} \sum_{j \in J} p_j(s) ds.$$

Thus, by taking into account the facts that y is positive on the interval $[t_{-1}, t_1]$, where $t_{-1} = \min_{j \in J} \min_{t \geq t_0} [t - \tau_j(t)]$ (clearly, $-\infty < t_{-1} \leq t_0$), and that $\sum_{j \in J} p_j(t_1) \geq \sum_{j \in J_0} p_j(t_1) > 0$, we conclude that x is positive on the interval $[t_0, t_1)$. We claim that x is also positive on the interval $[t_1, \infty)$.

Otherwise, there exists a point $T \geq t_1$ such that $x(T) = 0$, and $x(t) > 0$ for $t \in [t_0, T)$. Then (3.2) gives

$$0 = x(T) = \int_T^\infty \sum_{j \in J} p_j(s) x(s - \tau_j(s)) ds$$

and so

$$\sum_{j \in J} p_j(s) x(s - \tau_j(s)) = 0 \quad \text{for all } s \geq T.$$

Taking into account the fact that x is positive on $[t_0, T)$ as well as the fact that $\tau_j(T) > 0$ for $j \in J_0$ and that $\sum_{j \in J_0} p_j(T) > 0$, we have

$$0 = \sum_{j \in J} p_j(T) x(T - \tau_j(T)) \geq \sum_{j \in J_0} p_j(T) x(T - \tau_j(T)) \geq$$
$$\geq \left[\min_{j \in J_0} x(T - \tau_j(T)) \right] \sum_{j \in J_0} p_j(T) > 0.$$

But, this is a contradiction and so our claim is proved. □

Theorem 3.1. *Set*

$$t_0 = \min \left\{ s \geq 0 : \min_{j \in J} \min_{t \geq s} [t - \tau_j(t)] \geq 0 \right\}.$$

(*Clearly,* $t_{-1} \equiv \min_{j \in J} \min_{t \geq t_0} [t - \tau_j(t)] = 0$). *Suppose that there exist positive real numbers* γ_j *for* $j \in J$ *such that*

$$\exp \left[\sum_{i \in J} \gamma_i \int_{t - \tau_j(t)}^t p_i(s) ds \right] \leq \gamma_j \quad \text{for all } t \geq t_0 \text{ and } j \in J.$$

Also, define

$$t_1 = \min \left\{ s \geq 0 : \min_{j \in J} \min_{t \geq s} [t - \tau_j(t)] \geq t_0 \right\}$$

and assume that $t_1 > t_0$. (*Obviously,* $\min_{j \in J} \min_{t \geq t_1} [t - \tau_j(t)] = t_0$.) *Moreover, suppose that there exists a nonempty subset* J_0 *of* J *such that the functions* τ_j *for* $j \in J_0$, *and* $\sum_{j \in J_0} p_j$ *are positive on* $[t_1, \infty)$.

Then there exists a positive solution on $[t_1, \infty)$ *of the delay differential equation* (E_1), *which tends to zero at* ∞.

Proof. Define

$$y(t) = \exp \left[- \sum_{i \in J} \gamma_i \int_0^t p_i(s) ds \right] \quad \text{for } t \geq 0$$

and observe that y is positive on the interval $[0,\infty)$. By Lemma 3.1 it suffices to show that y is a solution on $[t_0,\infty)$ of the delay differential inequality (H_1). To this end we have for every $t \ge t_0$

$$
\begin{aligned}
&y'(t) + \sum_{j\in J} p_j(t) y(t-\tau_j(t)) = \\
&= -\Big[\sum_{i\in J} \gamma_i p_i(t)\Big] y(t) + \Big\{\sum_{j\in J} p_j(t) \exp\Big[\sum_{i\in J} \gamma_i \int_{t-\tau_j(t)}^{t} p_i(s)ds\Big]\Big\} y(t) = \\
&= \Big(\sum_{j\in J} p_j(t) \Big\{ -\gamma_j + \exp\Big[\sum_{i\in J} \gamma_i \int_{t-\tau_j(t)}^{t} p_i(s)ds\Big]\Big\}\Big) y(t) \le 0. \qquad \square
\end{aligned}
$$

Lemma 3.2. *Let $t_0 \ge 0$ and let y be a positive solution on $[t_0,\infty)$ of the advanced differential inequality* (H_2).

Then there exists a positive solution x on $[t_0,\infty)$ of the advanced differential equation (E_2) *such that $x(t) \le y(t)$ for all $t \ge t_0$.*

Proof. It follows from (H_2) that

$$
y(t) \ge y(t_0) + \int_{t_0}^{t} \sum_{j\in J} p_j(s) y(s+\tau_j(s))ds \quad \text{for all } t \ge t_0. \tag{3.4}
$$

Consider the set $\mathcal{X}$ of all continuous real-valued functions x on the interval $[t_0,\infty)$ such that $0 < x(t) \le y(t)$ for every $t \ge t_0$. Then by (3.4) we can see that the formula

$$
(Lx)(t) = y(t_0) + \int_{t_0}^{t} \sum_{j\in J} p_j(s) x(s+\tau_j(s))ds \quad \text{for } t \ge t_0
$$

is meaningful for any function x in $\mathcal{X}$ and that this formula defines an operator L of $\mathcal{X}$ into itself. This operator is monotone in the sense that, if x_1 and x_2 are two functions in $\mathcal{X}$ with $x_1(t) \le x_2(t)$ for $t \ge t_0$, then we also have $(Lx_1)(t) \le (Lx_2)(t)$ for $t \ge t_0$. Next, we define $x_0 = y$ and $x_\nu = Lx_{\nu-1}$ $(\nu = 1,2,\dots)$. Clearly, $x_0(t) \ge x_1(t) \ge x_2(t) \ge \cdots$ holds for every $t \ge t_0$ and so we can define $x(t) = \lim_{\nu\to\infty} x_\nu(t)$ for $t \ge t_0$. Then applying the Lebesgue dominated convergence theorem, we have $x = Lx$, i.e.,

$$
x(t) = y(t_0) + \int_{t_0}^{t} \sum_{j\in J} p_j(s) x(s+\tau_j(s))ds \quad \text{for every } t \ge t_0.
$$

This ensures that x is a solution on $[t_0,\infty)$ of the advanced equation (E_2), which is positive (on $[t_0,\infty)$) and such that $x(t) \le y(t)$ for $t \ge t_0$. $\square$

Theorem 3.2. *Suppose that there exist positive real numbers δ_j for $j \in J$ such that*

$$\exp\left[\sum_{i\in J}\delta_i\int_t^{t+\tau_j(t)}p_i(s)ds\right]\le\delta_j\quad \textit{for all } t\ge 0 \textit{ and } j\in J$$

and, when J is infinite,

$$\sum_{i\in J}\delta_i\int_0^t p_i(s)ds<\infty\quad \textit{for every } t\ge 0.$$

Then there exists a positive solution on $[0,\infty)$ of the advanced differential equation (E_2).

Proof. The function y defined by

$$y(t)=\exp\left[\sum_{i\in J}\delta_i\int_0^t p_i(s)ds\right]\quad \text{for } t\ge 0$$

is clearly positive on the interval $[0,\infty)$. Moreover, for every $t\ge 0$ we obtain

$$\begin{aligned}
&y'(t)-\sum_{j\in J}p_j(t)y(t+\tau_j(t))=\\
&=\left[\sum_{i\in J}\delta_i p_i(t)\right]y(t)-\left\{\sum_{j\in J}p_j(t)\exp\left[\sum_{i\in J}\delta_i\int_t^{t+\tau_j(t)}p_i(s)ds\right]\right\}y(t)=\\
&=\left(\sum_{j\in J}p_j(t)\left\{\delta_j-\exp\left[\sum_{i\in J}\delta_i\int_t^{t+\tau_j(t)}p_i(s)ds\right]\right\}\right)y(t)\ge 0
\end{aligned}$$

and hence y is a solution on $[0,\infty)$ of the advanced inequality (H_2). So, the proof can be completed by applying Lemma 3.2. □

4. Necessary Conditions for the Existence of Positive Solutions of Integrodifferential Equations and Inequalities

In this section the problem of the nonexistence of positive solutions of the integrodifferential equations (E_3) and (E_4) (or, more generally, of the integrodifferential inequalities (I_1) and (I_2)) will be treated. The main results here are Theorems 4.1 and 4.2 below.

Theorem 4.1. *Let $t_0\ge 0$. Assume that*

$$A\equiv\inf_{t\ge t_0+\tau_1}\left[q(t)\int_{\tau_0}^{t-t_0}K(s)ds\right]>0$$

for two points τ_0 *and* τ_1 *with* $0 < \tau_0 < \tau_1$. *Moreover, suppose that there exists a nonnegative integer* m *such that*

$$\int_{t^\star-\tau_0}^{t^\star} U_m(s)ds > \log\frac{4}{(A\tau_0)^2} \quad \textit{for some } t^\star \geq t_0+\tau_1+\tau_0/2,$$

where

$$U_0(t) = q(t)\int_0^{t-t_0} K(s)ds \quad \textit{for } t \geq t_0$$

and, when $m > 0$, *for* $i = 0, 1, \dots, m-1$

$$U_{i+1}(t) = q(t)\int_0^{t-t_0} K(s)\exp\left[\int_{t-s}^{t} U_i(\xi)d\xi\right]ds \quad \textit{for } t \geq t_0.$$

Then there is no solution on $[t_0, \infty)$ *of the integrodifferential inequality* (I_1) (*and, in particular, of the integrodifferential equation* (E_3)), *which is positive on* $[0, \infty)$.

Proof. Assume, for the sake of contradiction, that the integrodifferential inequality (I_1) admits a solution y on $[t_0, \infty)$, which is positive on $[0, \infty)$. Then (I_1) guarantees that $y'(t) \leq 0$ for every $t \geq t_0$ and so the solution y is decreasing on the interval $[t_0, \infty)$.

We first prove that

$$y'(t) + U_m(t)y(t) \leq 0 \quad \text{for all } t \geq t_0. \tag{4.1}$$

To this end, using the decreasing character of y on $[t_0, \infty)$, from (I_1) we obtain for any $t \geq t_0$

$$\begin{aligned} 0 &\geq y'(t) + q(t)\int_0^t K(t-s)y(s)ds = y'(t) + q(t)\int_0^t K(s)y(t-s)ds \geq \\ &\geq y'(t) + q(t)\int_0^{t-t_0} K(s)y(t-s)ds \geq y'(t) + q(t)\left[\int_0^{t-t_0} K(s)ds\right]y(t) \end{aligned}$$

and so we have

$$y'(t) + U_0(t)y(t) \leq 0 \quad \text{for all } t \geq t_0. \tag{4.2}$$

Thus (4.1) is satisfied when $m = 0$. Let us assume that $m > 0$. Then it follows from (4.2) that for $t \geq t_0$ and $0 \leq s \leq t - t_0$

$$\log\frac{y(t-s)}{y(t)} = -\int_{t-s}^{t}\frac{y'(\xi)}{y(\xi)}d\xi \geq \int_{t-s}^{t} U_0(\xi)d\xi$$

and consequently

$$y(t-s) \geq y(t)\exp\left[\int_{t-s}^{t} U_0(\xi)d\xi\right] \quad \text{for } t \geq t_0 \text{ and } 0 \leq s \leq t-t_0. \tag{4.3}$$

Furthermore, in view of (4.3), inequality (I_1) yields for $t \geq t_0$

$$0 \geq y'(t) + q(t)\int_0^t K(t-s)y(s)ds = y'(t) + q(t)\int_0^t K(s)y(t-s)ds \geq$$
$$\geq y'(t) + q(t)\int_0^{t-t_0} K(s)y(t-s)ds \geq$$
$$\geq y'(t) + q(t)\left\{\int_0^{t-t_0} K(s)\exp\left[\int_{t-s}^t U_0(\xi)d\xi\right]ds\right\}y(t).$$

Therefore

$$y'(t) + U_1(t)y(t) \leq 0 \quad \text{for all } t \geq t_0. \tag{4.4}$$

Hence (4.1) is proved when $m = 1$. In the case where $m > 1$, we can repeat the above procedure with (4.4) in place of (4.2) to conclude that (4.1) is finally satisfied.

Now from (4.1) we obtain for $t \geq t_0 + \tau_0$

$$\log\frac{y(t-\tau_0)}{y(t)} = -\int_{t-\tau_0}^t \frac{y'(s)}{y(s)}ds \geq \int_{t-\tau_0}^t U_m(s)ds$$

and hence

$$y(t-\tau_0) \geq y(t)\exp\left[\int_{t-\tau_0}^t U_m(s)ds\right] \quad \text{for every } t \geq t_0 + \tau_0. \tag{4.5}$$

Next, taking into account the fact that y is decreasing on $[t_0, \infty)$, from (I_1) we derive for $t \geq t_0 + \tau_1$

$$0 \geq y'(t) + q(t)\int_0^t K(s)y(t-s)ds \geq y'(t) + q(t)\int_{\tau_0}^{t-t_0} K(s)y(t-s)ds \geq$$
$$\geq y'(t) + \left[q(t)\int_{\tau_0}^{t-t_0} K(s)ds\right]y(t-\tau_0) \geq y'(t) + Ay(t-\tau_0),$$

i.e.,

$$y'(t) + Ay(t-\tau_0) \leq 0 \quad \text{for all } t \geq t_0 + \tau_1. \tag{4.6}$$

As in the proof of the Lemma in [8] (see also Lemma 1.6.1 in [21]), it follows from (4.6) that

$$y(t-\tau_0) \leq \frac{4}{(A\tau_0)^2}y(t) \quad \text{for every } t \geq t_0 + \tau_1 + \tau_0/2. \tag{4.7}$$

A combination of (4.5) and (4.7) leads to

$$\int_{t-\tau_0}^t U_m(s)ds \leq \log\frac{4}{(A\tau_0)^2} \quad \text{for all } t \geq t_0 + \tau_1 + \tau_0/2,$$

which is a contradiction. □

Theorem 4.2. *Let $\widehat{t}_0 \in \mathbb{R}$ and set $t_0 = \max\{0, \widehat{t}_0\}$. Moreover, let the assumptions of Theorem* 4.1 *be satisfied with r in place of q.*

Then there is no solution on $[\widehat{t}_0, \infty)$ of the integrodifferential inequality (I_2) (*and, in particular, of the integrodifferential equation* (E_4)), *which is positive on* $\mathbb{R}$.

Proof. Obviously, $t_0 \geq 0$. Assume that there exists a solution y on $[\widehat{t}_0, \infty)$ of the integrodifferential inequality (I_2), which is positive on $\mathbb{R}$. Then, for every $t \geq t_0$, we have

$$0 \geq y'(t) + r(t) \int_{-\infty}^{t} K(t-s)y(s)ds = y'(t) + r(t) \int_{-\infty}^{0} K(t-s)y(s)ds +$$
$$+ r(t) \int_{0}^{t} K(t-s)y(s)ds \geq y'(t) + r(t) \int_{0}^{t} K(t-s)y(s)ds.$$

This means that the function $y|[0,\infty)$ is a solution on $[t_0, \infty)$ of the integrodifferential inequality

$$y'(t) + r(t) \int_{0}^{t} K(t-s)y(s)ds \leq 0,$$

which is positive on $[0, \infty)$. By Theorem 4.1, this is a contradiction and hence our proof is complete. □

5. Sufficient Conditions for the Existence of Positive Solutions of Integrodifferential Equations

Theorems 5.1 and 5.2 below are the main results in this last section. Theorem 5.1 establishes conditions which guarantee the existence of positive solutions of the integrodifferential equation (E_3); similarly, Theorem 5.2 provides sufficient conditions for the existence of positive solutions of the integrodifferential equation (E_4).

To prove Theorems 5.1 and 5.2 we will apply Theorems A and B, respectively, which are known.

Theorem A (Philos [38]). *Let y be a positive solution on $[0, \infty)$ of the integrodifferential inequality* (I_1). *Moreover, let $t_0 > 0$ and suppose that K is not identically zero on $[0, t_0]$ and q is positive on $[t_0, \infty)$.*

Then there exists a solution x on $[t_0, \infty)$ of the integrodifferential equation (E_3), *which is positive on $[0, \infty)$ and such that*

$$x(t) \leq y(t) \quad \textit{for every } t \geq t_0, \quad \lim_{t\to\infty} x(t) = 0$$

and

$$x'(t) + q(t) \int_{0}^{t} K(t-s)x(s)ds \leq 0 \quad \textit{for } 0 \leq t < t_0.$$

Theorem B (Philos [38]). *Assume that K is not identically zero on $[0,\infty)$. Let y be a positive solution on $\mathbb{R}$ of the integrodifferential inequality (I_2). Moreover, let $t_0 \in \mathbb{R}$ and suppose that r is positive on $[t_0,\infty)$.*

Then there exists a solution x on $[t_0,\infty)$ of the integrodifferential equation (E_4), which is positive on $\mathbb{R}$ and such that

$$x(t) \le y(t) \quad for\ every\ t \in \mathbb{R}, \qquad \lim_{t\to\infty} x(t) = 0$$

and

$$x'(t) + r(t)\int_{-\infty}^{t} K(t-s)x(s)ds \le 0 \quad for\ t < t_0.$$

We will now state and prove Theorems 5.1 and 5.2.

Theorem 5.1. *Let λ be a positive continuous real-valued function on the interval $[0,\infty)$ such that*

$$\exp\left\{\int_{t-s}^{t} q(\xi)\left[\int_0^{\xi}\lambda(\sigma)K(\sigma)d\sigma\right]d\xi\right\} \le \lambda(s) \quad for\ all\ t\ge 0\ and\ 0\le s\le t.$$

Moreover, let $t_0 > 0$ and suppose that K is not identically zero on $[0,t_0]$ and q is positive on $[t_0,\infty)$.

Then there exists a solution on $[t_0,\infty)$ of the integrodifferential equation (E_3), which is positive on $[0,\infty)$ and tends to zero at ∞.

Proof. Define

$$y(t) = \exp\left\{-\int_0^t q(\xi)\left[\int_0^{\xi}\lambda(\sigma)K(\sigma)d\sigma\right]d\xi\right\} \quad \text{for } t\ge 0.$$

Clearly, y is positive on the interval $[0,\infty)$. By Theorem A it is enough to verify that y is a solution on $[0,\infty)$ of the integrodifferential inequality (I_1). For this purpose we have, for every $t \ge 0$,

$$\begin{aligned}
&y'(t) + q(t)\int_0^t K(t-s)y(s)ds = y'(t) + q(t)\int_0^t K(s)y(t-s)ds =\\
&= -q(t)\left[\int_0^t \lambda(\sigma)K(\sigma)d\sigma\right]y(t) +\\
&\quad + q(t)\left[\int_0^t K(s)\exp\left\{\int_{t-s}^{t} q(\xi)\left[\int_0^{\xi}\lambda(\sigma)K(\sigma)d\sigma\right]d\xi\right\}ds\right]y(t) =\\
&= q(t)\left[-\int_0^t \lambda(s)K(s)ds +\right.\\
&\qquad \left. + \int_0^t K(s)\exp\left\{\int_{t-s}^{t} q(\xi)\left[\int_0^{\xi}\lambda(\sigma)K(\sigma)d\sigma\right]d\xi\right\}ds\right]y(t) =
\end{aligned}$$

$$=q(t)\left(\int_0^t K(s)\left[-\lambda(s)+\exp\left\{\int_{t-s}^t q(\xi)\left[\int_0^\xi \lambda(\sigma)K(\sigma)d\sigma\right]d\xi\right\}\right]ds\right)y(t)\leq$$
$$\leq 0. \qquad \square$$

Theorem 5.2. *Assume that K is not identically zero on $[0,\infty)$. Assume also that*

$$\int_{-\infty}^0 r(\xi)d\xi < \infty$$

and let μ be a positive continuous real-valued function on the interval $[0,\infty)$ such that

$$\int_0^\infty \mu(\sigma)K(\sigma)d\sigma < \infty$$

and

$$\exp\left\{\left[\int_0^\infty \mu(\sigma)K(\sigma)d\sigma\right]\int_{t-s}^t r(\xi)d\xi\right\} \leq \mu(s) \quad \textit{for all } t \in \mathbb{R} \textit{ and } s \geq 0.$$

Moreover, let $t_0 \in \mathbb{R}$ and suppose that r is positive on $[t_0,\infty)$.

Then there exists a solution on $[t_0,\infty)$ of the integrodifferential equation (E_4)*, which is positive on $\mathbb{R}$ and tends to zero at ∞.*

Proof. Set

$$y(t) = \exp\left\{-\left[\int_0^\infty \mu(\sigma)K(\sigma)d\sigma\right]\int_{-\infty}^t r(\xi)d\xi\right\} \quad \text{for } t \in \mathbb{R}.$$

We observe that y is positive on the real line $\mathbb{R}$. So by Theorem B it suffices to show that y is a solution on $\mathbb{R}$ of the integrodifferential inequality (I_2). To this end we obtain, for every $t \in \mathbb{R}$,

$$y'(t) + r(t)\int_{-\infty}^t K(t-s)y(s)ds = y'(t) + r(t)\int_0^\infty K(s)y(t-s)ds =$$
$$= -r(t)\left[\int_0^\infty \mu(\sigma)K(\sigma)d\sigma\right]y(t) +$$
$$+ r(t)\left[\int_0^\infty K(s)\exp\left\{\left[\int_0^\infty \mu(\sigma)K(\sigma)d\sigma\right]\int_{t-s}^t r(\xi)d\xi\right\}ds\right]y(t) =$$
$$= r(t)\left[-\int_0^\infty \mu(s)K(s)ds +\right.$$
$$\left.+\int_0^\infty K(s)\exp\left\{\left[\int_0^\infty \mu(\sigma)K(\sigma)d\sigma\right]\int_{t-s}^t r(\xi)d\xi\right\}ds\right]y(t) =$$
$$= r(t)\left(\int_0^\infty K(s)\left[-\mu(s)+\exp\left\{\left[\int_0^\infty \mu(\sigma)K(\sigma)d\sigma\right]\int_{t-s}^t r(\xi)d\xi\right\}\right]ds\right)y(t)\leq$$
$$\leq 0. \qquad \square$$

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Authors' address:
Department of Mathematics
University of Ioannina
P.O. Box 1186
451 10 Ioannina
Greece