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A REGULARITY CRITERION FOR SEMIGROUP RINGS

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Abstract. An analogue of the Kunz–Frobenius criterion for the regularity of a local ring in a positive characteristic is established for general commutative semigroup rings.

Let S be a commutative semigroup (we always assume that S contains a neutral element), and K a field. For every $m \in \mathbb{Z}_+$ the assignment $x \mapsto x^m$, $x \in S$, induces a K-endomorphism π_m of the semigroup ring $R = K[S]$. Therefore we can consider R as an R-algebra via π_m, and especially as an R-module. Let $R^{[m]}$ denote R with its R-module structure induced by π_m. If S is finitely generated, then $R^{[m]}$ is obviously a finitely generated R-module.

In this note we want to give a regularity criterion for S in terms of the homological properties of $R^{[m]}$ that is analogous to Kunz's [1] characterization of regular local rings of a characteristic $p > 0$ in terms of the Frobenius functor. Our criterion, which generalizes the result of Gubeladze [2, 10.2], requires only a mild condition on S and we provide a 'pure commutative algebraic' proof. (In [2] the result was stated for seminormal simplicial affine semigroup rings and derived from the main result of [2] that K_1-regularity implies the regularity for such rings.)

Theorem 1. *Let S be a finitely generated semigroup, K a field, $R = K[S]$, and $m \in \mathbb{Z}_+$, $m > 0$. Suppose that S has no invertible element $\neq 1$ and is generated by irreducible elements. Then the following conditions are equivalent:*

(a) *$R^{[m]}$ has a finite projective dimension;*
(b) *$R^{[m]}$ is a free module;*
(c) *S is free, in other words, $S \cong \mathbb{Z}_+^n$ for some $n \in \mathbb{Z}_+$.*

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Proof. It is obvious that (c) implies (b) and (b) implies (a). Now assume that (a) is satisfied. We first reduce the problem to a question of local algebra.

Set $T = S \setminus \{1\}$. The ideal $\mathfrak{m} = TR$ is a maximal ideal of R. Indeed, $R/\mathfrak{m} \cong K$. Furthermore the only prime ideal of $\mathfrak{q}$ of R such that $\mathfrak{m} = \pi_m^{-1}(\mathfrak{q})$ is $\mathfrak{m}$. Therefore $\pi_m \otimes R_\mathfrak{m}$ is an endomorphism of $R_\mathfrak{m}$ that makes $R_\mathfrak{m}$ a finitely generated $R_\mathfrak{m}$-module of finite projective dimension. In particular, $R_\mathfrak{m}$ has the same depth considered as an $R_\mathfrak{m}$-module via $\pi_m \otimes R_\mathfrak{m}$ as it has in its natural $R_\mathfrak{m}$-module structure (for example, see [3, 1.2.26]. The Auslander–Buchsbaum formula [3, 1.3.3] thus implies that $R_\mathfrak{m}$ is a finite free module over itself via $\pi_m \otimes R_\mathfrak{m}$. The lemma below shows that $R_\mathfrak{m}$ is a regular local ring.

Let $x_1, \dots, x_n$ be the irreducible elements of S. We claim that their images in $R_\mathfrak{m}$ form a minimal system of generators of the maximal ideal $\mathfrak{m}R_\mathfrak{m}$. Indeed, consider a presentation

$$R^r \xrightarrow{\varphi} R^n \xrightarrow{\psi} \mathfrak{m} \to 0,$$

where the i-th element e_i of the natural basis of R^n is mapped to x_i. We must show that all the entries of the matrix φ are in $\mathfrak{m}$. Suppose on the contrary that there is a relation

$$a_1x_1 + \cdots + a_nx_n = 0$$

with, for example, $a_1 \notin \mathfrak{m}$. Then $a_1 = \alpha_1 + \alpha_2 s_2 + \cdots + \alpha_u s_u$ with $\alpha_i \in K$, $\alpha_i \neq 0$, and $s_2, \dots, s_u \in T$. Writing $a_2, \dots, a_m$ similarly, we see that there are only two possibilities, (i) $x_1 = s_i x_1$ for some i, or (ii) $x_1 = vx_j$ for some $v \in S$ and $j > 0$. Both cases are impossible because x_1 is irreducible.

However, $R_\mathfrak{m}$ is a regular local ring. Especially it is a factorial ring, in which the (images of the) x_i are pairwise non-associated prime elements. Therefore all the elements $x_1^{e_1} \cdots x_n^{e_n}$, $e_1, \dots, e_n \in \mathbb{Z}_+$ are pairwise different, and it follows that $S \cong \mathbb{Z}_+^n$. □

Remark 2. (a) If we omit the hypothesis that S be generated by irreducible elements, then the proof above shows just the following: the subsemigroup generated by $x_1, \dots, x_n \in S$ such that $x_1, \dots, x_n$ form a minimal system of generators of the ideal $\mathfrak{m}R_\mathfrak{m}$ is free of rank n.

(b) One can weaken the hypothesis of the theorem by requiring only that the group S_0 of invertible elements of S be a free abelian group. Then $T = S \setminus S_0$ generates a prime ideal $\mathfrak{p}$ in R, and part (c) of the theorem must be replaced by the condition that $S \cong \mathbb{Z}_+^n \times \mathbb{Z}^q$ for some $n, q \in \mathbb{Z}_+$.

The following lemma is just an abstract version of Herzog's argument [4] characterizing the modules of finite projective dimension in terms of the Frobenius functor.

Lemma 3. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. If there exists an endomorphism π of R with $\pi(\mathfrak{m}) \subset \mathfrak{m}^2$ and such that R is a flat R-module via π, then R is a regular local ring.*

Proof. According to the criterion of Auslander–Buchsbaum–Serre [3, 2.2.7] we must show that $k = R/\mathfrak{m}$ has finite projective dimenion as an R-module. Write R' for R considered as an R-module via π, and let $\mathcal{P}$ be the functor that takes an R-module M to $M \otimes R'$ considered as an R-module via the identification $R = R'$. We choose a minimal free resolution $\mathcal{F}$ of k,

$$\mathcal{F}\colon \cdots \longrightarrow F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow k \longrightarrow 0.$$

One has $\mathcal{P}(R) = R$, $\mathcal{P}(F_i) = F_i$, and $\mathcal{P}(\mathcal{F})$ is the complex that we obtain from $\mathcal{F}$ by replacing all entries in its matrices by their images under π. By hypothesis, $\mathcal{P}(\mathcal{F})$ is again exact, and the exactness is preserved by an e-fold iteration of this process. Especially, $\mathcal{P}^e(\mathcal{F})$ is a free resolution of $\mathcal{P}^e(k)$ for all $e > 0$.

Let $x_1, \dots, x_t \in \mathfrak{m}$ be a maximal R-sequence. Then $\bar{R} = R/(x_1, \dots, x_t)$ has projective dimension t, and so $\operatorname{Tor}_i^R(\bar{R}, \mathcal{P}^e(k)) = 0$ for all $i > t$ and $e > 0$. On the other hand, one can compute $\operatorname{Tor}_i^R(\bar{R}, \mathcal{P}^e(k))$ by tensoring $\mathcal{P}^e(\mathcal{F})$ with $\bar{R}$. Let B_i be the kernel of φ_i. Then for sufficiently large i and all $e > 0$ we have an exact sequence

$$0 \longrightarrow \bar{R} \otimes \mathcal{P}^e(B_{i+1}) \longrightarrow \bar{R} \otimes F_{i+1} \longrightarrow \bar{R} \otimes \mathcal{P}^e(B_i) \longrightarrow 0.$$

Since we have chosen a maximal R-sequence, depth $\bar{R} \otimes F_{i+1} = 0$ if $F_{i+1} \neq 0$. On the other hand, for e sufficiently large, $\mathcal{P}^e(B_{i+1}) \subset \mathfrak{m}^{2^e} \bar{R} \otimes F_{i+1}$ and $\mathcal{P}^e(B_i) \subset \mathfrak{m}^{2^e} \bar{R} \otimes F_i$ have a positive depth or are zero according to [4, Lemma 3.2]. This is a contradiction. $\square$

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