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# THE BASIC MIXED PROBLEM FOR AN ANISOTROPIC ELASTIC BODY

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ABSTRACT. The mixed boundary value problem is considered for an anisotropic elastic body under the condition that a boundary value of the displacement vector is given on some part of the boundary and a boundary value of the generalized stress vector on the remainder. Using the potential method and the theory of singular integral equations with discontinuous coefficients, the existence of a solution of the mixed boundary value problem is proved.

In [1,2], the basic plane mixed boundary value problem for isotropic elastic body was studied by reducing it to a singular integral equation with discontinuous coefficients that contains, together with an unknown function, a complex conjugate outside the characteristic part. This problem for an anisotropic elastic body was solved in [3] by means of the problem of linear conjugation with displacements both for several unknown functions with discontinuous coefficients and for systems of singular integral equations.

In [4, 5], the mixed problem for isotropic and anisotropic bodies was treated by the method of the potential theory and a system of singular integral equations. However this way of solving the problem is connected with the construction of Green's tensor.

In [6], again applying the method of the potential theory and systems of singular integral equations, but without resorting to Green's tensor, the mixed problem was solved for an anisotropic body.

In the present work we solve the mixed boundary value problem in a general statement for an anisotropic elastic body. Using the potential theory and some transformations, we reduce the problem to a singular integral equation similar to that constructed by D. I. Sherman in [1].

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## § 1. Statement of the Problem

Let an elastic anisotropic medium occupy a finite simply connected two-dimensional domain $D^+$ in the plane $x_1 0 x_2$ bounded by a closed curve $S$ of Hölder-continuous curvature.

By $D^-$ we denote an infinite domain that complements $D^+ \cup S$ to the whole plane. Assume that on the curve $S$ we have the arcs $S'_j = a_j b_j$, $j = 1, 2, \dots, p$, which have no common ends, whose positive directions coincide with the positive direction of $S$, thus leaving the domain $D^+$ on the left, and which are arranged one after another in that direction. The set of arcs $S'_j$ will be denoted by $S'$ and the set of arcs $S''_j = b_j a_{j+1}$, $(a_{p+1} \equiv a_1)$ by $S''$. The direction of the outward normal with respect to $D^+$ is chosen as positive.

A system of equations of statics of an anisotropic elastic body free from body forces is written in terms of displacement vector components as

$$\begin{aligned}
&A_{11}\frac{\partial^2 u_1}{\partial x_1^2} + 2A_{13}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{33}\frac{\partial^2 u_1}{\partial x_2^2} + \\
&\qquad + A_{13}\frac{\partial^2 u_2}{\partial x_1^2} + (A_{12}+A_{33})\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{23}\frac{\partial^2 u_2}{\partial x_2^2} = 0, \\
&A_{13}\frac{\partial^2 u_1}{\partial x_1^2} + (A_{12}+A_{33})\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{23}\frac{\partial^2 u_1}{\partial x_2^2} + \\
&\qquad + A_{33}\frac{\partial^2 u_2}{\partial x_1^2} + 2A_{23}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{22}\frac{\partial^2 u_2}{\partial x_2^2} = 0, \\
&\qquad\qquad x = (x_1, x_2) \in D^+ \subset R^2,
\end{aligned} \tag{1.1}$$

where $u = (u_1, u_2)^T$ is the displacement vector, $T$ denotes transposition and $A_{11}, A_{12}, \dots, A_{33}$ are the elastic constants satisfying the conditions

$$\begin{gathered}
A_{11} > 0, \quad A_{22} > 0, \quad A_{33} > 0, \quad A_{11}A_{22} - A_{12}^2 > 0, \\
A_{11}A_{33} - A_{13}^2 > 0, \quad A_{22}A_{33} - A_{23}^2 > 0, \\
\Delta = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} > 0,
\end{gathered}$$

which are obtained from the positive definiteness of potential energy.

We introduce the matrix differential operator $T(\partial_x, n) = [T_{k_j}(\partial_x, n)]_{2\times 2}$, where

$$\begin{aligned}
T_{11}(\partial_x, n) &= n_1(x)\Big(A_{11}\frac{\partial}{\partial x_1} + A_{13}\frac{\partial}{\partial x_2}\Big) + n_2(x)\Big(A_{13}\frac{\partial}{\partial x_1} + A_{33}\frac{\partial}{\partial x_2}\Big), \\
T_{12}(\partial_x, n) &= n_1(x)\Big(A_{13}\frac{\partial}{\partial x_1} + A_{12}\frac{\partial}{\partial x_2}\Big) + n_2(x)\Big(A_{33}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\Big),
\end{aligned}$$

$$T_{21}(\partial_x, n) = n_1(x)\Big(A_{13}\frac{\partial}{\partial x_1} + A_{33}\frac{\partial}{\partial x_2}\Big) + n_2(x)\Big(A_{12}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\Big),$$
$$T_{22}(\partial_x, n) = n_1(x)\Big(A_{33}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\Big) + n_2(x)\Big(A_{23}\frac{\partial}{\partial x_1} + A_{22}\frac{\partial}{\partial x_2}\Big);$$

$n_1(x)$ and $n_2(x)$ are the components of some unit vector $n$ at the point $x$. Now $T(\partial_x, n)u(x)$ will be the stress vector acting on an element of the arc passing through $x$, with the normal $n$.

Denote by

$$\overset{\varkappa}{T}(\partial_x, n)u(x) = T(\partial_x, n)u(x) + \varkappa \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial u(x)}{\partial s(x)} \tag{1.2}$$

the generalized stress vector, where $\varkappa$ is an arbitrary real constant, $\frac{\partial}{\partial s(x)} = n_1(x)\frac{\partial}{\partial x_2} - n_2(x)\frac{\partial}{\partial x_1}$ is a derivative with respect to the tangent [8].

We can see from (1.2) that for $\varkappa = 0$ the generalized stress vector coincides with the physical stress vector.

**Definition 1.** A vector $u$ defined in the domain $D^+$ is said to be regular if:

(a) $u \in C(\overline{D}^+) \cap C^2(D^+)$;

(b) the vector $\overset{\varkappa}{T}(\partial_x, n)u$ is continuously extendable to all points of the boundary $S$ except possibly the points $a_j, b_j$, $j = 1, 2, \dots, p$, and in the neighborhood of these points it has an integrable singularity.

**Definition 2.** A vector $u$ defined in the domain $D^-$ is said to be regular if it satisfies conditions (a) and (b) of Definition 1 in the domain $D^-$, and for sufficiently large $|x| = \sqrt{x_1^2 + x_2^2}$

$$u(x) = O(1), \quad \overset{\varkappa}{T}(\partial_x, n)u(x) = O(|x|^{-2}). \tag{1.3}$$

Now we shall consider *the Basic Mixed Problem*: Find in the domain $D^+$ a regular solution of system (1.1) by the boundary condition

$$\big\{\overset{\varkappa}{T}(\partial_t, n)u(t)\big\}^+ = F(t), \quad t \in S', \tag{1.4}$$
$$\{u(t)\}^+ = f(t), \quad t \in S'', \tag{1.5}$$

where $F = (F_1, F_2)^T$ and $f = (f_1, f_2)^T$ are the given vectors; the symbol $\{\cdot\}^+$ ($\{\cdot\}^-$) denotes the boundary value on $S$ from $D^+$ ($D^-$) of the vector contained in the curly brackets.

Suppose: (1) $f$ belongs to the Hölder class $H$ on $S''$; (2) $f'$ and $F$ belong to the class $H^*$ at the nodes $a_j, b_j$, $j = 1, 2, \dots, p$ (for the definition of the classes $H$ and $H^*$ see [9]).

Note that in stating the problem the parameter $\varkappa$ is temporarily assumed to be arbitrary. Below it will be specified under which values of $\varkappa$ the problem is solvable. As mentioned above, for $\varkappa = 0$ this problem was studied in [6] by means of the potential method and the existence of a unique solution was proved.

## § 2. Some Auxiliary Statements

It is known [10] that the characteristic equation of system (1.1)

$$a_{11}\alpha^4 - 2a_{13}\alpha^3 + (2a_{12} + a_{33})\alpha^2 - 2a_{23}\alpha + a_{22} = 0, \tag{2.1}$$

where

$$\begin{aligned} a_{11} &= \Delta^{-1}(A_{22}A_{33} - A_{23}^2), \quad a_{12} = \Delta^{-1}(A_{13}A_{23} - A_{12}A_{33}),\\ a_{13} &= \Delta^{-1}(A_{12}A_{23} - A_{13}A_{22}), \quad a_{22} = \Delta^{-1}(A_{11}A_{33} - A_{13}^2),\\ a_{23} &= \Delta^{-1}(A_{12}A_{13} - A_{11}A_{23}), \quad a_{33} = \Delta^{-1}(A_{11}A_{22} - A_{12}^2), \end{aligned}$$

admits only the complex roots $\alpha_k = a^{(k)} + ib^{(k)}$ $(b^{(k)} > 0)$, $\overline{\alpha}_k$, $k = 1, 2$.

The matrix of fundamental solutions of system (1.1) is of the form

$$\Gamma(z,t) = \operatorname{Im} \sum_{k=1}^{2} \begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} \ln \sigma_k, \tag{2.2}$$

where

$$\begin{aligned} A_k &= -\frac{2}{\Delta a_{11}}(A_{22}\alpha_k^2 + 2A_{23}\alpha_k + A_{33})d_k,\\ B_k &= \frac{2}{\Delta a_{11}}(A_{23}\alpha_k^2 + (A_{12} + A_{33})\alpha_k + A_{13})d_k,\\ C_k &= -\frac{2}{\Delta a_{11}}(A_{33}\alpha_k^2 + 2A_{13}\alpha_k + A_{11})d_k,\\ d_k^{-1} &= (-1)^k(\alpha_k - \overline{\alpha}_1)(\alpha_2 - \alpha_1)(\alpha_k - \overline{\alpha}_2), \quad k = 1, 2; \end{aligned} \tag{2.3}$$

here $z = x_1 + ix_2$ and $t = y_1 + iy_2$ are arbitrary points of the plane, and $\sigma_k = z_k - t_k$, $z_k = x_1 + \alpha_k x_2$, $t_k = y_1 + \alpha_k y_2$.

Note that in the sequel we shall use the notation $\Psi(z)$ for for the function $\Psi(x_1, x_2)$ of the variables $x_1$ and $x_2$.

By equalities (2.3) the coefficients $A_k$, $B_k$ and $C_k$ satisfy the equation

$$A_k C_k - B_k^2 = 0, \quad k = 1, 2, \tag{2.4}$$

which differs from the characteristic equation (2.1) in a constant multiplier.

Note further that $\Gamma(z,t)$ is a one-valued matrix whose every column is a solution of system (1.1) at every point of the plane except the point $z = t$ at which it has a logarithmic singularity.

Introduce the notation [7]

$$C = 2i\sum_{k=1}^{2} d_k, \quad A = 2i\sum_{k=1}^{2} \alpha_k d_k, \quad B = 2i\sum_{k=1}^{2} \alpha_k^2 d_k. \tag{2.5}$$

It can be easily verified that the real constants $A, B, C$ satisfy the conditions

$$B > 0, \quad C > 0, \quad BC - A^2 > 0. \tag{2.6}$$

From (2.5) we find that

$$\begin{aligned} d_k &= -\frac{i}{2}(Cs_k - Ar_k), \quad \alpha_k d_k = -\frac{i}{2}(Ap_k - Cq_k), \\ \alpha_k^2 d_k &= -\frac{i}{2}(Bp_k - Aq_k), \quad Ap_k - Cq_k = As_k - Br_k, \end{aligned} \tag{2.7}$$

where

$$r_k = \frac{(-1)^k}{\alpha_1 - \alpha_2}, \quad s_k = \frac{(-1)^k \alpha_1 \alpha_2}{\alpha_k(\alpha_1 - \alpha_2)}, \quad p_k = -\alpha_k r_k, \quad q_k = -\alpha_k s_k. \tag{2.8}$$

By virtue of (2.7) and (2.8) we readily obtain

$$\begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} = \frac{2id_k}{BC - A^2} \begin{bmatrix} \alpha_k^2 & -\alpha_k \\ -\alpha_k & 1 \end{bmatrix} \begin{bmatrix} C & A \\ A & B \end{bmatrix}, \tag{2.9}$$

$$\sum_{k=1}^{2} \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv E. \tag{2.10}$$

After applying the operator $\overset{\varkappa}{T}(\partial_t, n)$ to the matrix $\Gamma(z,t)$ given by (2.2) and taking into account (2.4), we have

$$\big[\overset{\varkappa}{T}(\partial_t, n)\Gamma(z,t)\big]^T = \operatorname{Im}\sum_{k=1}^{2} \begin{bmatrix} P_k & \Lambda_k \\ Q_k & S_k \end{bmatrix} \frac{\partial \ln \sigma_k}{\partial s(t)}, \tag{2.11}$$

where

$$\begin{aligned} \begin{bmatrix} P_k & \Lambda_k \\ Q_k & S_k \end{bmatrix} &= \begin{bmatrix} N_k & M_k \\ L_k & R_k \end{bmatrix} + \varkappa \begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ M_k &= -\frac{2}{a_{11}}(a_{11} d_k^2 - a_{13}\alpha_k + a_{12}) d_k, \quad N_k = -\alpha_k M_k, \\ R_k &= -\frac{2}{a_{11}\alpha_k}(a_{12}\alpha_k^2 - a_{23}\alpha_k + a_{22}) d_k, \quad L_k = -\alpha_k R_k. \end{aligned}$$

Each column of matrix (2.11) treated as a vector with respect to the point $z$ satisfies (1.1) everywhere except the point $z = t$, for any value of the parameter $\varkappa$. In the neighborhood of the point $z=t$ the terms of matrix (2.11) possess singularities (of order not higher than one).

Applying Viet's theorem to equation (2.1), we can represent the coefficients $N_k$, $L_k$, $M_k$, $R_k$ as

$$\begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} + 2\omega d_k \begin{bmatrix} \alpha_k & \alpha_k^2 \\ -1 & -\alpha_k \end{bmatrix}, \tag{2.12}$$

where $\omega = a_{12}/a_{11} - \operatorname{Re}(\alpha_1\alpha_2) \equiv a_{12}/a_{11} + b^{(1)}b^{(2)} - a^{(1)}a^{(2)}$.

By (2.7) we have

$$\begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} \begin{bmatrix} 1-i\omega A & -i\omega B \\ i\omega C & 1+i\omega A \end{bmatrix}, \tag{2.13}$$

or else by (2.9)

$$\begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix} = 2d_k \begin{bmatrix} \alpha_k^2 & -\alpha_k \\ -\alpha_k & 1 \end{bmatrix} \left\{ \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{i}{BC-A^2} \begin{bmatrix} C & A \\ A & B \end{bmatrix} \right\}. \tag{2.14}$$

Since $\alpha_k$ is the root of equation (2.1), the following relationships are established between the coefficients $A_k$, $B_k$, $C_k$ and $M_k$, $R_k$:

$$A_k = \frac{a_{11}}{2d_k} M_k^2, \quad B_k = \frac{a_{11}}{2d_k} M_k R_k, \quad C_k = \frac{a_{11}}{2d_k} R_k^2. \tag{2.15}$$

By virtue of (2.14) and (2.12) relations (2.15) imply

$$\begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} = \begin{bmatrix} N_k & M_k \\ L_k & R_k \end{bmatrix} \left\{ \omega a_{11} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{i a_{11}}{BC-A^2} \begin{bmatrix} C & A \\ A & B \end{bmatrix} \right\}. \tag{2.16}$$

By (2.13) it follows from (2.16) that

$$\sum_{k=1}^{2} \begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} = \frac{im}{BC-A^2} \begin{bmatrix} C & A \\ A & B \end{bmatrix}, \tag{2.17}$$

where $m = a_{11}[1-\omega^2(BC-A^2)]$. Note that $\omega > 0$ and $m > 0$ [7].

It is evident that for $\varkappa = 0$ from (2.11) we obtain the matrix

$$\big[\overset{0}{T}(\partial_t, n)\Gamma(z,t)\big]^T = \big[T(\partial_t, n)\Gamma(z,t)\big]^T = \operatorname{Im} \sum_{k=1}^{2} \begin{bmatrix} N_k & M_k \\ L_k & R_k \end{bmatrix} \frac{\partial \ln \sigma_k}{\partial s(t)}$$

whose every element by (2.13) and (2.10) will have a singularity of the type $|z-t|^{-1}$ at the point $z=t$.

Matrix (2.11) for $\varkappa = \varkappa_N$, where

$$\varkappa_N = \frac{\omega(BC-A^2)}{m}, \tag{2.18}$$

takes the form either

$$\big[\overset{\varkappa_N}{T}(\partial_t,n)\Gamma(z,t)\big]^T=\big[N(\partial_t,n)\Gamma(z,t)\big]^T=\operatorname{Im}\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)},\quad (2.19)$$

where $N(\partial_t,n)$ is the so-called pseudostress operator [7],

$$\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}=\frac{a_{11}}{m}\begin{bmatrix}N_k & M_k\\ L_k & R_k\end{bmatrix}\begin{bmatrix}1+i\omega A & i\omega C\\ i\omega B & 1-i\omega A\end{bmatrix}\qquad (2.20)$$

or

$$\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}=-\frac{i}{m}\begin{bmatrix}A_k & B_k\\ B_k & C_k\end{bmatrix}\begin{bmatrix}B & -A\\ -A & C\end{bmatrix}.\qquad (2.21)$$

We can see from (2.18) that $\varkappa_N>0$, since $\omega>0$, $m>0$, and it follows from (2.6) that $BC-A^2>0$.

By (2.13) or (2.17) it follows from (2.20) or (2.21) that

$$\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}=E.$$

Using the latter condition, we can show that when $z$ and $t$ are points of the Lyapunov curve, for $z=t$ the elements of matrix (2.19) contain only singularities integrable in an ordinary sense.

We shall say that $u=(u_1,u_2)^T$ and $v=(v_1,v_2)^T$ are the conjugate vectors or satisfy the generalized Cauchy–Riemann conditions [11] if the following conditions are fulfilled:

$$\begin{aligned}
&A_{11}\frac{\partial u_1}{\partial x_1}+A_{12}\frac{\partial u_2}{\partial x_2}+A_{13}\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)+\varkappa_N\frac{\partial u_2}{\partial x_2}=\\
&\quad=\frac{1}{m}\Big(B\frac{\partial v_1}{\partial x_2}-A\frac{\partial v_2}{\partial x_2}\Big),\\
&A_{12}\frac{\partial u_1}{\partial x_1}+A_{22}\frac{\partial u_2}{\partial x_2}+A_{23}\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)+\varkappa_N\frac{\partial u_1}{\partial x_1}=\\
&\quad=\frac{1}{m}\Big(A\frac{\partial v_1}{\partial x_1}-C\frac{\partial v_2}{\partial x_1}\Big),\\
&A_{13}\frac{\partial u_1}{\partial x_1}+A_{23}\frac{\partial u_2}{\partial x_2}+A_{33}\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)-\varkappa_N\frac{\partial u_2}{\partial x_1}=\\
&\quad=-\frac{1}{m}\Big(B\frac{\partial v_1}{\partial x_1}-A\frac{\partial v_2}{\partial x_1}\Big),\\
&A_{13}\frac{\partial u_1}{\partial x_1}+A_{23}\frac{\partial u_2}{\partial x_2}+A_{33}\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)-\varkappa_N\frac{\partial u_1}{\partial x_2}=\\
&\quad=-\frac{1}{m}\Big(A\frac{\partial v_1}{\partial x_2}-C\frac{\partial v_2}{\partial x_2}\Big).
\end{aligned}\qquad (2.22)$$

By virtue of equalities (2.22) we can readily show that the vector $u$ is a solution of system (1.1).

By transforming equalities (2.22) it is proved in [11] that the equalities obtained formally from (2.22) by replacing $u$ by $v$ and $v$ by $(-u)$ are also valid. Hence the vector $v$ is also a solution of system (1.1).

After multiplying the first equality from (2.22) by $n_1(x)$ and the third equality by $n_2(x)$ and summing the obtained equations, we obtain

$$\big[N(\partial_x,n)u(x)\big]_1=\frac{1}{m}\Big(B\frac{\partial v_1(x)}{\partial s(x)}-A\frac{\partial v_2(x)}{\partial s(x)}\Big).$$

Similarly, having multiplied the second equality from formula (2.22) by $n_2(x)$ and the fourth equality by $n_1(x)$ and summing the obtained equations, we have

$$\big[N(\partial_x,n)u(x)\big]_2=-\frac{1}{m}\Big(A\frac{\partial v_1(x)}{\partial s(x)}-C\frac{\partial v_2(x)}{\partial s(x)}\Big).$$

The latter two formulas finally yield

$$N(\partial_x,n)u(x)=\frac{1}{m}\begin{bmatrix} B & -A\\ -A & C\end{bmatrix}\frac{\partial v(x)}{\partial s(x)}. \tag{2.23}$$

The above arguments show that the equality

$$N(\partial_x,n)v(x)=-\frac{1}{m}\begin{bmatrix} B & -A\\ -A & C\end{bmatrix}\frac{\partial u(x)}{\partial s(x)} \tag{2.24}$$

is also valid.

Thus the conjugate vectors $u$ and $v$ satisfy conditions (2.23) and (2.24).

We can easily show that if $u$ and $v$ are conjugate, then on account of (2.23) and (2.24) we shall have

$$\overset{\varkappa}{T}(\partial_x,n)u(x)=\frac{1}{m}\begin{bmatrix} B & -A\\ -A & C\end{bmatrix}\frac{\partial v(x)}{\partial s(x)}+(\varkappa-\varkappa_N)\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}\frac{\partial u(x)}{\partial s(x)}. \tag{2.25}$$

If we introduce the notation $w=u+iv$, then (2.23) and (2.24) can be rewritten as a single equality

$$N(\partial_x,n)w(x)=-\frac{i}{m}\begin{bmatrix} B & -A\\ -A & C\end{bmatrix}\frac{\partial w(x)}{\partial s(x)}. \tag{2.26}$$

As is known, any solution $v$ of system (1.1) of the class $C^1(\overline{D}^+)\cap C^2(D^+)$ can be written as [12, Ch. 1]

$$v(z)=\frac{1}{2\pi}\int\limits_S\Big(\big[N(\partial_t,n)\Gamma(z,t)\big]^T\{v(t)\}^+-\Gamma(z,t)\big\{N(\partial_t,n)v(t)\big\}^+\Big)ds,\quad z\in D^+,$$

which after integrating by parts the second summand under the integral sign and taking into account (2.24) and (2.21), eventually gives

$$v(z)=\frac{1}{2\pi}\int\limits_S\bigg(\operatorname{Im}\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)}\{v(t)\}^+ - \\ -\operatorname{Re}\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)}\{u(t)\}^+\bigg)ds,\quad z\in D^+, \tag{2.27}$$

where $u$ is the conjugate to the vector $v$ of the class $C^1(\overline{D}^+)\cap C^2(D^+)$.

Analogously, if $u$ is any solution of system (1.1) of the class $C^1(\overline{D}^+)\cap C^2(D^+)$ and $v$ is its conjugate vector of the same class, then

$$u(z)=\frac{1}{2\pi}\int\limits_S\bigg(\operatorname{Im}\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)}\{u(t)\}^+ + \\ +\operatorname{Re}\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)}\{v(t)\}^+\bigg)ds,\quad z\in D^+. \tag{2.28}$$

Thus from (2.27) and (2.28) we obtain

$$w(z)=\frac{1}{2\pi i}\int\limits_S\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)}\{w(t)\}^+ds,\quad z\in D^+, \tag{2.29}$$

which is a generalization of the Cauchy integral formula.

Let us consider the potential

$$\widetilde{w}(g)(z)=\frac{1}{\pi}\int\limits_S\big[N(\partial_t,n)\Gamma(z,t)\big]^T g(t)ds, \tag{2.30}$$

where the density $g=(g_1,g_2)^T$ is an arbitrary real vector. Obviously, potential (2.30) is a solution of system (1.1) both in $D^+$ and in $D^-$ for an arbitrary integrable density $g$.

**Theorem 1.** *If $S\in C^{2+\alpha}$, $0<\alpha<1$, and $0<\beta<\alpha$, then:*

(a) $\widetilde{w}(g)\in C^{k+\beta}(\overline{D}^\pm)$, *if $g\in C^{k+\beta}(S)$ for $k=0,1,2$, and for arbitrary $g\in C^\beta(S)$ and $t_0\in S$, we have the equality*

$$\{\widetilde{w}(g)(t_0)\}^\pm=\pm g(t_0)+\frac{1}{\pi}\int\limits_S\big[N(\partial_t,n)\Gamma(t_0,t)\big]^T g(t)ds; \tag{2.31}$$

(b) *for arbitrary $g \in C^{1+\beta}(S)$ the vector $N(\partial_z, n)\widetilde{w}(g)(z)$ is continuously extendable to $S$ from $D^{\pm}$, and for arbitrary $t_0 \in S$ we have the equality*

$$\{N(\partial_{t_0}, n)\widetilde{w}(g)(t_0)\}^{\pm} = -\frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \mathcal{E}_{(k)} \int_S \frac{\partial \ln \sigma_k}{\partial s(t_0)} \frac{\partial g(t)}{\partial s(t)} ds, \quad (2.32)$$

*where*

$$\mathcal{E}_{(k)} = \frac{2d_k}{a_{11}} \begin{bmatrix} \alpha_k^2 & -\alpha_k \\ -\alpha_k & 1 \end{bmatrix} + \varkappa_N \begin{bmatrix} 2L_k & R_k - N_k \\ R_k - N_k & -2M_k \end{bmatrix} + \varkappa_N^2 \begin{bmatrix} C_k & -B_k \\ -B_k & A_k \end{bmatrix}.$$

Finally, let us quote Green's formula. If $u \in C^1(\overline{D}^+) \cap C^2(D^+)$ is a solution of system (1.1) in $D^+$ and $\varkappa$ is an arbitrary constant, then the formula

$$\int_{D^+} \overset{\varkappa}{T}(u,u) d\sigma = \int_S \{u\}^+ \{T(\partial_t, n)u\}^+ ds \quad (2.33)$$

is valid [7], where

$$\begin{aligned}\overset{\varkappa}{T}(u,u) = & A_{11}\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + A_{22}\Big(\frac{\partial u_2}{\partial x_2}\Big)^2 + 2A_{13}\frac{\partial u_1}{\partial x_1}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) + \\ & + 2A_{23}\frac{\partial u_2}{\partial x_2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) + 2(A_{12} + \varkappa)\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \\ & + \Big(A_{33} - \frac{\varkappa}{2}\Big)\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)^2 + \frac{\varkappa}{2}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)^2.\end{aligned}$$

It is proved [12, Ch. 1] that the quadratic form $\overset{\varkappa}{T}(u,u)$ is positive definite for $0 \le \varkappa < \alpha^*$, where $\alpha^*$ is the least positive root of the equation

$$\alpha^3 + 2(A_{12} - A_{33})\alpha^2 + (A_{12}^2 - A_{11}A_{22} + 4A_{13}A_{33} - 4A_{12}A_{33})\alpha + 2\Delta = 0.$$

Note that $\varkappa_N < \alpha^*$, and hence the quadratic form $\overset{\varkappa}{T}(u,u)$ is positive definite for $\varkappa = 0$ and $\varkappa = \varkappa_N$.

The equation $\overset{\varkappa}{T}(u,u) = 0$ admits, for $\varkappa = 0$, a solution $u(z) = (e_1 - e_3x_2, e_2 + e_3x_1)^T$, and for $0 < \varkappa < \alpha^*$, a solution $u(z) = (e_1, e_2)^T$, where $e_1$, $e_2$, $e_3$ are arbitrary real constants.

For the domain $D^-$, formula (2.33) (with the minus before the integral sign on the right-hand side) also holds if the vector $u$ satisfies additionally conditions (1.3) at infinity.

**Theorem 2.** *The homogeneous basic mixed problem* ($F = 0$, $f = 0$) *in the class of regular vector functions* (*in the sense of Definition* 1) *has, for $0 \le \varkappa < \alpha^*$, only a trivial solution.*

*Proof.* Note first that formula (2.33) can also be applied to the regular solution $u$ of system (1.1) in $D^+$. Indeed, denote by $D^+_*$ the domain which is obtained from $D^+$ by removing infinitesimal circumferences of infinitesimal radii with centers at the points $a_j, b_j$, $j = 1, 2, \dots, p$, of the contour $S$. Obviously, $u \in C^1(\overline{D}^+_*)$ and therefore we can apply (2.33) to the domain $D^+_*$ and then pass to the limit as the radii of the above-mentioned circumferences tend to zero. Taking into account that the vector $u$ is continuously extendable at all points of the contour $S$, and $\overset{\varkappa}{T}(\partial_x, n)u$ is continuously extendable at all points of the contour $S$, except possibly the points $a_j$, $b_j$, $j = 1, 2, \dots, p$, in whose neighborhood we have singularities less tthan 1, we find that the integrals on the right side extended to the arcs of these circumferences contained in $D^+$ will tend to zero, and formula (2.33) will appear to be valid for a regular solution of system (1.1) in $D^+$.

Let now $u_0$ be a regular solution of the homogeneous basic mixed problem ($F = 0$, $f = 0$). In that case, if we apply (2.33) to the vector $u_0$, then the integral on the right-hand side will be equal to zero, and, since the quadratic form $\overset{\varkappa}{T}(u_0, u_0)$ is positive definite, for $0 \le \varkappa < \alpha^*$ we shall have $\overset{\varkappa}{T}(u_0, u_0) = 0$. These arguments and the condition that on some part of $S''$ the boundary value of the vector $u_0$ is equal to zero imply that $u_0(z) = 0$, $z \in D^+$, which was to be proved. □

## § 3. Reducting the Basic Mixed Problem to Integral Equations

A solution of the problem is sought in the form $u(z) = \operatorname{Re} w(z)$, where $w(z)$ is represented in terms of a potential of type (2.29),

$$w(z) = \frac{1}{\pi i} \int\limits_S \sum_{k=1}^{2} \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} \frac{\partial \ln \sigma_k}{\partial s(t)} g(t) ds \tag{3.1}$$

whose density $g = (g_1, g_2)^T$ is an unknown Hölder class real vector.

Then on the basis of (2.25) and (2.26) the generalized stress vector will be of the form

$$\overset{\varkappa}{T}(\partial_z, n)u(z) = \frac{1}{m} \begin{bmatrix} B & -A \\ -A & C \end{bmatrix} \frac{\partial v(z)}{\partial s(z)} + (\varkappa - \varkappa_N) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial u(z)}{\partial s(z)}, \tag{3.2}$$

where this time

$$v(z) = \operatorname{Im} w(z) = \operatorname{Im} \frac{1}{\pi i} \int\limits_S \sum_{k=1}^{2} \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} \frac{\partial \ln \sigma_k}{\partial s(t)} g(t) ds. \tag{3.3}$$

By (3.2), the generalized principal stress vector acting from the side of the positive normal on an arbitrary arc $l$ confined within $D^+$, connecting the points $z_0$ and $z$ of $D^+$ (to within arbitrary constants of additive vectors) has the form

$$\frac{1}{m}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix} v(z) + (\varkappa - \varkappa_N)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} u(z) = \\ = \int_{z_0}^{z} \overset{\varkappa}{T}(\partial_t, n)u(t)ds + \text{const}. \tag{3.4}$$

By equalities (3.1), (3.3), (3.4), (2.31) and Theorem 1 the boundary conditions of problem (1.4) and (1.5) give, for the unknown vector $g$, a system of singular integral equations with discontinuous coefficients of the form

$$A(t_0)g(t_0) + \frac{B(t_0)}{\pi}\int_S \frac{g(t)dt}{t-t_0} + \frac{1}{\pi}\int_S K(t_0,t)g(t)ds = \\ = \Omega(t_0) + D(t_0), \quad t_0 \in S, \tag{3.5}$$

where

$$A(t_0) = \begin{cases} (\varkappa - \varkappa_N)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & t_0 \in S', \\ E, & t_0 \in S'', \end{cases}$$

$$B(t_0) = \begin{cases} -\dfrac{1}{m}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix}, & t_0 \in S', \\ 0, & t_0 \in S'', \end{cases}$$

$$K(t_0,t) = \frac{1}{m}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix} Q(t_0,t) + (\varkappa - \varkappa_N)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} R(t_0,t), \\ t_0 \in S',\ t \in S,$$

$$K(t_0,t) = R(t_0,t), \ \ t_0 \in S'', \ \ t \in S,$$

$$Q(t_0,t) = i\frac{\partial\theta}{\partial s(t)}E - \operatorname{Re}\sum_{k=1}^{2}\begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix}\frac{\partial}{\partial s(t)}\ln\Big(1 + \nu_k\frac{\overline{t} - \overline{t_0}}{t - t_0}\Big),$$

$$R(t_0,t) = \frac{\partial\theta}{\partial s(t)}E + \operatorname{Im}\sum_{k=1}^{2}\begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix}\frac{\partial}{\partial s(t)}\ln\Big(1 + \nu_k\frac{\overline{t} - \overline{t_0}}{t - t_0}\Big),$$

$$\nu_k = \frac{1 + i\alpha_k}{1 - i\alpha_k}, \quad \theta = \arg(t_0 - t),$$

$$\Omega(t_0) = \begin{cases} \int_{a_j}^{t_0} F(t)ds, & t_0 \in S'_j, \\ f(t_0), & t_0 \in S'', \end{cases} \qquad D(t_0) = \begin{cases} D^{(j)}, & t_0 \in S'_j, \\ 0, & t_0 \in S''; \end{cases}$$

$D^{(j)} = (D_1^{(j)}, D_2^{(j)})^T$ is an arbitrary real constant vector.

Multiplying the first equation of system (3.5) by an arbitrary constant $M$ for $t_0 \in S'$ and adding the obtained equaion to (3.5) we obtain

$$(\varkappa - \varkappa_N)(g_1(t_0) - Mg_2(t_0)) + \frac{1}{\pi m} \int\limits_S \frac{(A-MB)g_1(t) + (AM-C)g_2(t)}{t - t_0} dt +$$

$$+ \frac{1}{\pi} \int\limits_S \big\{[K_{11}(t_0,t)M + K_{21}(t_0,t)]g_1(t) + [MK_{12}(t_0,t) + K_{22}(t_0,t)]g_2(t)\big\} ds =$$

$$= \int\limits_{a_j}^{t_0} (F_2(t) + MF_1(t)) ds + D_2^{(j)} + MD_1^{(j)}, \quad t_0 \in S'_j, \tag{3.6}$$

where $K_{rq}(t_0,t)$, $r = 1,2$, $q = 1,2$, are the components of the matrix $K(t_0,t)$.

Choose $M$ by the condition that $C - AM = M(A - MB)$. This implies that

$$M_{1,2} = \frac{A \pm i\sqrt{BC - A^2}}{B}.$$

Let $M = (A + i\sqrt{BC - A^2})B^{-1}$ and substitute it into (3.6). Then we obtain

$$(\varkappa - \varkappa_N)\omega(t_0) - \frac{i\sqrt{BC - A^2}}{\pi m} \int\limits_S \frac{\omega(t)dt}{t - t_0} + \frac{1}{\pi} \int\limits_S K_1(t_0,t)\omega(t) ds +$$

$$+ \frac{1}{\pi} \int\limits_S \overline{K_2(t_0,t)\omega(t)} ds = \int\limits_{a_j}^{t_0} (F_2(t) + MF_1(t)) ds + D_2^{(j)} + MD_1^{(j)}, \tag{3.7}$$

where $\omega(t) = g_1(t) - Mg_2(t)$,

$$\begin{aligned} K_1(t_0,t) &= \frac{1}{\overline{M} - M} \Big\{ \overline{M}[K_{21}(t_0,t) + MK_{11}(t_0,t)] + \\ &\quad + MK_{12}(t_0,t) + K_{22}(t_0,t) \Big\}, \\ \overline{K_2(t_0,t)} &= \frac{1}{M - \overline{M}} \Big\{ M[K_{21}(t_0,t) + MK_{11}(t_0,t)] + \\ &\quad + MK_{12}(t_0,t) + K_{22}(t_0,t) \Big\}, \quad t_0 \in S', \quad t \in S. \end{aligned} \tag{3.8}$$

If now we multiply the second equation of system (3.5) by $M$ for $t_0 \in S''$ and add the obtained equation to (3.5), we obtain

$$\omega(t_0) + \frac{1}{\pi} \int\limits_S K_1(t_0,t)\omega(t) ds + \frac{1}{\pi} \int\limits_S \overline{K_2(t_0,t)\omega(t)} ds = f_1(t_0) - Mf_2(t_0), \tag{3.9}$$

where

$$
\begin{aligned}
K_1(t_0,t)=&\frac{1}{\overline{M}-M}\Big\{\overline{M}[K_{11}(t_0,t)-MK_{21}(t_0,t)]+\\
&+K_{12}(t_0,t)-MK_{22}(t_0,t)\Big\},\\
\overline{K_{22}(t_0,t)}=&\frac{1}{M-\overline{M}}\Big\{M[K_{11}(t_0,t)-MK_{21}(t_0,t)]+\\
&+K_{12}(t_0,t)-MK_{22}(t_0,t)\Big\},\ \ t_0\in S'',\ \ t\in S.
\end{aligned}
\tag{3.10}
$$

It is obvious that by combining equations (3.7) and (3.9) we shall have a singular integral equation with discontinuous coefficients that contains, along with the unknown function, its conjugate lying outside the characteristic part and is of the form

$$
\begin{aligned}
&a(t_0)\omega(t_0)+\frac{b(t_0)}{\pi i}\int\limits_S\frac{\omega(t)dt}{t-t_0}+\frac{1}{\pi}\int\limits_S K_1(t_0,t)\omega(t)ds+\\
&+\frac{1}{\pi}\int\limits_S\overline{K_2(t_0,t)\omega(t)}ds=\varphi(t_0)+D_0(t_0),\quad t_0\in S,
\end{aligned}
\tag{3.11}
$$

where

$$
a(t_0)=\begin{cases}\varkappa-\varkappa_N, & t_0\in S',\\ 1, & t_0\in S'',\end{cases}\qquad b(t_0)=\begin{cases}m^{-1}\sqrt{BC-A^2}, & t_0\in S',\\ 0, & t_0\in S'',\end{cases}
$$

$$
\varphi(t_0)=\begin{cases}\int\limits_{a_j}^{t_0}(F_2(t)+MF_1(t))ds, & t_0\in S'_j,\\ f_1(t_0)-Mf_2(t_0), & t_0\in S'',\end{cases}
$$

$$
D_0(t_0)=\begin{cases}D_2^{(j)}+MD_1^{(j)}, & t_0\in S'_j,\\ 0, & t_0\in S'',\end{cases}
$$

the functions $K_1(t_0,t)$ and $K_2(t_0,t)$ are defined by equalities (3.8) and (3.10), respectively.

The characteristic homogeneous equation corresponding to equation (3.11) is

$$
a(t_0)\omega(t_0)+\frac{b(t_0)}{\pi i}\int\limits_S\frac{\omega(t)dt}{t-t_0}=0.
$$

The homogeneous problem of conjugation corresponding to the above equation is written as

$$
\Phi^+(t)=G(t)\Phi^-(t),\quad t\in S,
\tag{3.12}
$$

where

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_S \frac{\omega(t)dt}{t-z},$$

$$G(t) = \frac{a(t)-b(t)}{a(t)+b(t)} = \begin{cases} (\varkappa-\delta_0)(\varkappa+\delta_1)^{-1}, & t_0 \in S', \\ 1, & t_0 \in S'', \end{cases}$$

$\delta_0$ and $\delta_1$ are the positive numbers [8]

$$\delta_0 = \frac{\sqrt{BC-A^2}}{a_{11}(1-\omega\sqrt{BC-A^2})}, \quad \delta_1 = \frac{\sqrt{BC-A^2}}{a_{11}(1+\omega\sqrt{BC-A^2})}.$$

Since $\alpha^* < \delta_0$ [12, Ch. 1], for $0 \le \varkappa < \alpha^* < \delta_0$ we have $(\varkappa-\delta_0)(\varkappa+\delta_1)^{-1} < 0$. By the general theory [9], all nodes $a_j, b_j$, $j = 1, 2, \dots, p$, are nonsingular, and the canonical solution $X(z)$ of the class of functions bounded at all nodes, i.e., of the class $h_{2p}$, of the problem of conjugation (3.12) is given by the formula

$$X(z) = \prod_{j=1}^{p}(z-a_j)^{\frac{1}{2}+i\beta}(z-b_j)^{\frac{1}{2}-i\beta}, \quad \beta = \frac{1}{2\pi}\ln\frac{\delta_0-\varkappa}{\delta_1+\varkappa}. \tag{3.13}$$

Under $X(z)$ is meant a branch which is holomorphic on the plane cut along $S'$ and satisfying the condition $\lim_{z\to\infty} z^{-p}X(z) = 1$.

It follows from (3.13) that the index of the class $h_{2p}$ of the problem of conjugation and hence of equation (3.11) is equal to $-p$, since the order $X(z)$ is equal to $p$ at infinity.

A solution $\omega(t)$ of equation (3.11) will be sought in the class $h_{2p}$.

Note that equation (3.11) does not actually differ from the equation obtained for the basec mixed problem in the isotropic case. Therefore if we take into account the restrictions imposed on the contour $S$ and the boundary values $f$ and $F$, then on the entire contour $S$, arguing as in [2], we can show that $\omega(t)$ belongs to the Hölder class, while the derivative $\omega'(t)$ to the class $H^*$.

Since the index of the class $h_{2p}$ of equation (3.11) is equal to $-p$, by the general theorem [2] we have $\nu - \nu' = -2p$, where $\nu$ is the number of linearly independent solutions of the class $h_{2p}$ of the homogeneous equation corresponding to (3.11), and $\nu'$ is the number of linearly independent solutions of the class $h'_{2p}$ of the adjoint homogeneous equation adjoint to $h_{2p}$.

Let us prove that $\nu = 0$ and hence $\nu' = 2p$. Indeed, let the homogeneous equation corresponding to equation (3.11) have a nontrivial solution $\omega_0(t)$

in the class $h_{2p}$. We construct the

$$w_0(z)=\frac{1}{\pi i}\int\limits_S\sum_{k=1}^{2}\begin{bmatrix}E_k & F_k\\ G_k & H_k\end{bmatrix}\frac{\partial\ln\sigma_k}{\partial s(t)}g_0(t)ds, \tag{3.14}$$

where

$$g_0(t)=\left(\frac{\overline{M}\omega_0(t)-M\overline{\omega_0(t)}}{\overline{M}-M},\frac{\omega_0(t)-\overline{\omega_0(t)}}{\overline{M}-M}\right)^T.$$

Then $u_0(z)=\operatorname{Re}w_0(z)$ will satisfy the homogeneous boundary conditions $\{u_0(t)\}^+=0$ for $t_0\in S''$ and $\{\overset{\varkappa}{T}(\partial_t,n)u_0(t)\}^+=0$ for $t_0\in S'$, except possibly the points $a_j,b_j$, $j=1,2,\dots,p$, at which the the generalized stress vector may have only an integrable singularity.

Assume that $0\le\varkappa<\alpha^*$. Then by Theorem 2 we conclude that $u_0(z)=0$, $z\in D^+$. Therefore $N(\partial_z,n)u_0(z)=0$, and by equality (2.32) and Theorem 1 we have $\{N(\partial_t,n)u_0(t)\}^+=\{N(\partial_t,n)u_0(t)\}^-$, which holds at all points of the contour $S$, except possibly the points $a_j,b_j$, $j=1,2,\dots,p$. Thus $u_0(z)$ satisfies system (1.1) in the domain $D^-$, the boundary condition $\{N(\partial_t,n)u_0(t)\}^-=0$ on $S$ and the decreasing conditions (1.3) at infinity. Since $\varkappa_N<\alpha^*$, we can easily show by formula (2.33) that $u_0(z)=0$, $z\in D^-$. Finally, from the equality $\{u_0(t)\}^+-\{u_0(t)\}^-=2g_0(t)$ it follows that $g_0(t)=0$, $t\in S$, which contradicts our assumption.

Thus the homogeneous equation corresponding to (3.11) has only a trivial solution in the class $h_{2p}$ and hence, as mentioned above, $\nu'=2p$. By the general theory [2], the condition of solvability of equation (3.11) in the class $h_{2p}$ has the form

$$\operatorname{Re}\int\limits_S(\varphi(t)+D_0(t))\sigma_j(t)dt=0,\quad j=1,2,\dots,2p, \tag{3.15}$$

where $\sigma_j$, $j=1,2,\dots,2p$, is a complete system of linearly independent solutions of the class $h'_{2n}$ of the adjoint homogeneous equation. Conditions (3.15) can be rewritten as

$$\sum_{r=1}^{p}(\gamma_{jr}D_1^{(r)}+\delta_{jr}D_2^{(r)})=\eta_j,\quad j=1,2,\dots,2p, \tag{3.16}$$

where

$$\gamma_{jr}=\operatorname{Re}M\int\limits_{S'_r}\sigma_j(t)dt,\quad \delta_{jr}=\operatorname{Re}\int\limits_{S'_r}\sigma_j(t)dt,\quad \eta_j=-\operatorname{Re}\int\limits_S\varphi(t)\sigma_j(t)dt.$$

Let us prove that with respect to the unknowns $D_1^{(r)}$ and $D_2^{(r)}$ the determinant of the system of algebraic equations (3.16) is different from zero.

Indeed, let $\varphi = 0$. Then in system (3.16) all $\eta_j = 0$ and it becomes the homogeneous system. If the determinant of the system is equal to zero, then the homogeneous system admits a solution different from zero. Let $D_{10}^{(r)}$ and $D_{20}^{(r)}$, $r = 1, 2, \dots, p$, be such a solution. Then equation (3.11) will be solvable in the class $h_{2p}$ for $\varphi = 0$ and $D_0(t) = D_{20}^{(r)} + MD_{10}^{(r)}$, $t \in S_r'$, $r = 1, 2, \dots, p$, and $D_0(t) = 0$, $t \in S''$. Let $\omega_0(t)$ be a solution of (3.11). Again constructing potential (3.14) and repeating the above arguments, we obtain $g_0(t) = 0$, and hence $D_{10}^{(r)} = D_{20}^{(r)} = 0$, $r = 1, 2, \dots, p$, which contradicts our assumption.

Consequently, the determinant of system (3.16) is different from zero, system (3.16) is always uniquely solvable in the class $h_{2p}$ and its solution leads to the solution of the considered problem.

Thus the following theorem is valid.

**Theorem 3.** *Under conditions* (1) *and* (2)*, the basic mixed problem* (1.4)*,* (1.5) *has, for* $0 \le \varkappa < \alpha^*$*, a unique solution which is representable in the form* $u(z) = \operatorname{Re} w(z)$*, where* $w(z)$ *is given by formula* (3.1)*, and the vector* $g$ *is defined uniquely by a solution of the class* $h_{2p}$ *of equation* (3.11).

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