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# BOUNDARY VALUE PROBLEMS OF THE THEORY OF ANALYTIC FUNCTIONS WITH DISPLACEMENTS

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ABSTRACT. Integral representations are constructed for functions holomorphic in a strip. Using these representations an effective solution of Carleman type problem is given for a strip.

## INTRODUCTION

In studying some problems of the theory of elasticity and mathematical physics there arise boundary value problems of the theory of analytic functions for a strip [1, 2, 3, 4] when a linear combination of function values is given at a point $t$ of the lower strip boundary and at a point $t+a$ of the upper boundary.

We refer this problem to Carleman type problem for a strip. To solve this problem, in §1 we construct integral representations which play the same role in its solution as a Cauchy type integral plays in solving a linear conjugation problem. In §2 a solution is obtained for a Carleman type problem for a strip with continuous coefficients and in §3 a solution is given for a Carleman type problem for a strip with a coefficient polynomially increasing at infinity. In §4 a conjugation problem with a displacement is investigated.

When the coefficient is a meromorphic function, a Carleman type homogeneous problem was solved by E. W. Barens in [5] by means of Euler's gamma-functions (provided that the poles and zeros of the coefficient are known). Later various particular cases were studied in [1] and [6].

## § 1. INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS IN A STRIP

Let a function $\Phi(z)$, $z=x+iy$, be holomorphic in a strip $\{a<y<b$, $-\infty<x<\infty\}$, continuous in a closed strip $\{a\le y<\le b$, $-\infty<x<\infty\}$

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and satisfy the condition $\Phi(z)e^{\mu|z|}\to 0$ for $|z|\to\infty$, $\mu\geq 0$. The class of functions satisfying these conditions will be denoted by $A_a^b(\mu)$.

Let

$$\Phi_k(z)\in A_0^\beta(\mu_k),\quad \mu_k<\frac{\pi\beta[3+(-1)^k]}{2(\alpha^2+\beta^2)},\quad k=1,2, \tag{1.1}$$

where $\alpha$ and $\beta$ are real numbers, $\beta>0$. Then the following formulas are valid:

$$\Phi_1(z)=\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\Phi_1(t)+\Phi_1(t+a)}{\sinh p(t-z)}\,dt,\quad 0<\mathcal{I}_m z<\beta, \tag{1.2}$$

$$\Phi_2(z)=\frac{\cosh pz}{2a}\int\limits_{-\infty}^{+\infty}\frac{\Phi_2(t)-\Phi_2(t+a)}{\cosh pt\sinh p(t-z)}\,dt+\Phi_2\Big(\frac{a}{2}\Big),\ 0<\mathcal{I}_m z<\beta, \tag{1.3}$$

where $p=\frac{\pi i}{a}$, $a=\alpha+i\beta$.

The above formulas are obtained using the theorem on residues.

If $\Phi_k(z)$ has a form

$$\Phi_k(z)=\Psi_k(z)+\sum_{j=1}^{n}A_j\Big(z-\frac{a}{2}\Big)^{-j},\quad \Psi_k(z)\in A_0^\beta(\mu_k),\quad k=1,2,$$

then we shall have

$$\Phi_1(z)=\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\Phi_1(t)+\Phi_1(t+a)}{\sinh p(t-z)}\,dt-\sum_{j=1}^{n}\frac{(-p)^jA_j}{j!}\Big(\frac{1}{\cosh pz}\Big)^{(j-1)},$$
$$0<\mathcal{I}_m z<\beta, \tag{1.4}$$

$$\Phi_2(z)=\frac{\cosh pz}{2a}\int\limits_{-\infty}^{+\infty}\frac{\Phi_2(t)-\Phi_2(t+a)}{\cosh pt\sinh p(t-z)}\,dt-\sum_{j=1}^{n}\frac{A_j(-p)^j}{j!}(\tanh pz)^{(j-1)}+$$
$$+\Phi_2\Big(\frac{a}{2}\Big),\ 0<\mathcal{I}_m z<\beta. \tag{1.5}$$

Let further $F_k$, $k=1,2$, be functions given on the real axis $L$ and having the form $F_k(x)=f_k(x)e^{\mu_k|x|}$, $f_k(\pm\infty)=0$, where $f_k$ are functions satisfying the Hölder condition everywhere on $L$, $\mu_k$ are numbers satisfying inequality (1.1).

Consider the integrals

$$\Phi_1(z) = \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)dt}{\sinh p(t-z)}, \quad 0 < \mathcal{I}_m z < \beta, \tag{1.6}$$

$$\Phi_2(z) = \frac{\cosh pz}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)dt}{\cosh pt \sinh p(t-z)}, \quad 0 < \mathcal{I}_m z < \beta. \tag{1.7}$$

It is obvious that the functions are holomorphic in a strip $0 < y < \beta$.

Using the Sohotski–Plemelj formulas we can show that the boundary values of $\Phi_1$ and $\Phi_2$ are expressed by the formulas

$$\begin{aligned} \Phi_1(t_0) &= \frac{F_1(t_0)}{2} + \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)dt}{\sinh p(t-t_0)}, \\ \Phi_1(t_0+a) &= \frac{F_1(t_0)}{2} - \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)dt}{\sinh p(t-t_0)}; \end{aligned} \tag{1.8}$$

$$\begin{aligned} \Phi_2(t_0) &= \frac{F_2(t_0)}{2} + \frac{\cosh pz}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)dt}{\cosh pt \sinh p(t-t_0)}, \\ \Phi_2(t_0+a) &= -\frac{F_2(t_0)}{2} + \frac{\cosh pt_0}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)dt}{\cosh pt \sinh p(t-t_0)}. \end{aligned} \tag{1.9}$$

It follows from Plemelj–Privalov's theorem that that the boundary values of $\Phi_1$ and $\Phi_2$ satisfy the Hölder condition on a finite part of the boundary.

Let us investigate the behavior of these functions in the neighbourhood of a point at infinity. We begin by considering the case with $\mu_k = 0$, $k = 1, 2$.

Rewrite formula (1.6) as

$$\begin{aligned} \Phi_1(z) = \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \Big[\frac{1}{\sinh p(t-z)} - \frac{a}{p} \frac{1}{(t-z)(t+a-z)}\Big] F_1(t)dt \,+ \\ + \frac{1}{2ap} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)}{t-z} dt - \frac{1}{2ap} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)dt}{t+a-z}, \quad 0 < \mathcal{I}_m z < \beta. \end{aligned}$$

Here the first term is holomorphic in the closed strip $0 \le \mathcal{I}_m z \le \beta$ and tends to zero at infinity. The second and the third term are analytic in the strip $0 < \mathcal{I}_m z < \beta$, vanish at infinity and their boundary values satisfy the Hölder condition, including points at infinity [7].

Therefore $\Phi_1 \in A_0^\beta(0)$.

Now let us consider the function $\Phi_2(z)$. Rewrite formula (1.7) as

$$\Phi_2(z) = \frac{1}{2a}\int\limits_{-\infty}^{+\infty} \frac{(\cosh pz - \cosh pt)F_2(t)dt}{\cosh pt \sinh p(t-z)} + \frac{1}{2a}\int\limits_{-\infty}^{+\infty} \frac{F_2(t)dt}{\sinh p(t-z)}.$$

As we have shown, the second term here belongs to the class $A_0^\beta(0)$. Denote the first term by $\mathcal{I}$ and rewrite it as

$$\begin{aligned}\mathcal{I} &= -\frac{1}{2a}\int\limits_{-\infty}^{+\infty} \frac{\sinh \frac{p}{2}(t+z)F(t)}{\cosh pt \sinh \frac{p}{2}(t-z)}dt = \\ &= -\frac{1}{2a}\int\limits_{-\infty}^{+\infty} \frac{\sinh \frac{p}{2}(2x-\tau+iy)F_2(x-\tau)}{\cosh p(x-\tau)\cosh \frac{p}{2}(\tau+iy)}d\tau = \\ &= -\frac{1}{2a}\Big(\int\limits_{-\infty}^{0} + \int\limits_{0}^{+\infty}\Big) \frac{\sinh \frac{p}{2}(2x-\tau+iy)F_2(x-\tau)}{\cosh p(x-\tau)\cosh \frac{p}{2}(\tau+iy)}d\tau.\end{aligned}$$

Let $x > 0$. Then the first integral will be bounded in the strip $0 \le \mathcal{I}_m z < \beta$, since $2x - \tau < 2(x-\tau)$.

Rewrite the second integral as

$$\begin{aligned}&\frac{1}{2a}\int\limits_{0}^{+\infty} \frac{\sinh \frac{p}{2}(2x-\tau+iy)F_2(x-\tau)}{\cosh p(x-\tau)\cosh \frac{p}{2}(\tau+iy)}d\tau = \\ &= \Big(\frac{1}{2a}\int\limits_{-\infty}^{0} + \frac{1}{2a}\int\limits_{0}^{x}\Big) \frac{\sinh \frac{p}{2}(x+t+iy)F_2(t)dt}{\cosh pt \cosh \frac{p}{2}(x-t+iy)}.\end{aligned}$$

The first term is bounded, since $x + t < x - t$. The second term can be written in the form

$$\begin{aligned}\frac{1}{2a}\int\limits_{0}^{x} \frac{\sinh \frac{p}{2}([(x-t+iy)+2t]F_2(t)dt}{\cosh pt \cosh \frac{p}{2}(x-t+iy)} &= \frac{1}{2a}\int\limits_{0}^{x} \tanh \frac{p}{2}(x-t+iy)F_2(t)dt + \\ &+ \frac{1}{2a}\int\limits_{0}^{x} \tanh ptF_2(t)dt. \qquad (1.10)\end{aligned}$$

Since the function $\tanh \frac{p}{2}z = \tanh \frac{|p|^2}{2\pi}(\beta+\alpha i)z$ is holomorphic in the strip $0 \le \mathcal{I}_m z \le \delta < \beta$ and $|\tanh \frac{p}{2}z| \to 1$, the estimate

$$|\Phi_2(z)| < |\Phi_0(x)| + \varepsilon|x| \qquad (1.11)$$

holds for the function $\Phi_2$ when $x$ are large in the closed strip $0 \le \mathcal{I}_m z \le \delta$. $\Phi_0(x)$ is bounded for $x > 0$ and $\varepsilon < 0$ is an arbitrarily small number. A similar estimate is also true for the case $x < 0$. In the same manner one can obtain an estimate of the form (1.11) in the strip $0 < \delta \le \mathcal{I}_m z \le \beta$ provided that the function $\Phi_2(z)$ is represented as

$$\Phi_2(z) = \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{\cosh pz + \cosh pt}{\cosh pt \sinh p(t-z)} F_2(t)dt - \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)dt}{\sinh p(t-z)}.$$

Now let us consider the case with $\mu_k > 0$, $k = 1, 2$. Rewrite (1.6) as follows:

$$\Phi_1(z) = \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{\cosh \mu_1 t \varphi_1(t)dt}{\sinh p(t-z)}, \quad \varphi_1(t) \equiv f_1(t)/\cosh \mu_1 t.$$

It is obvious that $\varphi_1(t)$ satisfies the Hölder condition in the neighbourhood of a point at infinity.

We write the function $\Phi_1$ in the form

$$\Phi_1(z) = \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{\varphi_1(t) \cosh[\mu_1(t-z) + \mu_1 z]}{\sinh p(t-z)} dt =$$

$$= \frac{\cosh \mu_1 z}{2a} \int\limits_{-\infty}^{+\infty} \frac{\cosh \mu_1 (t-z)}{\sinh p(t-z)} \varphi_1(t)dt + \frac{\sinh \mu_1 z}{2a} \int\limits_{-\infty}^{+\infty} \frac{\sinh(t-z)\mu_1}{\sinh p(t-z)} \varphi_1(t)dt.$$

Since $\mu_1 < \pi\beta/(\alpha^2 + \beta^2) = \mathrm{Re}\, p$, we have $\Phi_1 \in A_0^\beta(\mu_1)$. Taking this into account and applying the arguments used in investigating the behavior of the function $\Phi_2(z)$ in the case with $F_2(\pm\infty) = 0$, we show that

$$\Phi_2 \in A_0^\beta(\beta_2).$$

Let us formulate the results obtained above.

**Theorem 1.** *If the functions $F_k(x)e^{-\mu_k |x|}$, $(k = 1, 2)$, satisfy the Hölder condition everywhere on $L$ and $F_k(x)e^{-\mu_k |x|} \to 0$ for $|x| \to +\infty$, where $\mu_k$ are some numbers satisfying inequality* (1.1)*, then $\Phi_k \in A_0^\beta(\mu_k)$ for $\mu_1 \ge 0$, $\mu_2 > 0$, $\exp \Phi_2 \in A_0^\beta(\varepsilon)$ for $\mu_2 = 0$, where $\varepsilon$ is an arbitrarily small positive number.*

Formulas (1.8) and (1.9) imply

$$\Phi_1(t) + \Phi_1(t+a) = F_1(t), \quad t \in (-\infty, \infty), \tag{1.12}$$
$$\Phi_2(t) - \Phi_2(t+a) = F_2(t), \quad t \in (-\infty, \infty), \tag{1.13}$$

i.e., $\Phi_1(z)$ and $\Phi_2(z)$ defined by (1.6) and (1.7) are solutions of boundary value problems (1.12) and (1.13) of the class $A_0^\beta(\mu_k)$, $k = 1, 2$.

Clearly, if the function $\Phi_2(z)$ is a solution of problem (1.13), then the function $W(z) = c + \Phi_2(z)$ will also be a solution. We shall show that problems (1.12) and (1.13) do not have other solutions of the class $A_0^\beta(\mu_k)$, $k = 1, 2$. For this we should prove

**Theorem 2.** *If $F_2(t) \in L(-\infty, \infty)$, then for a solution of problem* (1.13) *of the class $A_0^\beta(0)$ to exist it is necessary and sufficient that the condition*

$$\int_{-\infty}^{\infty} F_2(t)dt = 0$$

*be fulfilled.*

*Proof.* We can rewrite formula (1.7) as

$$\Phi_2(z) = \frac{1}{2a}\int_{-\infty}^{\infty} \coth p(t-z)F_2(t)dt - \frac{1}{2a}\int_{-\infty}^{\infty} \tanh ptF_2(t)dt. \qquad (1.14)$$

It is obvious that the limits of $\Phi_2(z)$ exist for $x \to \pm\infty$, $0 \le y \le \beta$, and

$$C + \Phi_2(\pm\infty + iy) = \pm\frac{1}{2a}\int_{-\infty}^{\infty} F_2(t)dt - \frac{1}{2a}\int_{-\infty}^{\infty} \tanh ptF_2(t)dt + C. \quad (1.15)$$

Taking

$$C = \frac{1}{2a}\int_{-\infty}^{\infty} F_2(t)\tanh pt\, dt$$

and setting

$$\int_{-\infty}^{\infty} F_2(t)dt = 0, \qquad (1.16)$$

we find by virtue of (1.15) and (1.16) that the solution of problem (1.13) has the form

$$W(z) = \frac{1}{2a}\int_{-\infty}^{\infty} \coth p(t-z)F_2(t)dt \qquad (1.17)$$

and belongs to the class $A_0^\beta(0)$.

The necessity is proved by integrating equality (1.13) and applying the Cauchy theorem. □

It remains to prove

**Theorem 3.** *If the function* $\varphi \in A_0^\beta\left(\frac{\pi\beta(3+\lambda)}{2(\alpha^2+\beta^2)}\right)$, $\lambda = \pm 1$, *and satisfies the condition* $\varphi(z) = \lambda\varphi(x+a)$, *then it is constant and, for* $\lambda = -1$, *equal to zero.*

*Proof.* Let $\lambda = -1$ and

$$\Psi(z) = \frac{\varphi(z)}{\cosh pz} + \varphi\Big(\frac{a}{2}\Big)\frac{a}{\pi(z-\frac{a}{2})}. \tag{1.18}$$

The function $\Psi(z) \in A_0^\beta(0)$ and satisfies the condition

$$\Psi(x) - \Psi(x+a) = \frac{2a^2}{\pi}\varphi\Big(\frac{a}{2}\Big)\frac{1}{x^2 - a^2/4}. \tag{1.19}$$

Since $\Psi(z)$ is a solution of problem (1.19) of the class $A_0^\beta(0)$, the condition

$$\frac{2a^2}{\pi}\varphi\Big(\frac{a}{2}\Big)\int\limits_{-\infty}^{\infty}\frac{dx}{x^2-a^2/4} = 4ai\varphi\Big(\frac{a}{2}\Big) = 0$$

is fulfilled on account of Theorem 2. Thus $\Psi(z)$ is a solution of the homogeneous problem

$$\Psi(x) - \Psi(x+a) = 0, \quad -\infty < x < +\infty.$$

If we introduce the function

$$\Psi_1(z) = \frac{\Psi(z) - \Psi\left(\frac{a}{2}\right)}{\cosh pz},$$

then we shall have

$$\Psi_1(x) + \Psi_1(x+a) = 0, \quad -\infty < x < +\infty.$$

By applying the Fourier transform to the latter equality we obtain

$$\widehat{\Psi}_1(1 + e^{i\alpha t}) \equiv 0.$$

Hence we have $\widehat{\Psi}_1(t) \equiv 0$, $\Psi_1(z) = 0$. Therefore by (1.18) $\varphi(z) = 0$. We have thereby proved the theorem for $\lambda = -1$.

Let $\lambda = 1$. Then

$$\varphi(x) - \varphi(x+a) = 0.$$

The function

$$\Psi(z) = \varphi(z) - \varphi\Big(\frac{3}{4}a\Big) \tag{1.20}$$

also satisfies this condition and $\varphi\left(\frac{3}{4}a\right) = 0$.

We introduce the notation

$$\Psi_0(z)=\frac{\Psi(z)}{\cosh 2pz}+\frac{a}{2\pi}\frac{\Psi\left(\frac{a}{4}\right)}{z-\frac{a}{4}}.$$

Now, repeating the above arguments, we find that $\Psi\left(\frac{a}{4}\right)=0$, i. e., $\Psi_0(z)\in A_0^\beta(0)$ and satisfies the condition

$$\Psi_0(x)-\Psi_0(x+a)=0.$$

But, as shown above, in that case $\Psi_0(z)=\Psi(z)=0$ and therefore equality (1.20) implies

$$\varphi(z)=\varphi\left(\frac{3}{4}a\right),$$

which proves the theorem. □

## § 2. A Carleman Type Problem with a Continuous Coefficient for a Strip

Let us consider the following problem: find a function $\Phi$ of the class $A_0^\beta(\mu)$ by the boundary condition

$$\Phi(x)=\lambda G(x)\Phi(x+a)+F(x),\quad -\infty<x<+\infty, \tag{2.1}$$

where $a=\alpha+i\beta$, $\beta>0$, $\mu<\pi\beta(3+\lambda)/2(\alpha^2+\beta^2)$, $F$ and $G$ are the given functions satisfying the Hölder condition including a point at infinity, $G\neq 0$ and $F(\pm\infty)=0$, $G(-\infty)=G(\infty)=1$, the constant $\lambda$ takes the value 1 or $-1$.

The integer number $\varkappa=\frac{1}{2\pi}[\arg G(x)]_{-\infty}^{+\infty}$, where $[\arg G(x)]_{-\infty}^{+\infty}$ denotes an increment of the function $\arg G(x)$ when $x$ runs over the entire real axis from $-\infty$ to $\infty$, is called the index of the function $G(x)$. The index of $G_0(x)=G(x)[(x-a/2)/(x+a/2)]^\varkappa$ is equal to zero and therefore any branch of the function $\ln G_0(x)$ is continuous all over the real axis. We choose a branch that vanishes at infinity. By formulas (1.7) and (1.9) $G(x)$ can be represented as

$$G(x)=\frac{X(x)}{X(x+a)}, \tag{2.2}$$

where

$$\begin{aligned}&X(z)=\left(z-\frac{a}{2}\right)^\varkappa X_0(z),\\&X_0(z)=\exp\left(\frac{\cosh pz}{2a}\int_{-\infty}^{+\infty}\frac{\ln G_0(t)dt}{\cosh pt\sinh p(t-z)}\right).\end{aligned} \tag{2.3}$$

By virtue of Theorem 1 $X_0(z)$ and $[X_0(z)]^{-1} \in A_0^{\beta}(\varepsilon)$, where $\varepsilon$ is an arbitrarily small positive number.

Using (2.2), we rewrite condition (2.1) as

$$\frac{\Phi(x)}{X(x)} = \lambda \frac{\Phi(x+a)}{X(x+a)} + \frac{F(x)}{X(x)}, \quad -\infty < x < \infty. \tag{2.4}$$

The function $\Phi(z)/X(z)$ is holomorphic in the strip $0 < \mathcal{I}_m z < \beta$ except perhaps of the point $z = \frac{a}{2}$, at which it may have a pole of order $\varkappa$, for $\varkappa > 0$, and satisfies the condition

$$(\Phi(z)/X(z))e^{-\mu|z|} \to 0 \quad \text{for} \quad |x| \to \infty \quad \text{and} \quad 0 \le y \le \beta,$$

where $0 < \mu < \pi\beta(3+\lambda)/2(\alpha^2+\beta^2)$. By (1.4) and (1.5) condition (2.4) implies

$$\Phi(z) = \frac{X(z)}{2a} \int_{-\infty}^{+\infty} \frac{F(t)dt}{X(t)\sinh p(t-z)} + X(z)\varphi_1(z) \ \text{ for } \lambda = -1, \tag{2.5}$$

$$\Phi(z) = \frac{X(z)\cosh pz}{2a} \int_{-\infty}^{+\infty} \frac{F(t)dt}{X(t)\cosh pt \sinh p(t-z)} + $$
$$+ X(z)\varphi_2(z) \quad \text{for} \quad \lambda = 1, \tag{2.6}$$

where

$$\varphi_1(z) = \begin{cases} 0, & \varkappa \le 0, \\ \sum\limits_{k=1}^{\varkappa-1} C_k (1/\cosh pz)^{(k)}, & \varkappa > 0, \end{cases} \tag{2.7}$$

$$\varphi_2(z) = \begin{cases} 0, & \varkappa < 0, \\ \sum\limits_{k=1}^{\varkappa} C_k (\tanh pz)^{(k)}, & \varkappa \ge 0, \end{cases} \tag{2.8}$$

$C_k$ are arbitrary constants.

Let us investigate the behavior of the function

$$\varphi(z) = \frac{X(z)}{2a} \int_{-\infty}^{+\infty} \frac{F(t)dt}{X(t)\sinh p(t-z)}, \quad 0 \le \mathcal{I}_m z \le \beta, \tag{2.9}$$

in the neighbourhood of a point at infinity. The function $X(z)$ can be represented as

$$X(z) = \left(z - \frac{a}{2}\right)^{\varkappa} \exp \Gamma_1(z) \cdot \exp \Gamma_2(z),$$

where

$$\Gamma_1(z) = -\frac{1}{2a}\int\limits_{-\infty}^{\infty} \frac{\sinh\frac{p}{2}(z+t)\ln G_0(t)}{\cosh pt\cosh\frac{p}{2}(t-z)}dt,$$

$$\Gamma_2(z) = \frac{1}{2a}\int\limits_{-\infty}^{\infty} \frac{\ln G_0(t)dt}{\sinh p(t-z)}.$$

As we have shown, $\Gamma_2 \in A_0^\beta(0)$, i. e., $(\exp\Gamma_2(z)-1) \in A_0^\beta(0)$.

By differentiating the function $\Gamma_1(z)$ we obtain

$$\Gamma_1'(z) = \frac{1}{2a}\int\limits_{-\infty}^{\infty} \frac{\ln G_0(t)dt}{\cosh\frac{p}{2}(t-z)}, \quad 0 \le \mathcal{I}_m z \le \beta_0 < \beta.$$

It is easy to verify that $\Gamma_1'(z) \to 0$ for $|z| \to \infty$ and therefore for any there is a number $N$ such that

$$|\Gamma_1'(x+iy)| < \varepsilon \quad \text{for} \quad |x| > N, \quad 0 \le y \le \beta_0 < \beta. \tag{2.10}$$

We represent $\varphi(z)$ as

$$\varphi(z) = \frac{X(z)}{2a}\int\limits_{-N}^{N} \frac{F(t)}{X(t)}\cdot\frac{dt}{\sinh p(t-z)} +$$

$$+\frac{X(z)}{2a}\Bigg(\int\limits_{-\infty}^{N} + \int\limits_{N}^{\infty}\Bigg)\frac{F(t)dt}{X(t)\sinh p(t-z)}, \quad 0 \le \mathcal{I}_m z \le \beta_0.$$

It is easy to show that the first and the second term vanish for $x \to +\infty$. We shall show that the third term also tends to zero for $x \to +\infty$, $0 \le y \le \beta_0 < \beta$. This term will be denoted by $\mathcal{I}$.

$$\mathcal{I} = \int\limits_{N}^{\infty} \frac{\exp(\Gamma_2(z)-\Gamma_2(t))\exp(\Gamma_1(z)-\Gamma_1(t))(z-\frac{a}{2})^{\varkappa}F(t)}{(t-\frac{a}{2})^{\varkappa}\sinh p(t-z)}dt.$$

Assume that $\varkappa \ge 0$ and represent the function $\left(z-\frac{a}{2}\right)^{\varkappa}$ as

$$\left(z-\frac{a}{2}\right)^{\varkappa} = \varkappa!\sum_{n=1}^{\varkappa}\frac{(t-\frac{a}{2})^{\varkappa-n}(z-t)^n}{(\varkappa-n)!\,n!} + \left(t-\frac{a}{2}\right)^{\varkappa}. \tag{2.11}$$

Inequality (2.10) implies that

$$|\Gamma_1(z)-\Gamma_1(t)|\le\Big|\int\limits_t^z\Gamma'(s)ds\Big|\le\varepsilon|t-z|,\quad t<N,\ \ \varkappa>N,\ \ 0\le y\le\beta_0,$$

i.e., $\mathrm{Re}(\Gamma_1(t)-\Gamma_1(z))-\varepsilon|t-z|<0$. Thus we have

$$\Big|\exp[\Gamma_1(t)-\Gamma_1(z))-\varepsilon|t-z|-1\Big|<A|t-z|,\quad t>N,\ \ x>N.$$

The latter inequality and formula (2.11) imply

$$|\mathcal{I}|\le c\sum_{n=1}^{\varkappa}\int\limits_N^\infty\frac{e^{\varepsilon|x-t|}|x-t+iy|^n|F(t)|dt}{(\varkappa-n)!n!|\sinh p(x-t+iy)||t-\frac{a}{2}|^n}+$$

$$+c_1\int\limits_N^\infty\frac{e^{\varepsilon|x-t|}[A|x-t|+(1-e^{-\varepsilon|x-t|})]|F(t)|dt}{|\sinh p(x-t+iy)|}+\Big|\int\limits_N^\infty\frac{F(t)dt}{2|a|\sinh p(t-z)}\Big|.$$

where, as shown above, the third term is the modulus of a function of the class $A_0^\beta(0)$. Since $\varepsilon$ is an arbitrarily small number and $F(\infty)=F(-\infty)=0$, the first two terms are the convolutions of functions summable with functions tending to zero for $\to\infty$. Therefore they tend to zero for $x\to\infty$, $0\le y\le\beta_0<\beta$.

It can be shown in a similar manner that $\varphi(z)\to0$ for $x\to-\infty$, $0\le y\le\beta_0$, as well. It is not difficult to prove that the function $\varphi(z)$ tends to zero for $|x|\to\infty$, $\beta_0\le\mathcal{I}_mz\le\beta$. When $\varkappa<0$, one can use the same reasoning to show that $\varphi(z)\to0$ for $|x|\to\infty$, $0\le y\le\beta$, provided that $z$ and $t$ are exchanged in equality (2.11). Thus the function $\Phi$ represented by (2.5) tends to zero for $|x|\to+\infty$, $0\le y\le\beta$. Quite similarly, it is proved that for the function $\Phi$ defined by (2.6) we have $\Phi(z)e^{-\varepsilon|z|}\to0$ for $|x|\to\infty$, $0\le y\le\beta$.

For $\varkappa<0$ the function $X(z)$ has a pole of order $-\varkappa$ at the point $z=\frac{a}{2}$. In that case the solution exists only if the following conditions are fulfilled:

$$\int\limits_{-\infty}^\infty\frac{F(t)}{X(t)}\Big(\frac{1}{\cosh pt}\Big)^{(k)}dt=0,\quad k=0,\dots,(-\varkappa-1),\ \ \text{for}\ \ \lambda=-1,\qquad(2.12)$$

$$\int\limits_{-\infty}^\infty\frac{F(t)}{X(t)}\Big(\frac{e^{pt}}{\cosh pt}\Big)^{(k)}dt=0,\quad k=1,\dots,(-\varkappa-1),\ \ \text{for}\ \ \lambda=1.\qquad(2.13)$$

The results obtained can be formulated as

**Theorem 4.** *For $\lambda=-1$ and $\varkappa\ge0$ problem* (2.1) *is solvable in the class $A_0^\beta(0)$ and a general solution is given by* (2.5) *with formula* (2.7) *taken* *into account. If $\varkappa < 0$, then the problem is solvable if condition* (2.12) *is fulfilled. In these conditions problem* (2.1) *has a unique solution in the class* $A_0^\beta(0)$ *which is given by formula* (2.5) *for* $\varphi_1 = 0$.

**Theorem 5.** *if $\lambda = 1$ and $\varkappa \geq -1$, the problem* (2.1) *is solvable in the class* $A_0^\beta(\varepsilon)$ *and the solution is given by* (2.6) *with* (2.8) *taken into account; for $\varkappa < -1$ the solution exists provided that condition* (2.13) *is fulfilled. If these conditions are fulfilled, then problem* (2.1) *has a unique solution in the class* $A_0^\beta(\varepsilon)$. *This solution is given by* (2.6), *where* $\varphi_2 = 0$.

## § 3. A Carleman Type Problem with Unbounded Coefficients for a Strip

Problems of the elasticity theory can often be reduced to a Carleman type problem with coefficients polynomially increasing or decreasing at infinity. We shall consider such a case below.

We write the boundary condition of the problem in the form

$$\Phi(x) = P_n(x)G(x)\Phi(x+i\beta) + F(x), \quad -\infty < x < \infty, \tag{3.1}$$

where $G(x)$ and $F(x)$ satisfy the conditions discussed in §2, and $P_n(x)$ is a polynomial without real zeros. Condition (3.1) can be rewritten as

$$\Phi(x) = q[x^2 + 4\beta^2]^{[\frac{n}{2}]}(2\beta - ix)^{\delta(n)}G_0(x)\Phi(x+i\beta) + F(x), \tag{3.2}$$

where $\delta(n) = 0$ for even $n$ and $\delta(n) = 1$ for odd $n$; $q$ is a complex number; $G_0(x)$ is a Hölder class function including a point at infinity $G_0(-\infty) = G_0(\infty) = 1$.

As shown above, the function $G_0(x)$ can be represented as

$$G_0(x) = \frac{X_0(x)}{X_0(x+i\beta)}, \quad -\infty < x < \infty, \tag{3.3}$$

where

$$X_0(z) = \Big(z - \frac{i\beta}{2}\Big)^{\varkappa} \exp\bigg(\frac{\cosh pz}{2i\beta}\int\limits_{-\infty}^{\infty}\frac{\ln\,\big[G_0(t)(\frac{t+i\beta/2}{t-i\beta/2})^{\varkappa}\big]}{\cosh pt\sinh p(t-z)}dt\bigg). \tag{3.4}$$

Write the function $[x^2+4\beta^2]^{[\frac{n}{2}]}(2\beta - ix)^{\delta(n)}$ in form (3.3). We shall find solutions of the problems

$$X_1(x) = (2\beta + ix)X_1(x+i\beta), \quad -\infty < x < +\infty, \tag{3.5}$$

$$X_2(x+i\beta) = (2\beta - ix)X_2(x), \quad -\infty < x < +\infty. \tag{3.6}$$

Applying the Fourier transformation to conditions (3.5) and (3.6), we obtain the differential equations

$$(f_1(t)e^{\beta t})' = (1-2\beta e^{\beta t})f_1(t), \quad -\infty < t < +\infty,$$
$$f_2'(t) = (2\beta - e^{-\beta t})f_2(t), \quad -\infty < t < +\infty,$$

where $f_1(t)$ and $f_2(t)$ denote the Fourier transforms of the functions $X_1(x)$ and $X_2(x)$.

By performing the reverse Fourier transformation of the solutions of these equations we obtain the solutions of problems (3.5) and (3.6):

$$X_1(z) = \int\limits_{-\infty}^{+\infty} \exp\Big(-\frac{1}{\beta}e^{\beta t} + 3\beta z + itz\Big)dt, \quad 0 < \mathcal{I}_m z < \beta, \tag{3.7}$$

$$X_2(z) = \int\limits_{-\infty}^{+\infty} \exp\Big(-\frac{1}{\beta}e^{-\beta t} - 2\beta t + itz)dt, \quad 0 < \mathcal{I}_m z < \beta. \tag{3.8}$$

On substituting $e^{\beta t} = \beta\tau$, we have

$$\begin{aligned} X_1(z) &= \beta^2\beta^{\frac{iz}{\beta}} \int\limits_0^\infty e^{-\tau}\tau^{2+\frac{iz}{\beta}}\,d\tau = \beta^2\beta^{\frac{iz}{\beta}}\,\Gamma\Big(3+\frac{iz}{\beta}\Big), \\ X_2(z) &= \beta^{-\frac{iz}{\beta}} \int\limits_0^\infty e^{-\tau}\tau^{1-\frac{iz}{\beta}}\,d\tau = \beta\beta^{-\frac{iz}{\beta}}\,\Gamma\Big(2-\frac{iz}{\beta}\Big). \end{aligned} \tag{3.9}$$

We introduce the notation

$$X_3(z) = \Big[\frac{X_1(z)}{X_2(z)}\Big]^{[\frac{n}{2}]}(X_2(z))^{-\delta(n)}, \quad 0 < \mathcal{I}_m z < \beta. \tag{3.10}$$

Using Stirling's formulas [11], we obtain from (3.9) and (3.10) the following representations of the functions $X_1(z)$ and $X_2(z)$ in the neighbourhood of a point at infinity:

$$|X_1(z)| = C_1(y)e^{-\frac{\pi}{2\beta}|x|}|x|^{\frac{5}{2}-\frac{y}{\beta}}\Big(1+O\Big(\frac{1}{x}\Big)\Big), \quad 0 \le y \le \beta,$$
$$|X_2(z)| = C_2(y)e^{-\frac{\pi}{2\beta}|x|}|x|^{\frac{3}{2}+\frac{y}{\beta}}\Big(1+O\Big(\frac{1}{x}\Big)\Big), \quad 0 \le y \le \beta,$$

where $C_1(y)$, $C_2(y)$ are the bounded functions that do not vanish.

By virtue of these formulas, for sufficiently large values of $|z|$ (3.10) implies

$$|X_3(z)| = C(y)\big(|x|^{\frac{\beta-2y}{\beta}}\big)^{[\frac{n}{2}]}\big(e^{-\frac{\pi}{2\beta}|x|}|x|^{\frac{3}{2}+\frac{y}{\beta}}\big)^{-\delta(n)}\Big(1+O\Big(\frac{1}{x}\Big)\Big). \tag{3.11}$$

Using equalities (3.3) and (3.11), we rewrite condition (3.2) as

$$\frac{\Phi(x)}{X(x)} - q\frac{\Phi(x+i\beta)}{X(x+i\beta)} = \frac{F(x)}{X(x)}, \quad -\infty < x < \infty, \tag{3.12}$$

where $X(z) = X_0(z)X_3(z)$.

The function $\Phi(z)/X(z)$ is holomorphic in the strip $0 < \mathcal{I}_m z < \beta$ except perhaps for the point $z = i\beta/2$, where for $\varkappa > 0$ it may have a pole of order not higher than $\varkappa$, and satisfies the condition

$$\big(\Phi(z)/X(z)\big)e^{-\mu|z|} \to 0 \quad \text{for} \quad |z| \to \infty, \quad \mu < \frac{\pi}{2\beta} + \varepsilon.$$

Write $q$ in the form

$$q = \frac{X_4(x)}{X_4(x+i\beta)}, \quad X_4(z) = \exp\Big(\frac{iz}{\beta}\ln q\Big).$$

From (2.7) and (2.5) it follows that if $q$ is not a real positive number, then a general solution of problem (3.1) is given by the formula

$$\Phi(z) = \frac{X(z)}{2i\beta}\int\limits_{-\infty}^{\infty} \frac{\exp\big(\frac{\pi-\delta+i\gamma}{\beta}(z-t)\big)}{X(t)\sinh p(t-z)} F(t)dt + X(z)\varphi(z), \tag{3.13}$$

where $\gamma = \ln|q|$, $\delta = \arg q$, $0 < \delta < 2\pi$.

$$\varphi(z) = \sum_{j=0}^{\varkappa-1} C_j \frac{d^j}{dz^j}\Big(\exp\frac{(\pi-\delta+i\gamma)z}{\beta}/\cosh pz\Big). \tag{3.14}$$

For $\varkappa \geq 0$ the solution of problem (3.1) is given by formulas (3.13) and (3.14). Note that for $\varkappa \leq 0$ it is assumed that $\varphi(z) \equiv 0$. For $\varkappa < 0$ the function $X(z)$ has, at the point $z = \frac{i\beta}{2}$, a pole of order $-\varkappa$ and therefore the bounded solution exists in the finite part of the strip only if the conditions $\varphi(z) = 0$;

$$\int\limits_{-\infty}^{\infty} F(t)\Psi_j(t) = 0, \ \ \Psi_j(t) = \frac{d^j}{dt^j}\bigg(\frac{\exp(\frac{\delta-\pi-ij}{\beta})t}{\cosh pt}\bigg),$$
$$j = 0, \dots, (-1-\varkappa), \tag{3.15}$$

are fulfilled. Thus, like in §2, one can easily prove that in the case of even $n$ problem (3.1) has a solution $\Phi(z) \in A_0^\beta(0)$ for any $\delta \in (0, 2\pi)$, while in the case of odd $n$ it has a solution $\Phi(z) \in A_0^\beta\big(\frac{\pi-2\delta}{2\beta} + \varepsilon\big)$ for $\delta \in \big(0, \frac{\pi}{2}\big]$; $\Phi(z) \in A_0^\beta(0)$ for $\delta \in \big(\frac{\pi}{2}, \frac{3}{2}\pi\big)$; $\Phi(z) \in A_0^\beta\big(\frac{2\delta-3\pi}{2\beta} + \varepsilon\big)$ for $\delta \in \big[\frac{3}{2}\pi, 2\pi\big)$, where $\varepsilon > 0$ is an arbitrarily small number.

When $q > 0$, by substituting

$$\Phi(z) = X_4(x)\Psi(t)$$

condition (3.12) can be reduced to the condition

$$\frac{\Psi(x)}{X(x)} - \frac{\Psi(x+i\beta)}{X(x+i\beta)} = \frac{F(x)X_4(x)}{X(x)}, \quad -\infty < x < \infty. \tag{3.16}$$

By virtue of formula (3.15) a general solution of problem (3.1) has the form

$$\Phi(z) = \frac{X^*(z)}{2i\beta} \int_{-\infty}^{\infty} \frac{F(t)dt}{X^*(t)\sinh p(t-z)} + X^*(z)\varphi_2(z), \tag{3.17}$$

where $X^*(z) = X(z)\cosh pz X_4(z)$,

$$\varphi_2(z) = \begin{cases} \sum_{j=0}^{\varkappa-1} C_j \dfrac{d^j}{dz^j}(\tanh pz) + Cx, & \text{for} \quad \varkappa > 0, \\ C, & \text{for} \quad \varkappa = 0, \\ 0, & \text{for} \quad \varkappa \le -1, \end{cases} \tag{3.18}$$

$C$, $C_j$, $j = 0, \dots, (\varkappa - 1)$, are arbitrary constants. If $\varkappa < -1$, then the solution exists only provided that the condition

$$\int_{-\infty}^{\infty} \frac{F(t)}{X^*(z)} \cdot \frac{d^j}{dt^j}\Big(\frac{1}{\cosh pt}\Big) dt = 0, \quad j = 0, \dots, (-\varkappa - 2),$$

is fulfilled.

One can prove that $\Phi(z) \in A_0^\beta(\varepsilon)$ for an even $n$ and $\Phi(z) \in A_0^\beta(\pi/(2\beta)+\varepsilon)$ for odd $n$; here $\varepsilon$ is a small positive integer.

*Remark* 1. Formulas (3.8) and (3.9) can be obtained by applying formulas (3.3) and (3.4).

Indeed, if in formula (3.4) $G_0(t)$ is replaced by the function $(2\beta - ix)^{-1}$, then we shall have

$$X_2(z) = \exp\left(\frac{\cosh pz}{2i\beta} \int_{-\infty}^{\infty} \frac{\ln i - \ln(x + 2i\beta)}{\cosh px \sinh p(x-z)} dx\right). \tag{3.19}$$

By the function $\ln z$ we understand $\ln z = \ln|z| + \arg z$, $-\pi < \arg z < \pi$. After rewriting $\ln(x + 2i\beta)$ as

$$\ln(x+2i\beta) = \sum_{k=0}^{n} \big[\ln(x+i\beta(k+2)) - \ln(x+i\beta(k+3))\big] + \ln(x+i\beta(3+n))$$

and substituting this expression into (3.19), by virtue of (1.3) we obtain

$$\omega(z)=\frac{\cosh pz}{2i\beta}\int\limits_{-\infty}^{\infty}\frac{\ln i-\ln(x+2i\beta)}{\cosh px\sinh p(x-z)}dx=$$
$$=\sum_{k=0}^{n}\Big[\ln(x+i\beta(k+2))-\ln\Big(\frac{5i\beta}{2}+ki\beta\Big)\Big]+$$
$$+\frac{\cosh pz}{2i\beta}\int\limits_{-\infty}^{\infty}\frac{\ln(1+n)\beta)}{\cosh pt\sinh p(t-z)}dt+O\Big(\frac{1}{n}\Big).$$

If we perform some simple transformations and calculate the latter integral by the formula

$$\frac{\cosh pt}{2i\beta}\int\limits_{-\infty}^{\infty}\frac{\ln[(n+1)\beta]dx}{\cosh px\sinh p(x-z)}=\ln[(1+n)\beta]\Big(\frac{iz}{\beta}+\frac{1}{2}\Big).$$

then we shall have

$$\omega(z)=\sum_{k=1}^{n}\ln\Big[\Big(1+\frac{\zeta}{k}\Big)e^{-\frac{\zeta}{k}}\Big]-\zeta\Big(\ln(n+1)-\sum_{k=1}^{n}\frac{1}{k}\Big)-\ln\beta^{\zeta}-$$
$$-\frac{5}{2}\Big(\ln(n+1)-\sum_{k=1}^{n}\frac{1}{k}\Big)+\ln\zeta+C_n,\ \ \zeta=\frac{z+2i\beta}{i\beta}.$$

Passing to the limit as $n\to+\infty$, by virtue of (3.19) we obtain

$$X_2(z)=A\zeta\prod_{1}^{\infty}\Big(1+\frac{\zeta}{k}\Big)e^{-\frac{\zeta}{k}}e^{-c\zeta}\beta^{\zeta}=A\Gamma\Big(2-\frac{iz}{\beta}\Big)\beta^{2-\frac{iz}{\beta}}.$$

## § 4. On a Conjugation Boundary Value Problem with Displacements

As an application of the results obtained in §2 we shall consider one kind of a conjugation problem with displacements, when the boundary is a real axis. Denote by $S^+$ and $S^-$ the upper and the lower half-planes, respectively.

Consider the following problem:

Find a piecewise-holomorphic function bounded throughout the plane using the boundary condition

$$\Phi^+(x)=G(x)\Phi^-[\alpha(x)]+f(x),\quad -\infty<x<+\infty, \tag{4.1}$$

where $G(x)$ and $f(x)$ are the given functions satisfying the Hölder condition, $G(x) \neq 0$, $G(\infty) = G(-\infty) = 1$, $f(+\infty) = f(-\infty) = 0$,

$$\alpha(x) = \begin{cases} x, & x < 0, \\ bx, & x \geq 0, \end{cases}$$

$b$ is a constant.

If we denote by $\varkappa$ the index of the function $G(x)$, then $G(x)$ can be represented as [7]

$$G(x) = \frac{X^+(x)}{X^-(x)}, \quad X(z) = \begin{cases} \exp \omega(z), & z \in S^+, \\ \Big(\dfrac{z+i}{z-i}\Big)^{\varkappa} \exp \omega(z), & z \in S^-, \end{cases} \tag{4.2}$$

$$\omega(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{\ln G_0(t) dt}{t - z}, \quad G_0(x) = G(x) \Big(\frac{x+i}{x-i}\Big)^{\varkappa}.$$

On putting the value of $G(x)$ into (4.1), we obtain

$$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(\alpha(x))}{X^-(x)} = \frac{f(x)}{X^+(x)}, \quad -\infty < x < +\infty. \tag{4.3}$$

For $x < 0$ condition (4.3) takes the form

$$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(x)}{X^-(x)} = \frac{f(x)}{X^+(x)}. \tag{4.4}$$

A general solution of problem (4.4) can be written as

$$\Phi(z) = \frac{X(z)}{2\pi i} \int\limits_{-\infty}^{0} \frac{f(t) dt}{X^+(t)(t-z)} + X(z)\Phi_0(z). \tag{4.5}$$

The function $\Phi(z)$ is holomorphic on the plane cut along the positive semi-axis except perhaps for the neighbourhood of the point $z = -i$ at which it has a pole of order $\varkappa$ for $\varkappa > 0$.

For $\varkappa < 0$ the function $X(z)$ has a pole of order $-\varkappa$ at the point $z = -i$. Therefore for a bounded solution to exist it is necessary that the condition

$$\Phi_0^{(k)}(-i) + \frac{k!}{2\pi i} \int\limits_{-\infty}^{0} \frac{f(t) dt}{X^+(t)(t+i)^{k+1}} = 0, \quad k = 0, 1, \dots, (-\varkappa - 1), \tag{4.6}$$

be fulfilled.

If we put the value of $\Phi(z)$ into (4.3), then we have

$$\Phi_0^+(x) = G_1(x)\Phi_0^-(bx) + f_0(x), \quad 0 < x < \infty, \tag{4.7}$$

where $G_1(x)=\frac{X^-(bx)}{X^-(x)}$, $f_0(x)=\frac{f(x)}{X^+(x)}-A^+(x)+G_1(x)A^-(bx)$,

$$A(z)=\frac{1}{2\pi i}\int\limits_{-\infty}^{0}\frac{f(t)dt}{X^+(t)(t-z)}.$$

The function $z=e^\zeta$, $\zeta=\xi+i\eta$, maps the strip $0<\eta<2\pi$ onto the plane having a cut along the axis $x>0$.

On introducing the notation $\Phi_0(e^\zeta)=\Psi_0(\zeta)$, $0<\eta<2\pi$, we obtain

$$\Phi_0^+(x)=\Psi_0(\xi),\quad \Phi_0^-(bx)=\Psi_0(\xi+\ln b+2\pi i),\ \ -\infty<\xi<+\infty. \quad (4.8)$$

Thus problem (4.7) is reduced to the problem considered in §2

$$\Psi_0(\xi)=G^+(\xi)\Psi_0(\xi+\ln b+2\pi i)+F_0(\xi),\ \ -\infty<\xi<+\infty, \quad (4.9)$$

where $G^+(\xi)=G_1(e^\xi)$, $F_0(\xi)=f_0(e^\xi)$, $G^*(-\infty)=G^*(\infty)=1$,

$$\mathcal{J}_n dG^*=0,\quad F_0(+\infty)=0,\quad F_0(-\infty)=\frac{f(0)}{X^+(0)}.$$

Since for $\varkappa>0$ the function $\Phi_0(z)$ can have a pole of order $\varkappa$ at the point $z=-i$, we seek a solution $\Psi_0$ of problem (4.9) in the class of functions satisfying the condition

$$\Psi_0(\zeta)\Big(\frac{\zeta-\frac{3}{2}\pi i}{\zeta+\frac{3}{2}\pi i}\Big)^\varkappa\in A_0^\beta(\mu),\quad \mu<\frac{4\pi^2}{4\pi^2+\ln b}. \quad (4.10)$$

By virtue of formula (2.6) it is easy to show that a general solution of problem (4.9) is given by the formula

$$\Psi_0(\zeta)=\frac{X^*(\zeta)\cosh p\zeta}{2a}\int\limits_{-\infty}^{+\infty}\frac{F_0(t)dt}{X^+(t)\cosh pt\sinh p(t-\zeta)}+X^*(\zeta)\Psi(\zeta), \quad (4.11)$$

where $a=\ln b+2\pi i$, $p=\frac{\pi i}{a}$,

$$\psi(\zeta)=\begin{cases}\sum\limits_{k=0}^{\varkappa}c_k\coth^k p\Big(\zeta-\dfrac{3}{2}\pi i\Big), & \varkappa\ge 0,\\ c_{-1}, & \varkappa=-1,\\ 0, & \varkappa<-1,\end{cases}$$

$$X^*(\zeta)=\exp\bigg(\frac{\cosh p\zeta}{2a}\int\limits_{-\infty}^{+\infty}\frac{\ln G^*(t)dt}{\cosh pt\sinh p(t-\zeta)}\bigg).$$

Returning to the variable $z$, we obtain

$$\Psi_0(\zeta) = \frac{X_0(z)}{a} \int_0^{+\infty} \frac{t^{2p-1} f_0(t)dt}{(t^{2p} - z^{2p})X_0^+(t)} + X_0(z)(\varphi_0(z) - A), \qquad (4.12)$$

$$X_0(z) = \exp\left(\frac{1}{a}\int_0^{\infty} \frac{\ln G_1(t)t^{2p-1}}{t^{2p} - z^{2p}}dt\right), \quad A = \frac{1}{a}\int_0^{\infty} \frac{t^{2p-1} f_0(t)}{(t^{2p}+1)X_0^*(t)}dt.$$

With (4.5) and (4.12) taken into account we conclude that a general solution of problem (4.1) has the form

$$\Phi(z) = X(z)\left[\frac{1}{2\pi i}\int_{-\infty}^{0} \frac{f(t)dt}{X^+(t)(t-z)} + \frac{X_0(z)}{a}\int_0^{\infty} \frac{t^{2p-1} f_0(t)dt}{X_0^+(t)(t^{2p} - z^{2p})} + \right.$$
$$\left. + X_0(z)(\varphi_0(z) - A)\right], \qquad (4.13)$$

$$\varphi_0(z) = \begin{cases} \sum_{k=0}^{\varkappa} c_k \left(\frac{z^{2p} + (-i)^{2p}}{z^{2p} - (-i)^{2p}}\right)^k, & \varkappa \geq 0, \\ c_{-1}, & \varkappa = -1, \\ 0, & \varkappa < -1. \end{cases} \qquad (4.14)$$

The function $z^{2p}$ is holomorphic on the plane cut along the positive axis if by this function we mean the branch for which the limit as $z \to 1$ from the upper half-plane is equal to 1 while $t^{2p}$ denotes the function value, at the point $t$, of the upper edge of the cut.

For $\varkappa = -1$ the function $X_0(z)$ has a pole of first order at the point $z = -i$. In that case $\varphi_0(z) = C_{-1}$ and $X_0(-i) \neq 0$ and therefore the constant $c_1$ can be chosen so that for $z = -i$ the expression in square brackets on the right-hand side of (4.13) would vanish. Hence when $\varkappa \geq -1$ problem (4.1) has a bounded solution for an arbitrary right-hand side. When $\varkappa < -1$, for a bounded solution to exist it is necessary and sufficient that the conditions

$$\frac{d^k}{dz^k}\left[\frac{1}{2\pi i}\int_{-\infty}^{0} \frac{f(t)dt}{X^+(t)(t-z)} + \frac{X_0(z)}{a}\int_0^{\infty} \frac{t^{2p-1} f_0(t)dt}{X_0^+(t)(t^{2p} - z^{2p})} - AX_0(z)\right] = 0,$$
$$z = -i, \quad k = 1, \ldots, -\varkappa,$$

be fulfilled. Then the solution is given by formula (4.13).

For $b = 1$ we have $p = \frac{1}{2}$, $X_0(z) \equiv 1$, $f_0(t) \equiv f(t)$ and formulas (4.13) and (4.14) give a solution of the conjugation problem.

Conjugation problems with displacements are investigated in [8–10] in the case with $\alpha'(t)$ belonging to the Hölder class.

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