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FACTORIZATION SYSTEMS AND ADJUNCTIONS

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Abstract. Several results on the translation of factorization systems along adjunctions are proved. The first of these results is the answer to the question posed by G.Janelidze.

1. Introduction

As is known, there exists a close relation between full reflective subcategories and factorization systems in the categories satisfying some natural requirements. This relation is described by C. Cassidy, M. Hebert and G. M. Kelly [1] and given by an adjunction between the respective categories. The left adjoint sends the full reflective subcategory X of a finitely-well-complete category C to the pair

$$\big(I^{-1}(\mathrm{Iso}\, X), (H(\mathrm{Mor}\, X))^{\uparrow\downarrow}\big), \tag{1}$$

where I is the reflection and H is the inclusion functor,

$$\begin{tikzcd} C \arrow[rr, shift right, "I"'] && X \arrow[ll, shift right, "H"'] \end{tikzcd}\ , \tag{2}$$

$(H(\mathrm{Mor}\, X))^{\uparrow\downarrow}$ is the Galois closure of the class $H(\mathrm{Mor}\, X)$ for the relation $\downarrow$ ($e \downarrow m$ if and only if for each pair of morphisms i, j with $mi = je$, there exists a unique k for which $ke = i$ and $mk = j$). The main result of [1] is that pair (2) is in fact a factorization system and holds not only for full reflective subcategories, but also for every adjunction (2) with full and faithful right adjoint H.

The purpose of this paper is to find out whether the pair of classes

$$\big(I^{-1}(E), (H(M))^{\uparrow\downarrow}\big) \tag{3}$$

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is a factorization system, where (E, M) is not necessarily a trivial factorization system $(\mathrm{Iso}\, X, \mathrm{Mor}\, X)$ on X.

Section 2 deals with the above problem in the case of semi-left-exact adjunctions. It is proved that pair (3) is a factorization system for any factorization system (E, M) on X. This answers the question posed by G. Janelidze. It is proved that in that case the class $(H(M))^{\uparrow\downarrow}$ coincides with the class of all trivial coverings in the sense of Janelidze's Galois theory [2].

In Section 3 the situation where C is a finitely-well-complete category and X has a terminal object is considered. We show that for every prefactorization system (E, M) on X satisfying the condition (*) of [1] (which means that every morphism $x \longrightarrow 1$ admits an (E, M)-factorization) the pair of classes

$$\big(I^{-1}(\overset{o}{E}), (H(\overset{o}{M}))^{\uparrow\downarrow}\big)$$

is a factorization system; $(\overset{o}{E}, \overset{o}{M})$ is the reflective interior of (E, M) in the sense of [1]. As a corollary we obtain that in the above-mentioned case (3) is a factorization system for any reflective prefactorization system (E, M) with the property (*).

2. Factorization Systems Generated by Semi-Left-Exact Adjunctions

We use the term "factorization system" in the sense of P. J. Freyd and G. M. Kelly [3]. Namely, a factorization system on a category C is a pair of morphism classes E and M, each containing isomorphisms and being closed under composition, such that every morphism has a factorization $\alpha = me$ with $m \in M$ and $e \in E$, and such that $e \downarrow m$ for each $e \in E$ and $m \in M$. Recall that the notation $e \downarrow m$ means that for each commutative diagram

$$\begin{tikzcd} c_1 \arrow[r, "e"'] \arrow[d, "i"'] & c_2 \arrow[d, "j"] \\ c_3 \arrow[r, "m"] & c_4 \end{tikzcd}$$

there is a unique morphism $k : c_2 \to c_3$ for which $ke = i$ and $mk = j$. For each class S of morphisms $S^{\uparrow}$ denotes $\{r | r \downarrow s \text{ for all } s \in S\}$ and $S^{\downarrow}$ denotes $\{r | s \downarrow r \text{ for all } s \in S\}$. Every factorization system (E, M) satisfies the equalities $E^{\downarrow} = M$, $M^{\uparrow} = E$. A pair with this property is called a prefactorization system.

Let

$$\begin{tikzcd} C \arrow[r, shift right, "I"'] & X \arrow[l, shift right, "H"'] \end{tikzcd}, \tag{4}$$

be an adjunction A with right adjoint H and unit η. The following theorem is valid.

Theorem 2.1 (C. Cassidy, M. Hebert and G. M. Kelly [1]). *Let C be a finitely-well-complete category (i.e., C admits finite limits and all intersections of strong monomorphisms) and H be full and faithful. Then the pair of morphism classes*

$$\left(I^{-1}(\operatorname{Iso} X), (H(\operatorname{Mor} X))^{\uparrow\downarrow}\right)$$

is a factorization system.

Our aim is to find out whether the generalization of Theorem 2.1 holds, more exactly, whether the pair of classes

$$\left(I^{-1}(E), (H(M))^{\uparrow\downarrow}\right)$$

is a factorization system, where (E, M) is not necessarily a trivial one $(\operatorname{Iso} X, \operatorname{Mor} X)$ on X.

Lemma 2.2. $\left(I^{-1}(E), (H(M))^{\uparrow\downarrow}\right)$ *is a prefactorization system for any factorization system (E, M) on X.*

Proof. It is sufficient to show that $I^{-1}(E) = (H(M))^{\uparrow}$. The inclusion $I^{-1}(E) \subset (H(M))^{\uparrow}$ is clear, since each commutative diagram

$$\begin{tikzcd} c_1 \arrow[r, "\alpha"'] \arrow[d] & c_2 \arrow[d] \\ Hx_1 \arrow[r, "Hm"] & Hx_2 \end{tikzcd}$$

gives the commutative one

$$\begin{tikzcd} Ic_1 \arrow[r, "I\alpha"'] \arrow[d] & Ic_2 \arrow[d] \\ x_1 \arrow[r, "m"] & x_2 \end{tikzcd} .$$

To prove the inverse inclusion, consider any $\alpha \in (H(M))^{\uparrow}$ and the (E, M)-factorization of $I\alpha$

$$Ic_1 \xrightarrow{e} x \xrightarrow{m} Ic_2$$

with $e \in E$ and $m \in M$. We have the commutative diagram

$$\begin{tikzcd}
c_1 \arrow[r, "\alpha"'] \arrow[d, "\eta_{c_1}"'] & c_2 \arrow[ddl, "\delta"] \arrow[dd, "\eta_{c_2}"] \\
HIc_1 \arrow[d, "He"'] & \\
Hx \arrow[r, "Hm"] & HIc_2
\end{tikzcd} \tag{5}$$

for some morphism $\delta : c_2 \longrightarrow Hx$. (5) is transformed under the adjunction to the diagram

$$\begin{tikzcd}
Ic_1 \arrow[r, "I(\alpha)"] \arrow[d, "e"'] & Ic_2 \arrow[dl, "\overline{\delta}"'] \arrow[d, "\|"] \\
x \arrow[r, "m"] & Ic_2
\end{tikzcd} ,$$

where $\overline{\delta}$ is the image of δ. It follows that m is an isomorphism. $\square$

Let $\mathfrak{M}_A(M)$ denote the class of all trivial coverings in the sense of G. Janelidze's Galois theory, i.e,

$$\mathfrak{M}_A(M) = I^{-1}(M) \cap \left\{ \mu \,\middle|\, \begin{array}{ccc} & \xrightarrow{\eta} & \\ \mu\big\downarrow & & \big\downarrow HI\mu \\ & \xrightarrow{\eta} & \end{array} \quad \text{is a pullback} \right\}.$$

Lemma 2.3.

$$\mathfrak{M}_A(M) \subset (H(M))^{\uparrow\downarrow}. \tag{6}$$

Proof. Every commutative square

$$\begin{tikzcd}
c_1 \arrow[r, "\varepsilon"'] \arrow[d, "\gamma_1"'] & c_2 \arrow[d, "\gamma_2"] \\
c_3 \arrow[r, "\mu"] & c_4
\end{tikzcd} \tag{7}$$

with $\varepsilon \in I^{1}(E)$, $\mu \in \mathfrak{M}_A(M)$ induces the morphism $\delta : c_2 \to c_3$ for which the diagram

$$\begin{tikzcd}
& c_1 \arrow[rr, "\eta_{c_1}"] \arrow[dl, "\gamma_1"'] \arrow[dd, "\varepsilon"] & & HIc_1 \arrow[dl] \arrow[dd] \\
c_3 \arrow[rr, "\eta_{c_3}"] \arrow[dd, "\mu"'] & & HIc_3 \arrow[dd] & \\
& c_2 \arrow[rr, "\eta_{c_2}"] \arrow[dl, "\gamma_2"] & & HIc_2 \arrow[ul, "H\delta"'] \arrow[dl] \\
c_4 \arrow[rr, "\eta_{c_4}"'] & & HIc_4 &
\end{tikzcd}$$

is commutative. The front square is a pullback. Therefore the pair of morphisms $\gamma_2 : c_2 \longrightarrow c_4$ and $H(\delta)\eta_{c_2} : c_2 \longrightarrow HIc_3$ gives the morphism $\delta' : c_2 \longrightarrow c_3$ for which the respective diagram is commutative. A trivial calculation shows that δ' makes (7) commutative. If there is δ'' for which $\delta''\varepsilon = \gamma_1$ and $\mu\delta'' = \gamma_2$, then $I\delta' = I\delta''$. Hence δ''s and δ'''s compositions with μ and η_{c_3} are equal. Thus $\delta' = \delta''$. $\square$

Inclusion (6) is strict in general even if we restrict our consideration only to trivial factorization systems as is shown by Example 4.2 of [1]. The desired adjunction is

$$\begin{tikzcd} Ab \arrow[r, shift right, "I"'] & Grp_2 \arrow[l, shift right, "H"'] \end{tikzcd} ,$$

where Ab is the category of abelian groups, Grp_2 is the category of groups with exponent 2, H is a forgetful functor, $I(G) = G/2G$ and η_G is the projection $G \longrightarrow G/2G$.

Let C admit pullbacks. Recall some definitions. An adjunction A is called admissible if for each object C the right adjoint H^c in the induced adjunction

$$\begin{tikzcd} C \downarrow c \arrow[r, shift right, "I^c"'] & X \downarrow I(c) \arrow[l, shift right, "H^c"'] \end{tikzcd} \tag{8}$$

is full and faithful [2]. (Here the action of I^c coincides with that of I and H^c pulls-back the H-image of morphisms along respective η.) An adjunction A is called semi-left-exact if for each object C and each $\gamma : b \longrightarrow HIc$ the morphism $I(\gamma_0)$ is an isomorphism, where the square

$$\begin{tikzcd}
a \arrow[r, "\gamma_0"] \arrow[d, "\gamma_1"'] & b \arrow[d, "\gamma"] \\
c \arrow[r, "\eta_c"] & HIc
\end{tikzcd}$$

is a pullback [1]. In the case of fully faithful H an adjunction is admissible if and only if it is semi-left-exact.

Lemma 2.4. *Let an adjunction A be admissible. Then every morphism α of C has the factorization $\alpha = \mu\varepsilon$ with $\mu \in \mathfrak{M}_A(M)$ and $\varepsilon \in I^{-1}(E)$.*

Proof. Consider $\alpha : c_1 \longrightarrow c_2$ and the factorization of $I(\alpha)$

$$Ic_1 \xrightarrow{e} x \xrightarrow{m} Ic_2$$

with $e \in E$ and $m \in M$. We have the morphism β in the commutative diagram

$$\begin{tikzcd}
c_1 \arrow[rr, "\eta_{c_1}"] \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & & HIc_1 \arrow[d, "HI(\alpha)"'] \arrow[ddr, "He"] & \\
c_2 \arrow[rr, "\eta_{c_2}"] & \; \arrow[d] & HIc_2 & \\
& q \arrow[ul, "\mu"] \arrow[rr, "\psi"'] & & Hx \arrow[ul, "Hm"']
\end{tikzcd} \tag{9}$$

where the bottom square is a pullback. Observe, that $\mu = H^{c_2}(m)$, where H^{c_2} is the functor from (8). Since it is full and faithful, $I\mu \approx m \in M$ and the image $\overline{\psi}$ of ψ under the adjunction is an isomorphism. Hence the outer quardrangle in the diagram

$$\begin{tikzcd}
& & HIq \arrow[dl, "\approx"', "H(\overline{\psi})"] \arrow[ddl, "HI(\mu)"] \\
q \arrow[urr, "\eta_q"] \arrow[r] \arrow[d, "\mu"'] \arrow[dr, phantom, "\text{pb.}"] & Hx \arrow[d, "Hm"] & \\
c_2 \arrow[r, "\eta_{c_2}"'] & HIc_2 &
\end{tikzcd}$$

is a pullback. It follows that $\mu \in \mathfrak{M}_A(M)$. Further, (9) gives the commutative diagram

$$\begin{tikzcd} Ic_1 \arrow[r, "="] \arrow[d, "I\beta"'] & Ic_1 \arrow[d, "e"] \\ Iq \arrow[r, "\overline{\psi}", "\approx"'] & x \end{tikzcd}$$

from which we conclude that $I(\beta) \in E$. $\square$

Thus we obtain

Theorem 2.5. *Let* (4) *be a semi-left-exact adjunction and H be fully faithful. Then the pair $(I^{-1}(E), \mathfrak{M}_A(M))$ is a factorization system on C for every factorization system (E, M) on X.*

This theorem is the answer to the question posed by G. Janelidze. Observe, that the particular case of Theorem 2.5 for trivial factorization systems clearly follows also from Theorem 2.1 and the result of[1] asserting that

$$(H(\mathrm{Mor}\, X))^{\uparrow\downarrow} = \mathfrak{M}_A(\mathrm{Mor}\, X)$$

for semi-left-exact A. The latter assertion can be generalized.

Corollary 2.6. *Let H be fully faithul and A be semi-left-exact. Then $(H(M))^{\uparrow\downarrow} = \mathfrak{M}_A(M)$ for each factorization system (E, M) on X.*

Proof. It trivially follows from Lemma 2.2 and Theorem 2.5. $\square$

3. Case of Reflective Prefactorization Systems

In this section we give another generalization of Theorem 2.1. For that let us recall some definitions and results from [1].

Let C be a category with terminal object 1. Consider the partially ordered class of prefactorization systems on C satisfying the condition:
(*) every morphism $c \longrightarrow 1$ admits a (E, M)-factorization,
with order

$$(E_1, M_1) \leq (E_2, M_2) \quad \text{iff} \quad M_1 \subset M_2.$$

Consider this class as a category and denote it by $PS_*(C)$. There is an adjunction

$$\begin{tikzcd} FR(C) \arrow[r, shift right, "\Phi_c"'] & PS_*(C) \arrow[l, shift right, "\Psi_c"'] \end{tikzcd} ,$$

where $FR(C)$ denotes the category of full reflective subcategories of C (also ordered by the inclusion); $\Psi_c(E,M)$ is the full subcategory with objects c such that the unique morphism $c \longrightarrow 1$ belongs to M and

$$\Phi_c(R) = \big(r^{-1}(\operatorname{Iso} R), (\operatorname{Mor} R)^{\uparrow\downarrow}\big)$$

for full reflective subcategory

$$\begin{tikzcd} C \arrow[r, shift right, "r"'] & R \arrow[l, shift right, "i"'] \end{tikzcd} .$$

Observe, that by Theorem 2.1 the prefactorization system $\Phi_c(R)$ is a factorization system for any finitely-well-complete category C.

We have $\Psi_c\Phi_c = 1_{FR(C)}$. For each prefactorization system (E,M) the pair $\Phi_c\Psi_c(E,M)$ is called its reflective interior and denoted by $(\overset{\circ}{E}, \overset{\circ}{M})$. The structure of $\overset{\circ}{E}$ is described as :

$$\alpha \in \overset{\circ}{E} \quad \text{iff} \quad \beta\alpha \in E \quad \text{for some} \quad \beta \in E.$$

(E,M) is called reflective if $\overset{\circ}{E} = E$, $\overset{\circ}{M} = M$.

Theorem 3.1. *Let H be full and faithful in an adjunction*

$$\begin{tikzcd} C \arrow[r, shift right, "I"'] & X \arrow[l, shift right, "H"'] \end{tikzcd} ,$$

and let (E,M) be a reflective factorization system on X with the property $(*)$. Suppose C is finitely-well-complete and X has a terminal object 1. Then the pair

$$\big(I^{-1}(\overset{\circ}{E}), (E(\overset{\circ}{M}))^{\uparrow\downarrow}\big)$$

is a factorization system.

Proof. The pair

$$(E', M') = \Phi_c\big(H(\Psi_X(E,M))\big)$$

is a factorization system on C. Let us show that $E' = I^{-1}(\overset{\circ}{E})$.

Suppose r is the reflection

$$C \xrightarrow{r} H(\Psi_X(E,M)).$$

Consider any $\alpha : c_1 \to c_2$ in C. Let $Ic_i \xrightarrow{e_i} b_i \xrightarrow{m_i} 1$ be the factorization of $Ic_i \longrightarrow 1$ $(i = 1,2)$ with $e_i \in E$, $m_i \in M$. $r(\alpha)$ is $H(\delta)$, where δ is the diagonal morphism for which the diagram

$$\begin{tikzcd}
Ic_1 \arrow[r, "e_1"] \arrow[d, "I\alpha"'] & b_1 \arrow[dd, "m_1"] \arrow[ddl, dashed, "\delta"'] \\
Ic_2 \arrow[d, "e_2"'] & \\
b_2 \arrow[r, "m_2"] & 1
\end{tikzcd}$$

is commutative. Thus $r(\alpha)$ is an isomorphism if and only if δ is an isomorphism. It follows that $E' = I^{-1}\{\omega | \exists e \in E, e\omega \in E\} = I^{-1}(\mathring{E})$.

By construction $M' = (H(M_1))^{\uparrow\downarrow}$, where $M_1 = \{m |\operatorname{dom} m \to 1, \operatorname{codom} m \to 1$ are in $M\}$. Since $(H(M_1))^{\uparrow\downarrow} \subset (H(M_1^{\uparrow\downarrow}))^{\uparrow\downarrow}$ and $I^{-1}(\mathring{E}) = (H(M_1))^{\uparrow}$, we have $(I^{-1}(\mathring{E}))^{\downarrow} \subset (H(M_1^{\uparrow\downarrow}))^{\uparrow\downarrow}$. Moreover, for each $\varepsilon \in I^{-1}(E)$ and $\mu \in H(\mathring{M})$ clearly $\varepsilon \downarrow \mu$. Therefore $I^{-1}(\mathring{E}) \subset (H(M_1^{\uparrow\downarrow}))^{\uparrow}$, from which it follows that $(I^{-1}(\mathring{E}))^{\downarrow} \supset (H(M_1^{\uparrow\downarrow}))^{\uparrow\downarrow}$. Thus $(I^{-1}(\mathring{E}))^{\downarrow} = (H(M_1^{\uparrow\downarrow}))^{\uparrow\downarrow} = (H(\mathring{M}))^{\uparrow\downarrow}$. □

This theorem immediately implies

Corollary 3.2. *In the conditions of Theorem* 3.1, *let* (E, M) *be a reflective prefactorization system with* $(*)$. *Then*

$$\big(I^{-1}(E), (H(M))^{\uparrow\downarrow}\big)$$

is a factorization system.

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