

ON THE VECTOR PROCESS OBTAINED BY ITERATED INTEGRATION OF THE TELEGRAPH SIGNAL

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ABSTRACT. We analyse the vector process $(X_0(t), X_1(t), \dots, X_n(t), t > 0)$ where $X_k(t) = \int_0^t X_{k-1}(s) ds, k = 1, \dots, n,$ and $X_0(t)$ is the two-valued telegraph process.

In particular, the hyperbolic equations governing the joint distributions of the process are derived and analysed.

Special care is given to the case of the process $(X_0(t), X_1(t), X_2(t), t > 0)$ representing a randomly accelerated motion where some explicit results on the probability distribution are derived.

1. GENERAL RESULTS CONCERNING THE INTEGRATED TELEGRAPH SIGNAL

Let us consider the two-valued telegraph process

$$X_0(t) = X(0)(-1)^{N(t)}, \tag{1.1}$$

where $X(0)$ is a random variable which is independent of $N(t)$ and takes values $\pm a$ with equal probability. By $N(t)$ we denote the number of events of a homogeneous Poisson process up to time t (with rate λ).

Let also

$$X_1(t) = \int_0^t X_0(s) ds \tag{1.2}$$

and, in general,

$$X_k(t) = \int_0^t X_{k-1}(s) ds, \quad k = 1, 2, \dots, n. \tag{1.3}$$

When $n = 2$, the vector process $(X_0(t), X_1(t), X_2(t), t \geq 0)$ has a straightforward physical interpretation. Indeed, $X_2(t)$ represents the position of

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a particle with acceleration X_0 and velocity X_1 . The same interpretation is possible for the triple $(X_j(t), X_{j+1}(t), X_{j+2}(t), t \geq 0)$, where $j = 0, 1, \dots, n-2$.

Most of the current literature (see [1-4]) concerns the process $(X_0(t), X_1(t), t \geq 0)$ and our effort here is to examine the general situation, with special attention to the case $n = 2$.

Let us now introduce

$$\begin{cases} F(x_1, \dots, x_n, t) = \Pr\{X_1(t) \leq x_1, \dots, X_n(t) \leq x_n, X_0(t) = a\}, \\ B(x_1, \dots, x_n, t) = \Pr\{X_1(t) \leq x_1, \dots, X_n(t) \leq x_n, X_0(t) = -a\} \end{cases} \quad (1.4)$$

and denote by $f = f(x_1, \dots, x_n, t)$, $b = b(x_1, \dots, x_n, t)$ the corresponding densities. Our first result is the differential system governing f and b .

Theorem 1.1. *The densities f and b are solutions of*

$$\begin{cases} \frac{\partial f}{\partial t} = -a \frac{\partial f}{\partial x_1} - \sum_{j=2}^n x_{j-1} \frac{\partial f}{\partial x_j} + \lambda(b - f), \\ \frac{\partial b}{\partial t} = a \frac{\partial b}{\partial x_1} - \sum_{j=2}^n x_{j-1} \frac{\partial b}{\partial x_j} + \lambda(f - b). \end{cases} \quad (1.5)$$

Proof. For the sake of simplicity, we divide the time domain into intervals of length Δt and assume that changes of the process X_0 occur only at the endpoints of these intervals. The effect of this assumption disappears as $\Delta t \rightarrow 0$.

In order to be at (x_1, \dots, x_n) at time $t + \Delta t$ one of the following cases need to occur:

(i) $\Delta N = 0$ and the starting point for each process must be

$$\bar{x}_k = x_k - x_{k-1}\Delta t + \dots + (-1)^j x_{k-j} \frac{(\Delta t)^j}{j!} + \dots + (-1)^k a \frac{(\Delta t)^k}{k!},$$

where $k = 1, 2, \dots, n$ and $x_0 = 0$.

(ii) $\Delta N = 1$ and the starting point for each process is

$$\bar{x}_k = x_k - x_{k-1}\Delta t + \dots + (-1)^j x_{k-j} \frac{(\Delta t)^j}{j!} + \dots + (-1)^{k-1} a \frac{(\Delta t)^k}{k!}.$$

(iii) $\Delta N > 1$.

Restricting ourselves only to the first-order terms we have

$$\begin{aligned} & f(x_1, \dots, x_n, t + \Delta t) = \\ & = (1 - \lambda\Delta t)f(x_1 - a\Delta t, x_2 - x_1\Delta t, \dots, x_n - x_{n-1}\Delta t, t) + \\ & + \lambda\Delta t b(x_1 + a\Delta t, x_2 - x_1\Delta t, \dots, x_n - x_{n-1}\Delta t, t) + o(\Delta t). \end{aligned} \quad (1.6)$$

Expanding in Taylor series and passing to the limit as $\Delta t \rightarrow 0$, we obtain the first equation of system (1.5). An analogous treatment leads to the second equation of (1.5).

If $p = f + b$ and $w = f - b$, the above system can be rewritten as

$$\begin{cases} \frac{\partial p}{\partial t} = -a \frac{\partial w}{\partial x_1} - \sum_{j=2}^n x_{j-1} \frac{\partial p}{\partial x_j}, \\ \frac{\partial w}{\partial t} = -a \frac{\partial p}{\partial x_1} - \sum_{j=2}^n x_{j-1} \frac{\partial w}{\partial x_j} - 2\lambda w. \end{cases} \quad (1.7)$$

A rather surprising fact is that the equation governing p (to be obtained from (1.7)) is of third order for any $n \geq 2$. \square

Theorem 1.2. *The probability density $p = p(x_1, \dots, x_n, t)$ is a solution of*

$$\begin{aligned} \frac{\partial}{\partial x_1} \left[\frac{\partial^2 p}{\partial t^2} + \sum_{j=2}^n x_{j-1} \frac{\partial}{\partial x_j} \left(\frac{\partial p}{\partial t} + \sum_{r=2}^n x_{r-1} \frac{\partial p}{\partial x_r} \right) + 2\lambda \left(\frac{\partial p}{\partial t} + \sum_{j=2}^n x_{j-1} \frac{\partial p}{\partial x_j} \right) + \right. \\ \left. + \sum_{j=2}^n x_{j-1} \frac{\partial^2 p}{\partial t \partial x_j} - a^2 \frac{\partial^2 p}{\partial x_1^2} \right] = - \frac{\partial}{\partial x_2} \left(\frac{\partial p}{\partial t} + \sum_{j=2}^n x_{j-1} \frac{\partial p}{\partial x_j} \right). \end{aligned} \quad (1.8)$$

Proof. Deriving the first equation of (1.7) with respect to time t and inserting the other one derived with respect to x_1 , we obtain

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} &= -a \frac{\partial^2 w}{\partial x_1 \partial t} - \sum_{j=2}^n x_{j-1} \frac{\partial^2 p}{\partial t \partial x_j} = \\ &= -a \left[-a \frac{\partial^2 p}{\partial x_1^2} - \frac{\partial w}{\partial x_2} - \sum_{j=2}^n x_{j-1} \frac{\partial^2 w}{\partial x_j \partial x_1} - 2\lambda \frac{\partial w}{\partial x_1} \right] - \sum_{j=2}^n x_{j-1} \frac{\partial^2 p}{\partial t \partial x_j} = \\ &= a^2 \frac{\partial^2 p}{\partial x_1^2} + a \frac{\partial w}{\partial x_2} - \sum_{j=2}^n x_{j-1} \frac{\partial}{\partial x_j} \left(\frac{\partial p}{\partial t} + \sum_{r=2}^n x_{r-1} \frac{\partial p}{\partial x_r} \right) - \\ &\quad - 2\lambda \left(\frac{\partial p}{\partial t} + \sum_{j=2}^n x_{j-1} \frac{\partial p}{\partial x_j} \right) - \sum_{j=2}^n x_{j-1} \frac{\partial^2 p}{\partial t \partial x_j}. \end{aligned}$$

A further derivation with respect to x_1 then leads to equation (1.8). \square

Dealing with the third-order equation (1.8) implies substantial difficulties. In order to circumvent them we present the second-order equations governing the densities f and b .

For convenience we write $f = e^{-\lambda t}\bar{f}$, $b = e^{-\lambda t}\bar{b}$ and obtain from (1.5)

$$\begin{cases} \frac{\partial \bar{f}}{\partial t} = -a \frac{\partial \bar{f}}{\partial x_1} - \sum_{j=2}^n x_{j-1} \frac{\partial \bar{f}}{\partial x_j} + \lambda \bar{b}, \\ \frac{\partial \bar{b}}{\partial t} = a \frac{\partial \bar{b}}{\partial x_1} - \sum_{j=2}^n x_{j-1} \frac{\partial \bar{b}}{\partial x_j} + \lambda \bar{f}. \end{cases} \quad (1.9)$$

Some calculations now suffice to obtain from the above system the second-order equations

$$\begin{cases} \frac{\partial^2 \bar{f}}{\partial t^2} = -2 \sum_{j=2}^n x_{j-1} \frac{\partial^2 \bar{f}}{\partial x_j \partial t} + a^2 \frac{\partial^2 \bar{f}}{\partial x_1^2} + a \frac{\partial \bar{f}}{\partial x_2} - \\ \quad - \sum_{j=2}^n x_{j-1} \frac{\partial}{\partial x_j} \left(\sum_{r=2}^n x_{r-1} \frac{\partial \bar{f}}{\partial x_r} \right) + \lambda^2 \bar{f}, \\ \frac{\partial^2 \bar{b}}{\partial t^2} = -2 \sum_{j=2}^n x_{j-1} \frac{\partial^2 \bar{b}}{\partial x_j \partial t} + a^2 \frac{\partial^2 \bar{b}}{\partial x_1^2} - a \frac{\partial \bar{b}}{\partial x_2} - \\ \quad - \sum_{j=2}^n x_{j-1} \frac{\partial}{\partial x_j} \left(\sum_{r=2}^n x_{r-1} \frac{\partial \bar{b}}{\partial x_r} \right) + \lambda^2 \bar{b}. \end{cases} \quad (1.10)$$

Remark 1.1. If $\lambda \rightarrow \infty$ and $a \rightarrow \infty$ in such a way that $\frac{a^2}{\lambda} \rightarrow 1$, we obtain from (1.8) the equation

$$\frac{\partial p}{\partial t} + \sum_{j=2}^n x_{j-1} \frac{\partial p}{\partial x_j} = \frac{1}{2} \frac{\partial^2 p}{\partial x_1^2}, \quad (1.11)$$

which is satisfied by the probability law of the vector process $(X_1(t), \dots, X_n(t), t \geq 0)$, where X_1 is a standard Brownian motion and

$$X_k(t) = \int_0^t X_{k-1}(s) ds, \quad k = 2, \dots, n.$$

2. THE SPECIAL CASE $n = 2$

A deeper analysis is possible in the case of the vector process $(X_0(t), X_1(t), X_2(t), t \geq 0)$ representing a uniformly accelerated random motion.

In that case system (1.6) reads as

$$\begin{cases} \frac{\partial f}{\partial t} = -a \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} + \lambda(b - f), \\ \frac{\partial b}{\partial t} = a \frac{\partial b}{\partial x_1} - x_1 \frac{\partial b}{\partial x_2} + \lambda(f - b) \end{cases} \quad (2.1)$$

and equations (1.10) are reduced to the form

$$\begin{cases} \frac{\partial^2 \bar{f}}{\partial t^2} = -2x_1 \frac{\partial^2 \bar{f}}{\partial t \partial x_2} - x_1^2 \frac{\partial^2 \bar{f}}{\partial x_2^2} + a^2 \frac{\partial^2 \bar{f}}{\partial x_1^2} + a \frac{\partial \bar{f}}{\partial x_2} + \lambda^2 \bar{f}, \\ \frac{\partial^2 \bar{b}}{\partial t^2} = -2x_1 \frac{\partial^2 \bar{b}}{\partial t \partial x_2} - x_1^2 \frac{\partial^2 \bar{b}}{\partial x_2^2} + a^2 \frac{\partial^2 \bar{b}}{\partial x_1^2} - a \frac{\partial \bar{b}}{\partial x_2} + \lambda^2 \bar{b}. \end{cases} \quad (2.2)$$

By combining equations (2.2) it is easy to obtain (1.8) once again provided that functions f and b are inserted.

In our view the most interesting result concerning equations (2.2) is given in the next theorem.

Theorem 2.1. *The function*

$$\bar{f}(x_1, x_2, t) = q\left(x_2 - \frac{1}{2}x_1t + \frac{a^2t^2 - x_1^2}{4a}, x_2 - \frac{1}{2}x_1t - \frac{a^2t^2 - x_1^2}{4a}\right) \quad (2.3)$$

is a solution of the first equation of (2.2), provided that $q = q(u, w)$ is a solution of

$$(w - u) \frac{\partial^2 q}{\partial u \partial w} = \frac{\partial q}{\partial w} + \frac{\lambda^2}{2a} q. \quad (2.4)$$

Analogously,

$$\bar{b}(x_1, x_2, t) = g\left(x_2 - \frac{1}{2}x_1t + \frac{a^2t^2 - x_1^2}{4a}, x_2 - \frac{1}{2}x_1t - \frac{a^2t^2 - x_1^2}{4a}\right) \quad (2.5)$$

is a solution of the second equation of (2.2) provided that g is a solution of

$$(w - u) \frac{\partial^2 g}{\partial u \partial w} = -\frac{\partial g}{\partial w} + \frac{\lambda^2}{2a} g. \quad (2.6)$$

Proof. Since only simple calculations are involved, we omit the details. \square

Remark 2.1. It is interesting that equations (2.4) and (2.6) are reduced by the transformation $z = \sqrt{w - u}$ to the Bessel equation

$$\frac{\partial^2 q}{\partial z^2} + \frac{1}{z} \frac{\partial q}{\partial z} + \frac{2\lambda^2}{a} q = 0. \quad (2.7)$$

This result is due to the fact that $q = q(s)$ is a function depending only on x_1 through z . This is related to the well-known fact (see [1]) that the marginals

$$\int f(x_1, x_2, t) dx_2 \quad \text{and} \quad \int b(x_1, x_2, t) dx_2$$

are expressed in terms of Bessel functions of order zero with imaginary arguments and depending on $z = \sqrt{a^2t^2 - x_1^2}$.

To increase our insight into the vector process $(X_1(t), X_2(t), t > 0)$ we first note that possible values of X_1 at time t are within $[-at, at]$ and possible

values of X_2 are located in the interval $[-\frac{1}{2}at^2, \frac{1}{2}at^2]$. However, not all couples of the set

$$R = \left\{ x_1, x_2 : -at \leq x_1 \leq at, -\frac{1}{2}at^2 \leq x_2 \leq \frac{1}{2}at^2 \right\}$$

can be occupied. For example, it is impossible for the process X_1 to take values close to $-at$ and for X_2 to occupy positions near $\frac{1}{2}at^2$ (the interpretation of X_1 as velocity and X_2 as the current position of a moving particle can help here).

We now present the following result.

Theorem 2.2. *At time t the support of $(X_1(t), X_2(t))$ is the set*

$$S = \left\{ x_1, x_2 : -at \leq x_1 \leq at, \frac{1}{2}x_1t - \frac{a^2t^2 - x_1^2}{4a} \leq x_2 \leq \frac{1}{2}x_1t + \frac{a^2t^2 - x_1^2}{4a} \right\}, \quad (2.8)$$

which can be rewritten as

$$S = \left\{ x_1, x_2 : -\frac{1}{2}at^2 \leq x_2 \leq \frac{1}{2}at^2, at - \sqrt{2a^2t^2 - 4ax_2} \leq x_1 \leq -at + \sqrt{2a^2t^2 + 4ax_2} \right\}. \quad (2.9)$$

Proof. Assume that at time t , $X_1(t) = x_1$. If

$$T_+ = \int_0^t I_{\{X_0(s) > 0\}} ds = \text{meas}\{s < t : X_0(s) > 0\},$$

$$T_- = \int_0^t I_{\{X_0(s) < 0\}} ds = \text{meas}\{s < t : X_0(s) < 0\},$$

it is clear that $a(T_+ - T_-) = x_1$. In that case the farthest position to the right of the origin is reached when the entire rightward motion occurs initially, during time T_+ .

The final position is

$$\max X_2 = \frac{1}{2}aT_+^2 + aT_+T_- - \frac{1}{2}aT_-^2.$$

Since $T_+ + T_- = t$, we obtain

$$\max X_2 = \frac{1}{2}x_1t + \frac{a^2t^2 - x_1^2}{4a}.$$

Conversely, $\min X_2$ is reached when the leftward motion is performed initially.

In writing down (2.9), the sign must be chosen in such a way that for $x_2 = \pm\frac{1}{2}at^2$ we should have $x_1 = \pm at$. \square

All the information which can be read from the form of S coincides with the intuition. In particular, when $x_1 = \pm at$ we have $x_2 = \frac{1}{2}at^2$ and the closer x_1 is to the origin, the bigger the interval of possible values of X_2 becomes.

We note that, if $X_1(t) = x_1$, at time t the random variable $X_2(t)$ can take any value from the interval $[\frac{1}{2}x_1t - \frac{a^2t^2-x_1^2}{4a}, \frac{1}{2}x_1t + \frac{a^2t^2-x_1^2}{4a}]$.

To realize this we present

Theorem 2.3. *If $N(t) = m$, $X_0(0) = a$, T_i is the random time at which the i -th event of the driving Poisson process ($i \leq m$) occurs, then*

$$\begin{cases} X_1(t) = 2a \sum_{i=1}^m (-1)^{i-1} T_i + a(-1)^m t, \\ X_2(t) = a \sum_{i=1}^m (-1)^i T_i^2 + 2at \sum_{i=1}^m (-1)^{i-1} T_i + (-1)^m \frac{1}{2} at^2. \end{cases} \quad (2.10)$$

Proof. At the instant the m -th Poisson event takes place, the processes $(X_1(t), X_2(t), t \geq 0)$ take values $X_1(T_m)$ and $X_2(T_m)$ and thus, after some substitutions and simplifications, at $t < T_m$ we have (by induction)

$$\begin{aligned} X_2(t) &= X_2(T_m) + X_1(T_m)(t - T_m) + \frac{1}{2} a(-1)^m (t - T_m)^2 = \\ &= a \sum_{i=1}^m (-1)^i T_i^2 + 2at \sum_{i=1}^m (-1)^{i-1} T_i + (-1)^m \frac{1}{2} at^2. \quad \square \end{aligned}$$

Corollary 2.1. *It $N(t) = m$, $X_1(t) = x_1$, $X_0(0) = a$, possible positions which can be occupied at time t are given by*

$$\begin{aligned} X_2(t) &= a \sum_{i=1}^{m-1} (-1)^i T_i^2 + \frac{(-1)^m}{4a} \left(x_1 - a(-1)^m t + \right. \\ &\quad \left. + 2a \sum_{i=1}^{m-1} (-1)^{i-1} T_i \right)^2 + tx_1 - \frac{1}{2} (-1)^m at^2. \end{aligned} \quad (2.11)$$

Proof. From the first equation of (2.10) we get

$$T_m = \frac{x_1 - a(-1)^m t - 2a \sum_{i=1}^{m-1} (-1)^{i-1} T_i}{2a(-1)^m}$$

and rewriting the second equation as,

$$X_2(t) = a \sum_{i=1}^{m-1} (-1)^i T_i^2 + a(-1)^m T_m^2 + t(x_1 - a(-1)^m t) + \frac{1}{2} (-1)^m at^2,$$

after a substitution the desired result emerges. \square

Remark 2.2. If $N(t) = m$, $X_0(0) = a$, and the random times T_1, \dots, T_{m-1} are fixed, possible couples (x_1, x_2) of velocities and positions form a parabola.

Remark 2.3. If $X_2(t) = x_2$, $B_i = \frac{1}{2}x_1t - \frac{a^2t^2 - x_1^2}{4a}$, $B_s = \frac{1}{2}x_1t + \frac{a^2t^2 - x_1^2}{4a}$, then from (2.11) we derive the special cases.

$$\begin{aligned} m = 2 \quad x_2 &= T_1(at - x_1) + B_i, \\ m = 3 \quad x_2 &= B_s + (T_1 - T_2)(x_1 + at - 2aT_1), \\ m = 4 \quad x_2 &= B_i + (T_2 - T_1)(2aT_2 - (at - x_1)) + \\ &\quad + T_3(at - x_1 + 2a(T_1 - T_2)). \end{aligned} \quad (2.12)$$

These formulas permit us to show that, if $N(t) \geq 2$, at time t for fixed values of $X_1(t)$, possible positions (namely the values of $X_2(t)$) cover the whole interval $[B_i, B_s]$.

It must be observed that various curves formed by the couples (x_1, x_2) must be analysed taking into account the constraints concerning times T_i .

For example, if $m = 2$, we have $0 \leq T_1 \leq \frac{at+x_1}{2a}$. Thus if $T_1 = 0$, then $x_2 = B_i$ and the couples (x_1, x_2) form one of the parabolas bounding the set S .

Analogously, if $T_1 = \frac{at+x_1}{2a}$, then $x_2 = B_s$ and the other curve bounding S is obtained.

If $T_1 = \frac{at+x_1}{4a}$, we obtain the line of symmetry of S , whose equation is $x_2 = \frac{1}{2}x_1t$.

Remark 2.4. The direct calculation of

$$\Pr \{X_2(t) \in dx_2 \mid X_0(0) = a, X_1(t) = x_1, N(t) = m\} \quad (2.13)$$

presents considerable difficulties even when m is small.

We have been able to derive these results when $m = 2, 3$. From the formulas obtained it follows that general expressions of the distribution densities of $(X_1(t), X_2(t), t \geq 0)$ are of form (2.3) and (2.5) and are solutions of equations (2.4) and (2.6).

To evaluate (2.13) it is necessary to know the conditional distribution:

$$\Pr \{T_1 \in dt_1, \dots, T_{m-1} \in dt_{m-1} \mid N(t) = m, X_1(t) = x_1, X_0(0) = a\}. \quad (2.14)$$

In particular, we have obtained

Theorem 2.4.

$$\begin{aligned} \Pr \{T_1 \in dt_1 \mid X_1(t) = x_1, N(t) = 2, X_0(0) = a\} &= 2a/(at + x_1) \\ &\text{when } 0 < t_1 < (at + x_1)/2a, \end{aligned} \quad (2.15)$$

$$\Pr \{T_1 \in dt_1, T_2 \in dt_2 \mid X_1(t) = x_1, N(t) = 3, X_0(0) = a\} = \frac{4a^2}{a^2t^2 - x_1^2}$$

when $0 < t_1 < (at + x_1)/2a$ and $t_1 < t_2 < t_1 + (at - x_1)/2a$. (2.16)

Proof. When $m = 3$, from (2.12) we immediately have

$$\begin{aligned} & \Pr \{N(t) = 3, X_1(t) \leq x_1, X_0(0) = a\} = \\ &= \frac{3!}{t^3} \left\{ \int_0^{(at+x_1)/2a} dt_1 \int_{t_1}^{t_1+(at-x_1)/2a} dt_2 \int_{t_2}^{t_2-t_1+(at+x_1)/2a} dt_3 + \right. \\ &+ \left. \int_0^{(at+x_1)/2a} dt_1 \int_{t_1+(at-x_1)/2a}^t dt_2 \int_{t_2}^t dt_3 \right\} = \frac{(at+x_1)^2(2at-x_1)}{(2a)^2 t^3 a}. \end{aligned}$$

Furthermore, when $0 < t_1 < \frac{at+x_1}{2a}$, we have

$$\begin{aligned} & \Pr \{T_1 \in dt_1, T_2 \in dt_2 \mid X_1(t) = x_1, N(t) = 3, X_0(0) = a\} = \\ &= \begin{cases} \frac{3!}{t^3} \left(\frac{at+x_1}{2a} - t_1 \right) dt_1 dt_2 & \text{if } t_1 < t_2 < t_1 + \frac{at-x_1}{2a}, \\ \frac{3!}{t^3} (t - t_2) dt_1 dt_2 & \text{if } t > t_2 > t_1 + \frac{at-x_1}{2a}. \end{cases} \end{aligned}$$

distribution (2.16) follows from the above results. \square

On the basis of all the previous results we can present the following explicit formulas.

Theorem 2.5.

$$\Pr \{X_2(t) \in dx_2 \mid X_1(t) = x_1, N(t) = 2\} = \frac{dx_2}{B_s - B_i}, \quad B_i < x_2 < B_s, \quad (2.17)$$

$$\begin{aligned} & \Pr \{X_2(t) \in dx_2 \mid X_1(t) = x_1, N(t) = 3\} = \\ &= -\frac{dx_2}{B_s - B_i} \log \left(1 - \frac{x_2 - B_i}{B_s - B_i} \right), \quad B_i < x_2 < B_s. \end{aligned} \quad (2.18)$$

Proof. From the first formula of (2.12) we readily have

$$\begin{aligned} & \Pr \{X_2(t) < x_2 \mid X_1(t) = x_1, N(t) = 2\} = \\ &= \Pr \left\{ T_1 < \frac{x_2 - B_i}{at - x_1} \mid X_1(t) = x_1, N(t) = 2 \right\} = \frac{2a}{at + x_1} \int_0^{\frac{x_2 - B_i}{at - x_1}} dt_1, \end{aligned}$$

where the last step is justified by (2.15).

The derivation of (2.18) requires some additional details.

Taking into account the second formula of (2.12), we have

$$\Pr \{X_2(t) < x_2 \mid X_1(t) = x_1, N(t) = 3\} =$$

$$\begin{aligned}
&= \Pr \left\{ T_2 < T_1 + \frac{B_s - x_2}{x_1 + at - 2T_2} \mid X_1(t) = x_1, N(t) = 3 \right\} = \\
&= \iint_{\substack{0 < t_1 < \frac{x_2 - B_i}{at - x_1} \\ t_1 + \frac{B_s - x_2}{x_1 + at - 2at_1} < t_2 < t_1 + \frac{at - x_1}{2a}}} \frac{4a^2}{a^2 t^2 - x_1^2} dt_1 dt_2 = \\
&= \frac{x_2 - B_i}{B_s - B_i} + \frac{B_s - x_2}{B_s - B_i} \log \left(1 - \frac{x_2 - B_i}{B_s - B_i} \right). \quad \square
\end{aligned}$$

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