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# ASYMPTOTIC DISTRIBUTION OF EIGENFUNCTIONS AND EIGENVALUES OF THE BASIC BOUNDARY-CONTACT OSCILLATION PROBLEMS OF THE CLASSICAL THEORY OF ELASTICITY

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ABSTRACT. The basic boundary-contact oscillation problems are considered for a three-dimensional piecewise-homogeneous isotropic elastic medium bounded by several closed surfaces. Using Carleman's method, the asymptotic formulas for the distribution of eigenfunctions and eigenvalues are obtained.

**1.** After the remarkable papers of T. Carleman [1–2] the method based on the asymptotic investigation of the resolvent kernel (or of any other function of the considered operator) with a subsequenet use of Tauberian theorems has become quite popular. By generalizing Carleman's method (and combining it with the variational one) A. Plejel [3] derived the asymptotic formulas for the distribution of eigenfunctions and eigenvalues of the boundary value oscillation problems of classical elasticity. Mention should also be made of T. Burchuladze's papers [4–5], where the asymptotic formulas for the distribution of eigenfunctions of the boundary value oscillation problems are obtained for isotropic and anisotropic elastic bodies using integral equations and Carleman's method. Further progress in this direction was made by R. Dikhamindzhia [6]. He obtained the asymptotic formulas for the distribution of eigenfunctions and eigenvalues for two- and three-dimensional boundary value oscillation problems of couple-stress elasticity which generalize analogous formulas of classical elasticity. In his recent work M. Svanadze [7] derived the asymptotic formulas for oscillation boundary value problems of the linear theory of mixtures of two homogeneous isotropic elastic materials.

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**2.** Throughout the paper we shall use the following notation: $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ are points of $\mathbb{R}^3$; $|x - y| = \big(\sum_{k=1}^{3}(x_k - y_k)^2\big)^{1/2}$ is the distance between the points $x$ and $y$; $D_0 \subset \mathbb{R}^3$ is a finite domain bounded by closed surfaces $S_0, S_1, \dots, S_m$ of the class $\Omega_2(\alpha)$ $0 < \alpha \leq 1$ [8] with $S_0$ covering all other $S_k$ while the latter surfaces not covering each other; $S_i \cap S_k = \varnothing$ for $i \neq k$, $i, k = \overline{0, m}$; the finite domain bounded by $S_k$ $(k = \overline{1, m})$ will be denoted by $D_k$; $\overline{D}_0 = D_0 \cup \big(\overset{m}{\underset{k=0}{\cup}} S_k\big)$, $\overline{D}_k = D_k \cup S_k$, $k = \overline{1, m}$.

If $u$ and $v$ are the three-component real vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, then $uv = \sum_{i=1}^{3} u_i v_i$ is the scalar product of these vectors; $|u| = \big(\sum_{i=1}^{3} u_i^2\big)^{1/2}$. The matrix product is obtained by multipliing a row by a column; the sign $[\cdot]^T$ denotes the operation of transposition; if $A = \|A_{ij}\|_{3\times 3}$ is a $3 \times 3$ matrix, then $|A|^2 = \sum_{i,j=1}^{3} A_{ij}^2$. Any vector $u = (u_1, u_2, u_3)$ is treated as a $3 \times 1$ one-column matrix: $u = \|u_i\|_{3\times 1}$; $A_k = \|A_{jk}\|_{j=1}^{3}$ is the $k$-th column vector of the matrix $A$.

The vector $u = (u_1, u_2, u_3)$ will be called regular in $D_k$ if

$$u_i \in C^1(\overline{D}_k) \cap C^2(D_k), \quad i = 1, 2, 3.$$

A system of differential homogeneous equations of oscillation of classical elasticity for a homogeneous isotropic elastic medium has the form [8]

$$\mu \Delta u + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u + \rho \omega^2 u = 0, \tag{1}$$

where $u(x) = (u_1, u_2, u_3)$ is the displacement vector, $\Delta$ is the three-dimensional Laplace operator, $\rho = \mathrm{const} > 0$ is the medium density, $\omega$ is the oscillation frequency, $\lambda$ and $\mu$ are the elastic Lamé constants satisfying the natural conditions

$$\mu > 0, \quad 3\lambda + 2\mu > 0.$$

We introduce the matrix differential operator

$$A(\partial x) = \big\|A_{ij}(\partial x)\big\|_{3\times 3}, \quad A_{ij}(\partial_x) = \delta_{ij}\mu\rho^{-1}\Delta + (\lambda + \mu)\rho^{-1}\frac{\partial^2}{\partial x_i\, \partial x_j},$$

where $\delta_{ij}$ is the Kronecker symbol. Then equation (1) can be rewritten in the vector-matrix form

$$A(\partial_x)u + \omega^2 u = 0. \tag{2}$$

The matrix-differential operator

$$T(\partial_x, n(x)) = \big\|T_{ij}(\partial_x, n(x))\big\|_{3\times 3},$$

where

$$T_{ij}(\partial_x, n(x)) = \lambda\rho^{-1} n_i(x)\frac{\partial}{\partial x_j} + \lambda\rho^{-1} n_j(x)\frac{\partial}{\partial x_i} + \mu\rho^{-1}\delta_{ij}\frac{\partial}{\partial n(x)},$$

$n(x)$ is an arbitrary unit vector at the point $x$ (if $x \in S_k$, then $k = \overline{0,m}$ is the normal unit vector external with respect to the domain $D_0$) is called the stress operator.

It will be assumed that the domains $D_k$, $k = \overline{0,m_0}$ are filled with homogeneous isotropic elastic media with the Lamé constants $\lambda_k$, $\mu_k$ and density $\rho_k$, while the other domains $D_k$, $k = \overline{m_0+1,m}$ are hollow inclusions. When the operators $A$ and $T$ contain $\lambda_k$ and $\mu_k$ instead of $\lambda$ and $\mu$, we shall write $\overset{k}{A}$ and $\overset{k}{T}$, respectively.

We introduce the notation

$$u^+(z) = \lim_{D_0 \ni x \to z \in S_k} u(x), \quad k = \overline{0,m}; \quad u^-(z) = \lim_{D_k \ni x \to z \in S_k} u(x), \quad k = \overline{1,m_0}.$$

The notation $\big(T(\partial_z, n(z))u(z)\big)^{\pm}$ has a similar meaning.

**3.** A Kupradze matrix of fundamental solutions of the homogeneous equation of oscillation (2) has the form [8]

$$\Gamma(x-y,\omega^2) = \big\|\Gamma_{kj}(x-y,\omega^2)\big\|_{3\times 3},$$

where

$$\Gamma_{kj}(x-y,\omega^2) = \frac{\rho\delta_{kj}}{4\pi\mu}\frac{e^{ik_2 r}}{r} - \frac{1}{4\pi\omega^2}\frac{\partial^2}{\partial x_k\,\partial x_j}\frac{e^{ik_1 r} - e^{ik_2 r}}{r}, \tag{3}$$

$i$ is the imaginary unit, $r = |x-y|$, $k_1$ and $k_2$ are the nonnegative numbers defined by the equalities

$$k_1^2 = \frac{\rho\omega^2}{\lambda+2\mu}, \quad k_2^2 = \frac{\rho\omega^2}{\mu}. \tag{4}$$

Let $\varkappa_0$ be an arbitrary real fixed positive integer and $\varkappa > \varkappa_0$ be an arbitrary number. If in (3) we replace $\omega = i\varkappa$, then we obtain

$$\Gamma_{kj}(x-y,-\varkappa^2) = \frac{\rho\delta_{kj}}{4\pi\mu}\frac{e^{-\frac{\varkappa r}{c_2}}}{r} + \frac{1}{4\pi\varkappa^2}\frac{\partial^2}{\partial x_k\,\partial x_j}\frac{e^{-\frac{\varkappa r}{c_1}} - e^{-\frac{\varkappa r}{c_2}}}{r}, \tag{5}$$

where $c_1^2 = (\lambda+2\mu)\rho^{-1}$, $c_2^2 = \mu\rho^{-1}$. Since $\mu > 0$ and $3\lambda + 2\mu > 0$, we have $\lambda + \mu > 0$ and $\lambda + 2\mu > \mu$. Hence $c_1 > c_2$ and $c_1^{-1} < c_2^{-1}$. Let $\delta < \frac{1}{2c_1}$ be an arbitrary positive integer. Then (5) can be rewritten as

$$\Gamma_{kj}(x-y,-\varkappa^2) = \frac{\rho\delta_{kj}}{4\pi\mu}e^{-\alpha\varkappa r}\frac{e^{-\delta_1\varkappa r}}{r} + \\ + \frac{1}{4\pi\varkappa^2}\frac{\partial^2}{\partial x_k\,\partial x_j}e^{-\alpha\varkappa r}\Big(\frac{e^{-\delta\varkappa r}}{r} - \frac{e^{-\delta_1\varkappa r}}{r}\Big), \tag{6}$$

where $\delta_1 = \frac{1}{c_2} - \alpha = \frac{1}{c_2} - \frac{1}{c_1} + \delta > 0$.

Moreover,

(1) $\dfrac{\partial}{\partial x_j}\Big(e^{-\alpha\varkappa r}\dfrac{e^{-\delta_1\varkappa r}}{r}\Big)=\dfrac{e^{-\alpha\varkappa r}}{r^2}e^{-\delta_1\varkappa r}\Big(-1-\dfrac{1}{c_2}\varkappa r\Big)\dfrac{\partial r}{\partial x_j}$;

(2) $\dfrac{\partial^2}{\partial x_k\,\partial x_j}\Big(e^{-\alpha\varkappa r}\dfrac{e^{-\delta_1\varkappa r}}{r}\Big)=\dfrac{e^{-\alpha\varkappa r}}{r^3}e^{-\delta_1\varkappa r}\Big(3+\dfrac{3}{c_2}\varkappa r+$
$+\dfrac{1}{c_2^2}\varkappa^2r^2\Big)\dfrac{\partial r}{\partial x_j}\dfrac{\partial r}{\partial x_k}+\dfrac{e^{-\alpha\varkappa r}}{r^3}e^{-\delta_1\varkappa r}\Big(-1-\dfrac{1}{c_2}\varkappa r\Big)\delta_{kj}\dfrac{\partial r}{\partial x_k}$,

(3) the functions $(\varkappa r)^n e^{-\delta_1\varkappa r}$, $n=0,1,2,\dots$ are bounded in the interval $\varkappa\in[0,+\infty)$.

Taking into account the above arguments, from (6) we obtain the estimates

$$\Big|\frac{\partial^n\Gamma_{pq}(x-y,-\varkappa^2)}{\partial x_1^i\,\partial x_2^j\,\partial x_3^k}\Big|\le\frac{\text{const}}{r^{n+1}}e^{-\alpha\varkappa r},\tag{7}$$
$$i+j+k=n;\quad n=0,1,2,\dots;\quad p,q=\overline{1,3}.$$

**4.** Let $x,y\in D_k$, $k=\overline{0,m_0}$ and $l_y$ be the distance from the point $y$ to the boundary of $D_k$.

Denote $\rho_y(x)=\max\{r,l_y\}$. and introduce an auxiliary matrix

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)=\Big[1-\Big(1-\frac{r^m}{\rho_y^m(x)}\Big)^n\Big]\overset{k}{\Gamma}(x-y,-\varkappa^2),\tag{8}$$

where $\overset{k}{\Gamma}(x-y,-\varkappa^2)$ is the Kupradze matrix of fundamental solutions for the operator $\overset{k}{A}(\partial x)-\varkappa^2\mathcal{I}$ ($\mathcal{I}$ is the $3\times3$ unit matrix).

We denote by $B(y,l_y)$ a sphere of radius $l_y$ and center at the point $y$, and by $C(y,l_y)$ its boundary. It is easy to verify that $\big(1-\frac{r^m}{\rho_y^m(x)}\big)^n$ vanishes together with its derivatives up to the $(n-1)$-th order inclusive when the point $x\in B(y,l_y)$ tends to the point of the boundary $C(y,l_y)$. For $x\in D_k\backslash\overline{B(y,l_y)}$ we have

$$1-\Big(1-\frac{r^m}{\rho_y^m(x)}\Big)^n=1$$

and

$$\lim_{B(y,l_y)\ni x\to z\in C(y,l_y)}\Big[1-\Big(1-\frac{r^m}{\rho_y^m(x)}\Big)^n\Big]=1.$$

Thus

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)=\overset{k}{\Gamma}(x-y,-\varkappa^2)$$

for $x\in D_k\backslash B(y,l_y)$, while, in passing the boundary $C(y,l_y)$, the function $\overset{\widehat{k}}{\Gamma}$ and its derivatives up to the $(n-1)$th order inclusive remain continuous.

We write $\overset{\widehat{k}}{\Gamma}$ as

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)=\overset{k}{\Gamma}(x-y,-\varkappa^2)\big(nr^m/\rho_y^m(x)+\cdots\big).$$

It is easy ti find that $\overset{\widehat{k}}{\Gamma}$ and its derivatives up to the $(m-2)$-th order inclusive are continuous for $x=y$, while for $x\in B(y,l_y)$ we have, by virtue of (7), the estimates

$$\Big|\frac{\partial^s\overset{\widehat{k}}{\Gamma}_{pq}(x-y,-\varkappa^2)}{\partial x_1^i\,\partial x_2^j\,\partial x_3^k}\Big|\leq\frac{\text{const}\,e^{-\alpha\varkappa r}}{l_y^m}r^{m-s-1}, \tag{9}$$
$$p,q=\overline{1,3},\quad i+j+k=s;\quad m\geq s+1.$$

**5.** We determine the limit

$$\lim_{x\to y}\Big[\overset{k}{\Gamma}(x-y,-\varkappa^2)-\overset{k}{\Gamma}(x-y,-\varkappa_0^2)\Big],\quad x,y\in D_k,\quad k=\overline{0,m_0}.$$

Taking into consideration the expansion

$$\frac{e^{-\frac{\varkappa r}{c_1}}}{r}=\frac{1}{r}-\frac{\varkappa}{c_1}+\frac{\varkappa^2}{2!\,c_1^2}r-\frac{\varkappa^3}{3!\,c_1^3}r^2+\cdots,$$

we obtain

$$(1)\quad\frac{e^{-\frac{\varkappa r}{c_2}}}{r}-\frac{e^{-\frac{\varkappa_0 r}{C_2}}}{r}=\frac{\varkappa_0-\varkappa}{c_2}+\frac{\varkappa^2-\varkappa_0^2}{2!\,c_2^2}r+\cdots,$$

$$(2)\quad\frac{e^{-\frac{\varkappa r}{c_1}}}{r}-\frac{e^{-\frac{\varkappa r}{c_2}}}{r}=\varkappa\Big(-\frac{1}{c_1}+\frac{1}{c_2}\Big)+\varkappa^2\Big(\frac{1}{2!\,c_1^2}-\frac{1}{2!\,c_2^2}\Big)r+$$
$$+\varkappa^3\Big(-\frac{1}{3!\,c_1^3}+\frac{1}{3!\,c_2^3}\Big)r^2+\cdots,$$

$$(3)\quad\frac{\partial^2}{\partial x_k\,\partial x_j}\frac{e^{-\frac{\varkappa r}{c_1}}-e^{-\frac{\varkappa r}{c_2}}}{r}=\varkappa^2\Big(\frac{1}{2!\,c_1^2}-\frac{1}{2!\,c_2^2}\Big)\frac{\partial^2 r}{\partial x_k\,\partial x_j}+$$
$$+\varkappa^3\Big(\frac{1}{6c_2^3}-\frac{1}{6c_1^3}\Big)\frac{\partial^2 r^2}{\partial x_k\,\partial x_j}+\cdots,$$

$$(4)\quad\frac{\partial^2 r^2}{\partial x_k\,\partial x_j}=2\delta_{kj}.$$

By virtue of the above relations we have

$$\lim_{x\to y}\Big[\overset{k}{\Gamma}_{pq}(x-y,-\varkappa^2)-\overset{k}{\Gamma}_{pq}(x-y,-\varkappa_0^2)\Big]=$$
$$=\frac{(\varkappa_0-\varkappa)\rho_k^{3/2}}{4\pi\mu_k^{3/2}}\delta_{pq}+\frac{(\varkappa_0-\varkappa)\rho_k^{3/2}}{12\pi}\Big[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}-\frac{1}{\mu_k^{3/2}}\Big]\delta_{pq}=$$

$$= \frac{(\varkappa_0-\varkappa)\rho_k^{3/2}}{12\pi}\left[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{\mu_k^{3/2}}\right]\delta_{pq}, \quad p,q=\overline{1,3}. \tag{10}$$

**6.** Our further investigation will be carried out for the first problem. The other problems are treated analogously.

We apply the term "Green's tensor of the first basic boundary-contact problem of the operator $\overset{k}{A}(\partial x)-\varkappa_0^2\mathcal{I}$" to the $3\times 3$ matrix $G(x,y,-\varkappa_0^2)=\overset{k}{G}(x,y,-\varkappa_0^2)$, $x\in D_k$, $y\in D$ $\big(D=\bigcup\limits_{k=0}^{m_0}D_k\big)$, $x\neq y$, $k\in\overline{0,m_0}$ which satisfies the following conditions:

(1) $\forall x\in D_k$, $\forall y\in D$, $x\neq y$ :
$\overset{k}{A}(\partial x)\overset{k}{G}(x,y,-\varkappa_0^2)-\varkappa_0^2\overset{k}{G}(x,y,-\varkappa_0^2)=0$, $k=\overline{0,m_0}$,

(2) $\forall z\in S_k$, $\forall y\in D:\overset{\circ}{G}{}^+(x,y,-\varkappa_0^2)=\overset{k}{G}-(x,y,-\varkappa_0^2)$,
$\big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{G}(z,y,-\varkappa_0^2)\big)^+=\big(\overset{k}{T}(\partial_z,n(z))\overset{k}{G}(z,y,-\varkappa_0^2)\big)^-$, $k=\overline{1,m_0}$,

(3) $\forall z\in S_k$, $\forall y\in D:\overset{\circ}{G}{}^+(z,y,-\varkappa_0^2)=0$, $k=0,m_0+1,\dots,m$;

(4) $\overset{k}{G}(x,y,-\varkappa_0^2)=\overset{k}{\Gamma}(x-y,-\varkappa_0^2)-\overset{k}{g}(x,y,-\varkappa_0^2)$, $x\in D_k$, $y\in D$, $k=\overline{0,m_0}$,

where $\overset{k}{g}(x,y,-\varkappa_0^2)$ is a regular in $D_k$ solution of the following problem:

(1) $\forall x\in D_k,\forall y\in D:\overset{k}{A}(\partial x)\overset{k}{g}(x,y,-\varkappa_0^2)-\varkappa_0^2\overset{k}{g}(x,y,-\varkappa_0^2)=0$, $k=\overline{0,m_0}$;

(2) $\forall z\in S_k$, $\forall y\in D:\overset{\circ}{g}{}^+(z,y,-\varkappa_0^2)-\overset{k}{g}{}^-(z,y,-\varkappa_0^2)=$
$=\overset{\circ}{\Gamma}(z-y,-\varkappa_0^2)-\overset{k}{\Gamma}(z-y,-\varkappa_0^2)$,
$\big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{g}(z,y,-\varkappa_0^2)\big)^+-\big(\overset{k}{T}(\partial_z,n(z))\overset{k}{g}(z,y,-\varkappa_0^2)\big)^-=$
$=\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}(z-y,-\varkappa_0^2)-\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}(z-y,-\varkappa_0^2)$, $k=\overline{1,m_0}$;

(3) $\forall z\in S_k$, $\forall y\in D:\overset{\circ}{g}{}^+(z,y,-\varkappa_0^2)=\overset{\circ}{\Gamma}(z-y,-\varkappa_0^2)$,
$k=0,m_0+1,\dots,m$;

The solvability of this problem is shown in [8] and thereby the existence of $G(x,y,-\varkappa_0^2)$ is proved. As is known [8], $G(x,y,-\varkappa_0^2)$ possesses a symmetry property of the form

$$G(x,y,-\varkappa_0^2)=G^T(y,x,-\varkappa_0^2). \tag{11}$$

Moreover, we have the estimates [9]

$$\left.\begin{aligned}\forall(x,y)\in D_k\times D_k:G_{pq}(x,y,-\varkappa_0^2)=O(|x-y|^{-1}),\\ \frac{\partial}{\partial x_j}G_{pq}(x,y,-\varkappa_0^2)=O(|x-y|^{-2}),\quad m,n,j=\overline{1,3};\quad k=\overline{0,m_0}.\end{aligned}\right\} \tag{12}$$

**7.** Let $u(x)=\overset{k}{u}(x)$ and $v(x)=\overset{k}{v}(x)$, $x\in D_k$, be arbitrary (regular) vectors of the class $C^1(\overline{D}_k)\cap C^2(D_k)$, $k=\overline{0,m_0}$. Then the following Green formula is valid [8]:

$$\sum_{k=0}^{m_0}\int\limits_{D_k}(\overset{k}{v}\overset{k}{A}\overset{k}{u}+\overset{k}{E}(\overset{k}{v},\overset{k}{u}))dx=\int\limits_{S}\overset{\circ}{v}{}^+(\overset{\circ}{T}\overset{\circ}{u})^+ds+$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\big(\overset{\circ}{v}{}^+(\overset{\circ}{T}\overset{\circ}{u})^+-\overset{k}{v}{}^-(\overset{k}{T}\overset{k}{u})^-\big)ds,\tag{13}$$

where $S=S_0\underset{k=m_0+1}{\overset{m}{\cup}}S_k$,

$$\overset{k}{E}(\overset{k}{v},\overset{k}{u})=\frac{\rho_k^{-1}(3\lambda_k+2\mu_k)}{3}\operatorname{div}\overset{k}{v}\operatorname{div}\overset{k}{u}+\frac{\rho_k^{-1}\mu_k}{2}\sum_{p\neq q}\Big(\frac{\partial\overset{k}{v}_p}{\partial x_q}+$$
$$+\frac{\partial\overset{k}{v}_q}{\partial x_p}\Big)\Big(\frac{\partial\overset{k}{u}_p}{\partial x_q}+\frac{\partial\overset{k}{u}_q}{\partial x_p}\Big)+\frac{\rho_k^{-1}\mu_k}{3}\sum_{p,q}\Big(\frac{\partial\overset{k}{v}_p}{\partial x_p}-\frac{\partial\overset{k}{v}_q}{\partial x_q}\Big)\Big(\frac{\partial\overset{k}{u}_p}{\partial x_p}-\frac{\partial\overset{k}{u}_q}{\partial x_q}\Big).\tag{14}$$

It follows from (14) that $\overset{k}{E}(\overset{k}{v},\overset{k}{u})=\overset{k}{E}(\overset{k}{u},\overset{k}{v})$ and $\overset{k}{E}(\overset{k}{v},\overset{k}{v})\ge 0$.

For the regular in $D_k$, $k=\overline{0,m_0}$ vector $u(x)$ the following general integral representation is valid [8]:

$$\forall y\in D_k:\ u_j(y)=-\sum_{k=0}^{m_0}\int\limits_{D_k}\overset{k}{\Gamma}_j(x-y,-\varkappa^2)(\overset{k}{A}(\partial x)\overset{k}{u}(x)-\varkappa^2\overset{k}{u}(x))dx+$$
$$+\int\limits_{S}\big[\overset{k}{\Gamma}_j(z-y,-\varkappa^2)(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{u}(z))^+-$$
$$-\overset{\circ}{u}{}^+(z)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS+$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\big[\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{u}(z))^+-$$
$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)(\overset{k}{T}(\partial_z,n(z))\overset{k}{u}(z))^-\big]d_zS-$$
$$-\sum_{k=1}^{m_0}\int\limits_{S_k}\big[\overset{\circ}{u}{}^+(z)(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)-$$
$$-\overset{k}{u}{}^-(z)(\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS,\quad j=1,2,3.\tag{15}$$

8. To establish the asymptotic behavior of eigenfunctions and eigenvalues we have to estimate the regular part of Green's tensor $g(x,y,-\varkappa^2)$ as $\varkappa\to\infty$. To this end we consider the functional

$$L[u]=\sum_{k=0}^{m_0}\int_{D_k}\big(\overset{k}{E}(\overset{k}{u},\overset{k}{u})+\varkappa^2\overset{k}{u}{}^2\big)dx-2\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{u}{}^+(z)\overset{\circ}{T}(\partial_z,n(z))\times$$

$$\times\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)-\overset{k}{u}{}^-(z)(\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS, \tag{16}$$

where $j=1,2,3$ is the fixed number and $y$ is an arbitrary fixed point in $D_k$, $k=\overline{0,m_0}$, which is defined in the class of regular in $D_k$, $k=\overline{0,m_0}$, vector functions satisfying the conditions:

(1) $\forall z\in S_k:\ \overset{\circ}{u}{}^+(z)-\overset{k}{u}{}^-(z)=\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)$,
$(\overset{\circ}{T}\overset{\circ}{u}(z))^+-(\overset{k}{T}\overset{k}{u}(z))^-=\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)-$
$\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)$, $k=\overline{1,m_0}$,

(2) $\forall z\in S_k:\ \overset{\circ}{u}{}^+(z)=\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)$,  $k=0,m_0+1,\dots,m$.

**Theorem 1.** *The functional $L$ takes a minimal value for $u=g_j(x,y,-\varkappa^2)$.*

*Proof.* Let $u$ be an arbitrary vector from the domain of definition of the functional $L$, and let $v=u-g_j$. Then, with (14) taken into account, (16) implies

$$L[u]=L[v+g_j]=\sum_{k=0}^{m_0}\int_{D_k}\big(\overset{k}{E}(\overset{k}{v}+\overset{k}{g}_j,\overset{k}{v}+\overset{k}{g}_j)+\varkappa^2(\overset{k}{v}+\overset{k}{g}_j)^2\big]dx-$$

$$-2\sum_{k=1}^{m_0}\int_{S_k}\big[(\overset{\circ}{v}+\overset{\circ}{g}_j)\overset{\circ}{T}\overset{\circ}{\Gamma}_j-(\overset{k}{v}{}^-+\overset{k}{g}_j{}^-)\overset{k}{T}\overset{k}{\Gamma}_j\big]ds=$$

$$=\sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{v})+2\overset{k}{E}(\overset{k}{v},\overset{k}{g}_j)+\overset{k}{E}(\overset{k}{g}_j,\overset{k}{g}_j)+\varkappa^2(\overset{k}{v}{}^2+2\overset{k}{v}\overset{k}{g}_j+\overset{k}{g}_j{}^2)\big]dx-$$

$$-\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{v}{}^+\overset{\circ}{T}\overset{\circ}{\Gamma}_j-\overset{k}{v}{}^-\overset{k}{T}\overset{k}{\Gamma}_j\big]ds-2\sum_{k=1}^{m_0}\int_{S_k}\big(\overset{\circ}{g}_j{}^+\overset{\circ}{T}\overset{\circ}{\Gamma}_j-\overset{k}{g}_j{}^-\overset{k}{T}\overset{k}{\Gamma}_j\big]ds=$$

$$=L[g_j]+\sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{v})+\varkappa^2\overset{k}{v}{}^2\big]dx+2\sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{g}_j)+$$

$$+\varkappa^2\overset{k}{v}\overset{k}{g}_j\big]dx-2\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{v}{}^+\overset{\circ}{T}\overset{\circ}{\Gamma}_j-\overset{k}{v}{}^-\overset{k}{T}\overset{k}{\Gamma}_j\big]ds. \tag{17}$$

Using the Green formula (13) for $v = u - g_j$, $u = g_j$ and taking into account that $\overset{k}{A}\overset{k}{g}_j = \varkappa^2 \overset{k}{g}_j$, $\overset{\circ}{v}{}^+(z) = 0$ for $z \in S_k$, $k = 0, m_0+1, \dots, m$, we obtain

$$\sum_{k=0}^{m_0} \int_{D_k} \big[\overset{k}{E}(\overset{k}{v}, \overset{k}{g}_j) + \varkappa^2 \overset{k}{v}\overset{k}{g}_j\big]dx = \sum_{k=1}^{m_0} \int_{S_k} \big[\overset{\circ}{v}{}^+(\overset{\circ}{T}\overset{\circ}{g}_j)^+ - \overset{k}{v}{}^-(\overset{k}{T}\overset{k}{g}_j)^-\big]ds. \tag{18}$$

Now, since $\overset{\circ}{v}{}^+(z) = \overset{k}{v}{}^-(z)$, $(\overset{\circ}{T}\overset{\circ}{g}_j)^+ - (\overset{k}{T}\overset{k}{g}_j)^- = \overset{\circ}{T}\overset{\circ}{\Gamma}_j - \overset{k}{T}\overset{k}{\Gamma}_j$ for $z \in S_k$, $k = \overline{1, m_0}$ (17) by virtue of (18) takes the form

$$L[u] = L[g_j] + \sum_{k=0}^{m_0} \int_{D_k} \big[\overset{k}{E}(\overset{k}{v}, \overset{k}{v}) + \varkappa^2 \overset{k}{v}{}^2\big]dx \geq L[g_j]. \qquad \square$$

**Theorem 2.** *The estimate*

$$\big|g_{jj}(y, y, -\varkappa^2) - g_{jj}(y, y, -\varkappa_0^2)\big| \leq \frac{\text{const}}{l_y^{1+\delta}}, \quad y \in D, \quad \delta > 0, \tag{19}$$

*holds for the function* $g_{jj}(y, y, -\varkappa^2)$.

*Proof.* We write formula (15) for $u_j(x) = g_{jj}(x, y, -\varkappa^2)$ and $\Gamma_j(x-y, -\varkappa^2) = G_j(x-y, -\varkappa^2)$. Then taking into account the boundary and contact conditions for $g$ and $G$, we get

$$\begin{aligned} &\forall (x,y) \in D_k : g_{jj}(x, y, -\varkappa^2) = \\ &= -\int_S \overset{\circ}{\Gamma}_j(z-x, -\varkappa^2)(\overset{\circ}{T}(\partial_z, n(z))\overset{\circ}{G}_j(z, y, -\varkappa^2))^+ d_zS - \\ &+ \sum_{k=1}^{m_0} \int_{S_k} \big[\overset{\circ}{G}_j{}^+(z, x, -\varkappa^2)\overset{\circ}{T}(\partial_z, n(z))\overset{\circ}{\Gamma}_j(z-y, -\varkappa^2) - \\ &- \overset{k}{G}_j{}^-(z, x, -\varkappa^2)\overset{k}{T}(\partial_z, n(z))\overset{k}{\Gamma}_j(z-y, -\varkappa^2)\big]d_zS - \\ &- \sum_{k=1}^{m_0} \int_{S_k} \big[\overset{\circ}{\Gamma}_j(z-x, -\varkappa^2)(\overset{\circ}{T}(\partial_z, n(z))\overset{\circ}{G}_j(z, y, -\varkappa^2))^+ - \\ &- \overset{k}{\Gamma}_j(z-x, -\varkappa^2)(\overset{k}{T}(\partial_z, n(z))\overset{k}{G}_j(z, y, -\varkappa^2))^-\big]d_zS. \end{aligned} \tag{20}$$

Using formula (13) for $u = v$, (16) can be rewritten as

$$L[u] = -\sum_{k=0}^{m_0} \int_{D_k} \overset{k}{u}\big[\overset{k}{A}(\partial_x)\overset{k}{u}(x) - \varkappa^2 \overset{k}{u}(x)\big]dx + \int_S \overset{\circ}{u}{}^+(z)(\overset{\circ}{T}\overset{\circ}{u}(z))^+ d_zS +$$

$$+\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{u}{}^+(z)(\overset{\circ}{T}\overset{\circ}{u}(z))^+ - \overset{k}{u}{}^-(z)(\overset{k}{T}\overset{k}{u}(z))^-\big]d_zS -$$
$$-2\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{u}{}^+(z)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2) -$$
$$-\overset{k}{u}{}^-(z)\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS. \tag{21}$$

which for $u(x)=g_j(x,y,-\varkappa^2)=\Gamma_j(x-y,-\varkappa^2)-G_j(x,y,-\varkappa^2)$ implies

$$L[g_j]=\int_S \overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)d_zS -$$
$$-\int_S \overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)\big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{G}_j(z,y,-\varkappa^2)\big)^+d_zS +$$
$$+\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2) -$$
$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS +$$
$$+\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{G}_j{}^+(z,y,-\varkappa^2)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2) -$$
$$-\overset{k}{G}_j{}^-(z,y,-\varkappa^2)\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS -$$
$$-\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{G}_j(z,y,-\varkappa^2))^+ -$$
$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)(\overset{k}{T}(\partial_z,n(z))\overset{k}{G}_j(z,y,-\varkappa^2))^-\big]d_zS. \tag{22}$$

On the basis of (22), from (20) we obtain

$$g_{jj}(y,y,-\varkappa^2)=L[g_j]-\int_S \overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)d_zS +$$
$$+\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2) -$$
$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS. \tag{23}$$

The vector $\overset{\widehat{k}}{\Gamma}_j(x-y,-\varkappa^2)$ defined by (8) belongs to the domain of definition of the functional $L$ and, since $g_j(x,y,-\varkappa^2)$ imparts a minimal value to the functional $L$, it is obvious that

$$L[g_j] \leq L[\widehat{\Gamma}_j].$$

Now (23) implies

$$g_{jj}(y,y,-\varkappa^2) \leq L[\widehat{\Gamma}_j] - \int_S \overset{\circ}{\Gamma}_j \overset{\circ}{T} \overset{\circ}{\Gamma}_j ds + $$
$$+ \sum_{k=1}^{m_0} \int_{S_k} (\overset{\circ}{\Gamma}_j \overset{\circ}{T} \overset{\circ}{\Gamma}_j - \overset{k}{\Gamma}_j \overset{k}{T} \overset{k}{\Gamma}_j) ds, \quad y \in D_k. \tag{24}$$

By virtue of the properties of $\widehat{\Gamma}$, from (21) we obtain

$$L[\widehat{\Gamma}_j] = -\int_{B(y,l_y)} \widehat{\Gamma}_j (A\widehat{\Gamma}_j - \varkappa^2 \widehat{\Gamma}_j) dx + \int_S \overset{\circ}{\Gamma}_j \overset{\circ}{T} \overset{\circ}{\Gamma}_j ds -$$
$$- \sum_{k=1}^{m_0} \int_{S_k} (\overset{\circ}{\Gamma}_j \overset{\circ}{T} \overset{\circ}{\Gamma}_j - \overset{k}{\Gamma}_j \overset{k}{T} \overset{k}{\Gamma}_j) ds. \tag{25}$$

By (25) and (24) we have

$$g_{jj}(y,y,-\varkappa^2) \leq -\int_{B(y,l_y)} \widehat{\Gamma}_j (A\widehat{\Gamma}_j - \varkappa^2 \widehat{\Gamma}_j) dx, \;\; y \in D_k, \;\; k = \overline{0, m_0}. \tag{26}$$

Taking into account estimates (9), for $m = 5$ we obtain

$$\left|\widehat{\Gamma}_{mj}(x,y,-\varkappa^2)\right| \leq \frac{\text{const}}{l_y} \quad \text{for} \quad s = 0,$$
$$\left|\varkappa^2 \widehat{\Gamma}_{ij}(x,y,-\varkappa^2)\right| \leq \varkappa^2 \frac{\text{const}\, e^{-\alpha\varkappa r}}{l_y^5} r^4 =$$
$$= \frac{\text{const}}{l_y^5} r^2 (\varkappa^2 r^2) e^{-\alpha\varkappa r} \leq \frac{\text{const}}{l_y^3} \quad \text{for} \quad s = 0,$$
$$\left|A\widehat{\Gamma}_j(x,y,-\varkappa^2)\right| \leq \frac{\text{const}}{l_y^3} \quad \text{for} \quad s = 2.$$

Hence (26) implies

$$g_{jj}(y,y,-\varkappa^2) \leq \frac{\text{const}}{l_y^4} \cdot \frac{4}{3}\pi l_y^3 \leq \frac{\text{const}}{l_y} \leq \frac{\text{const}}{l_y^{1+\delta}}, \tag{27}$$

where $\delta > 0$ is an arbitrary integer.

Let us estimate $g_{jj}(y,y,-\varkappa^2)$ from below. For this we introduce the notation:

$$M[u]=\sum_{k=0}^{m_0}\int\limits_{D_k}\big[\overset{k}{E}(\overset{k}{u},\overset{k}{u})+\varkappa^2\overset{k}{u}{}^2\big]dx,\quad M_0[u]=\sum_{k=0}^{m_0}\int\limits_{D_k}\big[\overset{k}{E}(\overset{k}{u},\overset{k}{u})+\varkappa_0^2\overset{k}{u}{}^2\big]dx,$$

$$N[u]=\sum_{k=1}^{m_0}\int\limits_{S_k}\big[\overset{\circ}{u}{}^+(z)\overset{\circ}{T}\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)-\overset{k}{u}{}^-(z)\overset{k}{T}\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zS.$$

Since $\varkappa_0^2\le\varkappa^2$, we have

$$L[g_j(x,y,-\varkappa^2)]=\min L[u]=\min(M[u]-2N[u])\ge\min(M_o[u]-2N[u]).$$

Let the vector function $\varphi(x,y)$ impart a minimal value to the functional $M_0[u]-2N[u]$. Then $\varphi(x,y)$ is a regular in $D_k$ $(k=\overline{0,m_0})$ solution of the following problem:

(1) $\forall x\in D_k,\forall y\in D:\overset{k}{A}(\partial x)\overset{k}{\varphi}(x,y)-\varkappa_0^2\overset{k}{\varphi}(x,y)=0,\ k=\overline{0,m_0};$

(2) $\forall z\in S_k,\forall y\in D:\overset{\circ}{\varphi}{}^+(z,y)-\overset{k}{\varphi}{}^-(z,y)=\overset{\circ}{\Gamma}(z-y,-\varkappa^2)-\overset{k}{\Gamma}(z-y,-\varkappa^2),$ $(\overset{\circ}{T}\overset{\circ}{\varphi}(z,y))^+-(\overset{k}{T}\overset{k}{\varphi}(z,y))^-=\overset{\circ}{T}\overset{\circ}{\Gamma}(z-y,-\varkappa^2)-\overset{k}{T}\overset{k}{\Gamma}(z-y,-\varkappa^2);$

(3) $\forall z\in S_k,\forall y\in D:\overset{\circ}{\varphi}{}^+(z,y)=\overset{\circ}{\Gamma}(z-y,-\varkappa^2),\ k=0,m_0+1,\dots,m.$

Rewriting formula (15) for $\varphi(x,y)$ with $\Gamma=G$, we obtain

$$\forall(x,y)\in D_k:\varphi(x,y)=-\int\limits_S\overset{\circ}{\Gamma}_j(z-x,-\varkappa^2)\overset{\circ}{T}\overset{\circ}{G}_j(z,y,-\varkappa_0^2)d_zs+$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\big[\overset{\circ}{G}_j(z,x,-\varkappa_0^2)\overset{\circ}{T}\overset{\circ}{\Gamma}_j(z,y,-\varkappa^2)-\overset{k}{G}_j{}^-(z,x,-\varkappa_0^2)\times$$
$$\times\overset{k}{T}\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zs-\sum_{k=1}^{m_0}\int\limits_{S_k}\big[\overset{\circ}{\Gamma}_j(z-x,-\varkappa^2)(\overset{\circ}{T}\overset{\circ}{G}_j(z,y,-\varkappa_0^2))^+-$$
$$-\overset{k}{\Gamma}_j(z-x,-\varkappa^2)(\overset{k}{T}\overset{k}{G}_j(z,y,-\varkappa_0^2))\big]d_zs.\tag{28}$$

By (7) and (12) and the theorem on kernel composition [10] it follows from (28) that

$$\forall(x,y)\in\overline{D}_k\times D_k:\ |\varphi(x,y)|\le\frac{\text{const}}{r_{xy}},\ k=\overline{0,m_0},\ r_{xy}=|x-y|.\tag{29}$$

In that case

$$L[g_j(x,y,-\varkappa^2)\ge M_0[\varphi]-2N[\varphi]\ge-2N[\varphi]=$$

$$=-2\sum_{k=1}^{m_0}\int\limits_{S_k}\big[\overset{\circ}{\varphi}{}^{+}(z,y)\overset{\circ}{T}\overset{\circ}{\Gamma}_j(z-y,-\varkappa^2)-\overset{k}{\varphi}{}^{-}(z,y)\overset{k}{T}\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\big]d_zs. \quad (30)$$

Now taking into account that

$$\forall (z,y)\in S_k\times D_k:\ |\varphi(z,y)|\le\frac{\text{const}}{r_{zy}}\le\frac{1}{l_y},$$

$$\forall (z,y)\in S_k\times D_k:\quad |T\Gamma_j(z-y,-\varkappa^2)|\le\frac{\text{const}}{r_{zy}^2}=\frac{\text{const}}{r_{zy}^{\delta}r_{zy}^{2-\delta}}\le$$

$$\le\frac{1}{l_y^{\delta}}\cdot\frac{1}{r_{zy}^{2-\delta}},\quad \delta>0,$$

from (30) we obtain

$$L[g_j]\ge-\frac{\text{const}}{l_y^{1+\delta}},\quad \delta>0.$$

By virtue of representation (23) we can easily conclude that the estimate

$$\forall y\in D:\quad g_{jj}(y,y,-\varkappa^2)\ge-\frac{\text{const}}{l_y^{1+\delta}} \quad (31)$$

holds.

(27) and (31) imply (19). $\square$

**9.** Consider the first boundary-contact problem on eigenvalues: In the domain $D_k$ $(k=\overline{0,m_0})$, find a regular vector $w(x)=\overset{k}{w}(x)$, $x\in D_k$, which is a nontrivial solution of the equations

$$\forall x\in D_k:\quad \overset{k}{A}(\partial x)\overset{k}{w}(x)+\gamma\overset{k}{w}(x)=0,\quad k=\overline{0,m_0},$$

satisfying the contact conditions

$$\forall z\in S_k:\quad \overset{\circ}{w}{}^{+}(z)=\overset{k}{w}{}^{-}(z),\quad (\overset{\circ}{T}\overset{\circ}{w}(z))^{+}=(\overset{k}{T}\overset{k}{w}(z))^{-},\quad k=\overline{1,m_0},$$

and the boundary condition

$$\forall z\in S_k:\quad \overset{\circ}{w}{}^{+}(z)=0,\quad k=0,m_0+1,\dots,m.$$

We denote this problem by $I_\gamma$. In the same manner as in [8] we can show that problem $I_\gamma$ is equivalent to a system of integral equations

$$w(x)=(\gamma+\varkappa_0^2)\int\limits_D G(x,y,-\varkappa_0^2)w(y)dy. \quad (32)$$

By virtue of (11) and (12) equation (32) is an integral equation with a symmetric kernel of the class $L_2(D)$. Hence it follows that there exists a countable system of eigenvalues $(\gamma_n+\varkappa_0^2)_{n=1}^\infty$ and the corresponding orthonormal in $D$ system of eigenvectors $\{w^{(n)}(x)\}_{n=1}^\infty=\{\overset{k}{w}{}^{(n)}(x)\}_{n=1}^\infty$, $x\in D_k$, $k=\overline{0,m_0}$, of equation (32). Therefore $(\gamma_n)_{n=1}^\infty$ and $\{w^{(n)}(x)\}_{n=1}^\infty$ are respectively the eigenvalues and eigenvectors of problem $I_\gamma$. As established in [8], all $\gamma_n>0$. Moreover, it is proved in [11] that the system $\{w^{(n)}(x)\}_{n=1}^\infty$ is complete in $L_2(D)$. The properties of the volume potential [8] imply that eigenvectors are regular.

**10.** Hardy and Littlewood's theorem of the Tauber type [12] plays the major part in deriving asymptotic formulas.

**Theorem 3.** *If a nondecreasing function $\Phi(t)$ is Stieltjes summable, and for $x\to\infty$ the asymptotic representation*

$$\int\limits_0^\infty \frac{d\Phi(t)}{(x+t)^l}\sim\frac{p}{x^m},$$

*where $l,m,p$ satisfy the conditions $0<m<l$, $p\neq 0$, is fulfilled, then*

$$\Phi(t)\sim\frac{p\Gamma(l)}{\Gamma(m)\Gamma(l-m+1)}t^{l-m}.$$

*Here $\Gamma$ is the Euler function.*

Write the kernel expansion in terms of the eigenfunctions

$$G(x,y,-\varkappa^2)=\sum_{n=1}^\infty\frac{w^{(n)}(x)\times w^{(n)}(y)}{\gamma_n+\varkappa^2},\tag{33}$$

$$G(x,y,-\varkappa_0^2)=\sum_{n=1}^\infty\frac{w^{(n)}(x)\times w^{(n)}(y)}{\gamma_n+\varkappa_0^2},\tag{34}$$

where $x,y\in D_k$, $k=\overline{0,m_0}$, and the symbol $\times$ denotes the matrix product of a column vector by a row vector (dyadic product) $w^{(n)}(x)\times w^{(n)}(y)=\|w_i^{(n)}(x)\cdot w_j^{(n)}(y)\|$ $i,j=1,2,3$. From (33) and (34) we obtain

$$G(x,y,-\varkappa^2)-G(x,y,-\varkappa_0^2)=(\varkappa_0^2-\varkappa^2)\sum_{n=1}^\infty\frac{w^{(n)}(x)\times w^{(n)}(y)}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}.\tag{35}$$

Passage in equality (35) to the limit as $x\to y$ results in

$$(\varkappa_0^2-\varkappa^2)\sum_{n=1}^\infty\frac{[w_j^{(n)}(y)]^2}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}=\lim_{x\to y}\big[\Gamma_{jj}(x-y,-\varkappa^2)-$$
$$-\Gamma_{jj}(x-y,-\varkappa_0^2)\big]-\lim_{x\to y}\big[g_{jj}(x,y,-\varkappa^2)-g_{jj}(x,y,-\varkappa_0^2)\big],\tag{36}$$

$$x,y\in D_k,\quad k=\overline{0,m_0},\quad j=\overline{1,3}.$$

Taking into account (10) and (19), we find from (36) that

$$\sum_{n=1}^{\infty}\frac{[w_j^{(n)}(y)]^2}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}\sim\frac{\rho_k^{3/2}}{12\pi(\varkappa+\varkappa_0)}\left[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{\mu_k^{3/2}}\right],\quad(37)$$

$$y\in D_k,\quad k=\overline{0,m_0},\quad j=\overline{1,3}.$$

Consider the function

$$\Phi_j(t)=\sum_{\gamma_n\le t}\frac{[w_j^{(n)}(y)]^2}{\gamma_n+\varkappa_0^2}\quad y\in D_k,\quad k=\overline{0,m_0},\quad j=\overline{1,3}.$$

It is easy to observe that

$$\int_0^{\infty}\frac{d\Phi_j(t)}{t+\varkappa^2}=\sum_{n=1}^{\infty}\frac{[w_j^{(n)}(y)]^2}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}.$$

By (37) we have

$$\int_0^{\infty}\frac{d\Phi_j(t)}{t+\varkappa^2}\sim\frac{A_k}{\varkappa},\tag{38}$$

where

$$A_k=\frac{\rho_k^{3/2}}{12\pi}\left[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{\mu_k^{3/2}}\right].$$

By Theorem 3 with regard for (38) we obtain

$$\Phi_j(t)=\sum_{\gamma_n\le t}\frac{[w_j^{(n)}(y)]^2}{\gamma_n+\varkappa_0^2}\sim\frac{A_k\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{3})}t^{1/2}=\frac{2}{\pi}A_kt^{1/2}.\tag{39}$$

Obviously,

$$\sum_{\gamma_n\le t}[w_j^{(n)}(y)]^2=\int_0^t(\xi+\varkappa_0^2)d\Phi_j(\xi)=(\xi+\varkappa_0^2)\Phi_j(\xi)\big|_0^t-\int_0^t\Phi_j(\xi)d\xi.$$

Taking into account (39), we obtain

$$\sum_{\gamma_n\le t}[w_j^{(n)}(y)]^2\sim\frac{2}{3\pi}A_kt^{3/2},\quad y\in D_k,\quad k=\overline{0,m_0},\quad j=\overline{1,3}.$$

If we sum these relations with respect to $j=1,2,3$, then we shall have

$$\sum_{\gamma_n\le t}[w_j^{(n)}(y)]^2\sim\frac{2}{\pi}A_kt^{3/2},\quad y\in D,$$

or

$$\sum_{\gamma_n\leq t}[\overset{k}{w}_j{}^{(n)}(y)]^2\sim\frac{\rho_k^{3/2}}{6\pi^2}\Big[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{\mu_k^{3/2}}\Big]t^{3/2},\ y\in D_k,\ k=\overline{0,m_0}. \quad (40)$$

The above expression provides the asymptotic distribution of eigenfunctions.

**11.** Taking into account (10), from (36) we obtain

$$(\varkappa^2-\varkappa_0^2)\sum_{n=1}^{\infty}\frac{[w^{(n)}(y)]^2}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}=3(\varkappa-\varkappa_0)A_k+ \\ +\sum_{j=1}^{3}\big[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa^2)\big]. \quad (41)$$

Denote

$$\varphi(y,\varkappa)=3(\varkappa-\varkappa_0)A_k+\sum_{j=1}^{3}\big[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa^2)\big].$$

Now (41) yields

$$\frac{\varphi(y,\varkappa)}{\varkappa^2-\varkappa_0^2}=\sum_{n=1}^{\infty}\frac{[w^{(n)}(y)]^2}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}\leq\sum_{n=1}^{\infty}\frac{[w^{(n)}(y)]^2}{(\gamma_n+\varkappa_0^2)^2}. \quad (42)$$

By virtue of the Bessel inequality we have

$$\sum_{n=1}^{\infty}\frac{[w^{(n)}(x)]^2}{(\gamma_n+\varkappa_0^2)^2}\leq\int_D|G(x,y,-\varkappa_0^2)|^2dy,\quad x\in D_k,\quad k=\overline{0,m_0}. \quad (43)$$

By estimate (8) it follows from (42) that the sum of the series

$$\sum_{n=1}^{\infty}\frac{[w^{(n)}(x)]^2}{(\gamma_n+\varkappa_0^2)^2}$$

exists and is uniformly bounded in $\overline{D}_k$. This and (42) imply

$$\forall y\in D_k:\ |\varphi(y,\varkappa)|\leq\text{const}(\varkappa^2-\varkappa_0^2),\quad k=\overline{0,m_0}. \quad (44)$$

Integrating equality (42) in $D$ and using the fact that the vectors $[w^{(n)}(x)]_{n=1}^{\infty}$ are orthonormal in $D$, we obtain

$$\int_D\varphi(y,\varkappa)dy=(\varkappa^2-\varkappa_0^2)\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}, \quad (45)$$

$$\int_D\varphi(y,\varkappa)dy=\int_D 3(\varkappa-\varkappa_0)A_k dy+\int_D\sum_{j=1}^{3}\big[g_{jj}(y,y,-\varkappa_0^2)-$$

$$-g_{jj}(y,y,-\varkappa^2)\big]dy = 3(\varkappa-\varkappa_0)\sum_{k=0}^{m_0} A_k \operatorname{mes} D_k +$$
$$+\sum_{j=1}^{3}\int_D \big[g_{jj}(y,y,-\varkappa_0^2) - g_{jj}(y,y,-\varkappa^2)\big]dy. \tag{46}$$

By (46), from (45) we have

$$\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)} - \frac{3}{\varkappa+\varkappa_0}\sum_{k=0}^{m_0} A_k \operatorname{mes} D_k =$$
$$= \frac{1}{\varkappa^2-\varkappa_0^2}\sum_{j=1}^{3}\int_D \big[g_{jj}(y,y,-\varkappa_0^2) - g_{jj}(y,y,-\varkappa^2)\big]dy. \tag{47}$$

Denote by $D_{k\eta}$ that part of $D_k$, $k=\overline{0,m_0}$, whose points lie from the boundary $D_k$ at a distance less than $\eta$. $D_\eta = \bigcup\limits_{k=0}^{m_0} D_{k\eta}$. Then

$$\int_D \big[g_{jj}(y,y,-\varkappa_0^2) - g_{jj}(y,y,-\varkappa^2)\big]dy = \int_{D\setminus D_\eta} \big[g_{jj}(y,y,-\varkappa_0^2) -$$
$$-g_{jj}(y,y,-\varkappa^2)\big]dy + \int_{D_\eta}\varphi(y,\varkappa)dy - \int_{D_\eta} 3(\varkappa-\varkappa_0)A_k dy.$$

This and (47) imply

$$\Bigg|\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)} - \frac{3}{\varkappa+\varkappa_0}\sum_{k=0}^{m_0} A_k \operatorname{mes} D_k\Bigg| \le$$
$$\le \frac{1}{\varkappa^2-\varkappa_0^2}\Bigg|\int_{D_\eta}\varphi(y,\varkappa)dy\Bigg| + \frac{1}{\varkappa^2-\varkappa_0^2}\sum_{j=1}^{3}\Bigg|\int_{D\setminus D_\eta}\big[g_{jj}(y,y,-\varkappa_0^2) -$$
$$-g_{jj}(y,y,-\varkappa)\big]dy\Bigg| + \frac{3}{\varkappa+\varkappa_0}\Bigg|\int_{D_m} A_k dy\Bigg|. \tag{48}$$

The following estimates are valid:

$$\frac{3}{\varkappa+\varkappa_0}\Bigg|\int_{D_\eta} A_k dy\Bigg| = \frac{3}{\varkappa+\varkappa_0}\Bigg|\sum_{k=0}^{m_0}\int_{D_{k\eta}} A_k dy\Bigg| \le \operatorname{const}\frac{3}{\varkappa+\varkappa_0}\eta, \tag{49}$$
$$\frac{1}{\varkappa^2-\varkappa_0^2}\Bigg|\int_{D_\eta}\varphi(y,\varkappa)dy\Bigg| \le \frac{\operatorname{const}}{\varkappa^2-\varkappa_0^2}(\varkappa^2-\varkappa_0^2)\eta = \operatorname{const}\eta, \tag{50}$$

$$\frac{1}{\varkappa^2-\varkappa_0^2}\sum_{j=1}^{3}\bigg|\int\limits_{D\setminus D_\eta}\big[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa)\big]\,dy\bigg|\le\frac{\text{const}}{\varkappa^2-\varkappa_0^2}\,\frac{1}{\eta^\delta}. \quad (51)$$

The validity of (49) is obvious. (50) and (51) hold by virtue of (44) and (19), respectively. It is important to remark here that the constants appearing in (49), (50) and (51) do not depend on $\varkappa$ and $y$. Consider the function

$$\Phi(t)=\sum_{\gamma_n\le t}\frac{1}{\gamma_n+\varkappa_0^2}.$$

It is easy to verify that

$$\int\limits_0^\infty\frac{d\Phi(t)}{t+\varkappa^2}=\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}.$$

By of (49), (50) and (51) we find from (48) that

$$\int\limits_0^\infty\frac{d\Phi(t)}{t+\varkappa^2}\sim\frac{3\sum_{k=0}^{m_0}A_k\,\text{mes}\,D_k}{\varkappa}$$

for $\eta=\frac{1}{\varkappa^2-\varkappa_0^2}$. According to Theorem 3 we have

$$\Phi(t)\sim\frac{6\sum_{k=0}^{m_0}A_k\,\text{mes}\,D_k}{\pi}t^{1/2}. \quad (52)$$

Denoting

$$N(t)=\sum_{\gamma_n\le t}1,$$

for a number of eigenvalues not higher than $t$, we get

$$N(t)=\int\limits_0^t(\xi+\varkappa_0^2)d\Phi(\xi)=(\xi+\varkappa_0^2)\Phi(\xi)\big|_0^t-\int\limits_0^t\Phi(\xi)d\xi.$$

Hence with regard for (52) we obtain

$$N(t)\sim\frac{2}{\pi}\sum_{k=0}^{m_0}A_k\,\text{mes}\,D_k\cdot t^{3/2},$$

or, finally,

$$N(t)\sim\frac{1}{6\pi^2}\sum_{k=0}^{m_0}\bigg[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{\mu_k^{3/2}}\bigg]\rho_k^{3/2}\,\text{mes}\,D_k\cdot t^{3/2}. \quad (53)$$

Thus the results of this paper can be formulated as

**Theorem 4.** *The asymptotic distribution of eigenvector-functions and eigenvalues of the basic boundary-contact problems of oscillation of classical elasticity is given by formulas* (40) *and* (53)*, respectively.*

*Remark* 1. The results obtained remain valid (a) when the boundary conditions on $S_k$ $(k = 0, m_0 + 1, \dots, m)$ are replaced by those of the second basic problem [8] (i.e., with elastic stresses given on the boundary) and (b) when the conditions of the first problem are given on some part of the boundary $S_k$ $(k = 0, m_0 + 1, \dots, m)$, and the conditions of the second problem on the remainder of the boundary (a mixed problem).

Thus we have considered the case of the main contact conditions. The results remain valid for other admissible contact conditions [13].

*Remark* 2. If $\lambda_k = \lambda$, $\mu_k = \mu$, $\rho_k = \rho$ $(k = \overline{0, m_0})$ i.e., if a homogeneous elastic medium $D_0 \cup (\bigcup\limits_{k=1}^{m_0} D_k) \cup (\bigcup\limits_{k=1}^{m_0} S_k)$ is considered, then, as can readily be verified, $[w^{(n)}(x)]_{n=1}^{\infty}$ and $(\gamma_n)_{n=1}^{\infty}$ will be the eigenvector-functions and eigenvalues of equation (2) the corresponding boundary value problem and the obtained asymptotic formulas will coincide with the well-known formulas of Weyl, Plejel, Burchuladze and others.

*Remark* 3. Note that the above asymptotic formulas with certain modifications are also fulfilled for two-dimensional problems. For example, in the two-dimensional case formula (53) takes the form

$$N(t) \sim \frac{1}{4\pi} \sum_{k=0}^{m_0} \left[ \frac{1}{\lambda_k + 2\mu_k} + \frac{1}{\mu_k} \right] \rho_k \operatorname{mes} D_k \cdot t,$$

where $\operatorname{mes} D_k$ is a space of the domain $D_k$.

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