

**ON THE ABSOLUTE SUMMABILITY OF SERIES WITH  
RESPECT TO BLOCK-ORTHONORMAL SYSTEMS**

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ABSTRACT. Theorems determining Weyl's multipliers for the summability almost everywhere by the  $|c, 1|$  method of the series with respect to block-orthonormal systems are proved. In particular, it is stated that if the sequence  $\{\omega(n)\}$  is the Weyl multiplier for the summability almost everywhere by the  $|c, 1|$  method of all orthogonal series, then there exists a sequence  $\{N_k\}$  such that  $\{\omega(n)\}$  will be the Weyl multiplier for the summability almost everywhere by the  $|c, 1|$  method of all series with respect to the  $\Delta_k$ -orthonormal systems.

The present paper deals with the summability almost everywhere (a.e.) by the  $|c, \alpha|$  method of series with respect to block-orthonormal systems. Under the summability by the  $|c, \alpha|$  method of the series

$$\sum_{n=1}^{\infty} a_n$$

is understood the convergence of the series

$$\sum_{n=1}^{\infty} |\sigma_{n+1}^{(\alpha)} - \sigma_n^{(\alpha)}|,$$

where

$$\sigma_n^{(\alpha)} = \frac{1}{A_n^\alpha} \sum_{k=1}^{\infty} A_{n-k}^\alpha a_k$$

are the Cesàro  $(c, \alpha)$ -means.

The problem of the summability a.e. by the  $|c, \alpha|$  method of orthogonal series was considered by P.L. Ul'yanov [1]. In particular, he proved that if

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the condition

$$\sum_{n=1}^{\infty} \frac{1}{n \omega(n)} < \infty, \quad (1)$$

is fulfilled for a positive nondecreasing sequence  $\{\omega(n)\}$ , then the convergence of the series

$$\sum_{n=1}^{\infty} a_n^2 \omega(n)$$

guarantees the summability a.e. on  $(0, 1)$  by the  $|c, \alpha|$  method ( $\alpha > \frac{1}{2}$ ) of the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \quad (2)$$

for every orthonormal system from  $L^2(0, 1)$ .

If however

$$\sum_{n=1}^{\infty} \frac{1}{n \omega(n)} = \infty,$$

then there exists an even function  $f(x) \in \bigcap_{p \geq 1} L^p[0, 2\pi]$  such that its Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} c_n \cos nx$$

converges a.e. on  $[0, 2\pi]$  and for every fixed  $\alpha > 0$  is not  $|c, \alpha|$  summable a.e. on  $[0, 2\pi]$  though

$$\sum_{n=1}^{\infty} c_n^2 \omega(n) < \infty.$$

**Definition 1 (see [2]).** Let  $\{N_k\}$  be an increasing sequence of natural numbers,  $\Delta_k = (N_k, N_{k+1}]$ ,  $k = 1, 2, \dots$ , and  $\{\varphi_n\}$  be a system of functions from  $L^2(0, 1)$ . The system  $\{\varphi_n\}$  will be called a  $\Delta_k$ -orthonormal system ( $\Delta_k$ -ONS) if:

- (1)  $\|\varphi_n\|_2 = 1$ ,  $n = 1, 2, \dots$ ;
- (2)  $(\varphi_i, \varphi_j) = 0$  for  $i, j \in \Delta_k$ ,  $i \neq j$ ,  $k \geq 1$ .

**Definition 2 (see [1]).** A positive nondecreasing sequence  $\{\omega(n)\}$  will be called the Weyl multiplier for the summability a.e. of series with respect to the  $\Delta_k$ -ONS  $\{\varphi_n\}$  if the condition

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty \quad (3)$$

guarantees the summability a.e. by the  $|c, \alpha|$  method of the corresponding series (2).

Below we shall quote the theorem showing that if the sequence  $\{\omega(n)\}$  is the Weyl multiplier for the summability a.e. by the  $|c, 1|$  method of all orthogonal series (2), then it will be the Weyl multiplier for the summability a.e. by the  $|c, 1|$  method of all series (2) with respect to the  $\Delta_k$ -ONS for the increasing sequence of natural numbers  $\{N_k\}$ .

**Theorem 1.** *If a positive nondecreasing sequence  $\{\omega(n)\}$  is the Weyl multiplier for the summability a.e. by the  $|c, 1|$  method of all orthonormal series (2), then there exists an increasing sequence of natural numbers  $\{N_k\}$  such that  $\{\omega(n)\}$  is the Weyl multiplier for the summability a.e. by the  $|c, 1|$  method of all series (2) with respect to the  $\Delta_k = (N_k, N_{k+1}]$ -ONS.*

*Proof.* We prove this theorem by the Wang–Ul’yanov’s scheme (see [1]) modifying it accordingly. Let the positive nondecreasing sequence  $\{\omega(n)\}$  be the Weyl multiplier for the summability a.e. by the  $|c, 1|$  method of all orthogonal series (2). Then condition (1) is fulfilled.

As is known (see [1]), for the positive nondecreasing on  $[n_0, +\infty)$  function  $\omega(x)$  the series

$$\sum_{m=n_0}^{\infty} \frac{1}{m \omega(m)} \quad \text{and} \quad \sum_{m=n_0^2}^{\infty} \frac{1}{m \omega(\sqrt{m})}$$

converge or diverge simultaneously. Therefore, taking into account (1), we have

$$\sum_{n=1}^{\infty} \frac{1}{n \omega(\sqrt[4]{n})} < \infty.$$

Then

$$\sum_{n=1}^{\infty} \frac{R(n)}{n \omega(\sqrt[4]{n})} < \infty, \quad (4)$$

where

$$R(n) = \frac{\left( \sum_{k=2}^{\infty} \frac{1}{k \omega(\sqrt[4]{k})} \right)^{\frac{1}{2}}}{\left( \sum_{k=n+1}^{\infty} \frac{1}{k \omega(\sqrt[4]{k})} \right)^{\frac{1}{2}}}.$$

Obviously,  $R(1) = 1$ ,  $R(n) < R(n+1)$  and  $\lim_{n \rightarrow \infty} R(n) = +\infty$ .

Define the sequence  $k(n)$  by the recursion formula

$$k(1) = 0, \quad k(n+1) = \begin{cases} k(n) + 1 & \text{if } R(n+1) \geq k(n) + 1, \\ k(n) & \text{if } R(n+1) < k(n) + 1, \end{cases} \quad n \geq 1.$$

Thus we obtain the nondecreasing sequence of nonnegative integers for which

$$k(n) \leq R(n), \quad n = 1, 2, \dots \quad (5)$$

Note that for the sequence  $k(n)$  there exists an increasing sequence of natural numbers  $\{N_k\}$  (it is assumed that  $N_0 = 0$ ) which is defined by the formula

$$k(n) = \max\{k : N_k < n\}.$$

Then, taking into account (4) and (5), we find that

$$\sum_{n=1}^{\infty} \frac{k(n)}{n \omega(\sqrt[4]{n})} < \infty. \quad (6)$$

Let  $\{\varphi_n\}$  be a block-orthonormal system with  $\Delta_k = (N_k, N_{k+1}]$  and condition (3) be fulfilled. Then for the corresponding series (2) we have

$$\sigma_n(x) - \sigma_{n-1}(x) = \frac{1}{n(n-1)} \sum_{i=1}^{\infty} a_i(i-1) \varphi_i(x), \quad n \geq 2.$$

Denoting by  $c$  the absolute positive constants which, generally speaking, may have different values in different inequalities and using (6), we find that

$$\begin{aligned} \sum_{n=2}^{\infty} \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)| dx &\leq \sum_{n=2}^{\infty} \left( \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx \right)^{\frac{1}{2}} \leq \\ &\leq c \sum_{n=2}^{\infty} \frac{1}{n^2} \left( \int_0^1 \left| \sum_{i=1}^n a_i(i-1) \varphi_i(x) \right|^2 dx \right)^{\frac{1}{2}} \leq \\ &\leq c \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \int_0^1 \left| \sum_{i=1}^{N_{k(n)}} a_i(i-1) \varphi_i(x) \right|^2 + \int_0^1 \left| \sum_{i=N_{k(n)+1}^n a_i(i-1) \varphi_i(x) \right|^2 dx \right)^{\frac{1}{2}} \leq \\ &\leq c \sum_{n=1}^{\infty} \frac{1}{n^2} \left( k(n) \sum_{i=1}^{N_{k(n)}} a_i^2 i^2 + \sum_{i=N_{k(n)+1}^n a_i^2 i^2 \right)^{\frac{1}{2}} \leq \\ &\leq c \sum_{n=1}^{\infty} \frac{1}{n^2} \left( k(n) \sum_{i=1}^n i^2 a_i^2 \right)^{\frac{1}{2}} \leq c \sum_{n=1}^{\infty} \frac{\sqrt{k(n)}}{n^2} \left[ \left( \sum_{i=1}^{\lfloor \sqrt[4]{n} \rfloor} i^2 a_i^2 \right)^{\frac{1}{2}} + \right. \\ &\quad \left. + \left( \sum_{i=\lfloor \sqrt[4]{n} \rfloor + 1}^n i^2 a_i^2 \right)^{\frac{1}{2}} \right] \leq c \left( \sum_{n=1}^{\infty} \frac{\sqrt{k(n)} (\sqrt{n} \sqrt[4]{n})^{\frac{1}{2}}}{n^2} + \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \frac{\sqrt{k(n)} n \omega(\sqrt[4]{n})^{\frac{1}{2}}}{n^2 n \omega(\sqrt[4]{n})^{\frac{1}{2}}} \left( \sum_{i=[\sqrt[4]{n}]+1}^n i^2 a_i^2 \right)^{\frac{1}{2}} \leq \\
& \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{8}}} + \left( \sum_{n=1}^{\infty} \frac{\omega(\sqrt[4]{n})}{n^3} \sum_{i=[\sqrt[4]{n}]+1}^n i^2 a_i^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \frac{k(n)}{k \omega(\sqrt[4]{n})} \right)^{\frac{1}{2}} \right) \leq \\
& \leq c + c \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{i=1}^{\infty} i^2 a_i^2 \omega(i) \right)^{\frac{1}{2}} \leq c + c \left( \sum_{i=1}^{\infty} a_i^2 \omega(i) \right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

whence by Levy's theorem

$$\sum_{n=2}^{\infty} |\sigma_n(x) - \sigma_{n-1}(x)| < \infty \quad \text{a.e. on } (0,1). \quad \square$$

The theorem below makes it possible to determine the Weyl multipliers for the summability a.e. of the series (2) with respect to the  $\Delta_k$ -ONS for regularly increasing sequences  $\{N_k\}$ .

**Theorem 2.** *Let an increasing sequence of natural numbers  $\{N_k\}$  be given, for which the condition*

$$\sum_{k=n}^{\infty} \frac{1}{N_k^2} = O\left(\frac{n}{N_n^2}\right) \quad (n \rightarrow \infty) \quad (7)$$

is fulfilled, and let

$$k(n) = \max\{k : N_k < n\}.$$

If for the positive nondecreasing sequence  $\{\omega(n)\}$  condition (1) is fulfilled, then for every  $\Delta_k$ -ONS  $\{\varphi_n\}$  the condition

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) k(n) < \infty \quad (8)$$

guarantees the summability a.e. by the  $|c,1|$  method of the corresponding series (2).

*Proof.* Let conditions (1), (7) and (8) be fulfilled. Then for the corresponding series (2) we have

$$\begin{aligned}
& \sum_{n=2}^{\infty} \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)| dx \leq \sum_{n=2}^{\infty} \left( \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx \right)^{\frac{1}{2}} \leq \\
& \leq c \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \int_0^1 \left| \sum_{i=1}^{N_{k(n)}} a_i(i-1) \varphi_i(x) \right|^2 dx + \int_0^1 \left| \sum_{i=N_{k(n)}+1}^n a_i(i-1) \varphi_i(x) \right|^2 dx \right)^{\frac{1}{2}} \leq
\end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{n=1}^{\infty} \frac{1}{n^2} \left( k(n) \sum_{i=1}^{\infty} i^2 a_i^2 \right)^{\frac{1}{2}} \leq c \sum_{n=1}^{\infty} \frac{\sqrt{k(n)}}{n^2} \left[ \left( \sum_{i=n}^{[\sqrt[4]{n}]} i^2 a_i^2 \right)^{\frac{1}{2}} + \right. \\
&\quad \left. + \left( \sum_{i=[\sqrt[4]{n}]+1}^n i^2 a_i^2 \right)^{\frac{1}{2}} \right] \leq c \left( \sum_{n=1}^{\infty} \frac{\sqrt{k(n)} n^{\frac{3}{8}}}{n^2} + \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \frac{\sqrt{k(n)}}{n^2} \frac{(n\omega(\sqrt[4]{n}))^{\frac{1}{2}}}{(n\omega(\sqrt[4]{n}))^{\frac{1}{2}}} \left( \sum_{i=[\sqrt[4]{n}]+1}^n i^2 a_i^2 \right)^{\frac{1}{2}} \right) \leq \\
&\leq c + c \left( \sum_{n=1}^{\infty} \frac{k(n)}{n^3} \sum_{i=1}^{\infty} i^2 a_i^2 \omega(i) \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \frac{1}{n\omega(\sqrt[4]{n})} \right)^{\frac{1}{2}} \leq \\
&\leq c + c \left( \sum_{i=1}^{\infty} i^2 a_i^2 \omega(i) \sum_{i=1}^{\infty} \frac{k(n)}{n^3} \right)^{\frac{1}{2}} = \\
&= c + c \left( \sum_{i=1}^{\infty} i^2 a_i^2 \omega(i) \left( \sum_{n=i}^{N_{k(i)+1}} \frac{k(i)}{n^3} + \sum_{j=k(i)+1}^{\infty} j \sum_{n=N_j+1}^{N_{j+1}} \frac{1}{n^3} \right) \right)^{\frac{1}{2}} \leq \\
&\leq c + c \left( \sum_{i=1}^{\infty} i^2 a_i^2 \omega(i) \left( \frac{k(i)}{i^2} + (k(i)+1) \sum_{n=N_{k(i)+1}+1}^{\infty} \frac{1}{n^3} + \right. \right. \\
&\quad \left. \left. + \sum_{j=k(i)+1}^{\infty} \frac{1}{N_j^2} \right) \right)^{\frac{1}{2}} \leq c + c \left( \sum_{i=1}^{\infty} i^2 a_i^2 \omega(i) k(i) \right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

whence by Levy's theorem

$$\sum_{n=2}^{\infty} |\sigma_n(x) - \sigma_{n-1}(x)| < \infty \quad \text{a.e. on } (0, 1). \quad \square$$

*Remark 1.* In Theorem 2, the Weyl multipliers defined by conditions (1) and (8) can be assumed to be exact on the set of sequences  $\{N_k\}$  with condition (7) in the sense that if condition (1) is violated, then one can construct a sequence  $\{N_k\}$  for which condition (7) is fulfilled and also there exists a trigonometric series

$$\sum_{n=1}^{\infty} b_n \cos nx,$$

which is nonsummable by the  $|c, \alpha|$  method for almost all  $x \in [0, 2\pi]$  (for every fixed  $\alpha > 0$ ) though

$$\sum_{n=1}^{\infty} b_n^2 \omega(n) k(n) < \infty.$$

Indeed, let the condition

$$\sum_{n=1}^{\infty} \frac{1}{n \omega(n)} = \infty$$

be fulfilled for the sequence  $\{\omega(n)\}$ .

We construct an increasing sequence of natural numbers  $\{N_k\}$  in such a way that the condition

$$k = O\left(\sum_{n=1}^{N_k} \frac{1}{n \omega(n)}\right)^{\beta}, \quad 0 < \beta \leq \frac{1}{2},$$

be fulfilled and the sequence  $\frac{N_k}{k}$  be increasing.

Clearly, condition (7) is fulfilled (see [3], Remark 2).

Take

$$s_k = \sum_{n=1}^k \frac{1}{n \omega(n)}, \quad k = 1, 2, \dots,$$

and

$$c_m = \frac{1}{\sqrt{m} \omega(m) (s_m)^{\beta + \frac{1}{2}}} \quad m = 1, 2, \dots$$

Then for arbitrary  $\varepsilon_m = \pm 1$  we have

$$\begin{aligned} & \sum_{m=1}^{\infty} (\varepsilon_m c_m)^2 \omega(m) k(m) = \sum_{m=1}^{\infty} \frac{k(m)}{m \omega(m) (s_m)^{2\beta+1}} = \\ &= \sum_{k=0}^{\infty} \sum_{m=N_k+1}^{N_{k+1}} \frac{k(m)}{m \omega(m) (s_m)^{2\beta+1}} \leq c \sum_{k=0}^{\infty} \sum_{m=N_k+1}^{N_{k+1}} \frac{(s_{N_k})^{\beta}}{m \omega(m) (s_m)^{2\beta+1}} \leq \\ &\leq c \sum_{k=0}^{\infty} \sum_{m=N_k+1}^{N_{k+1}} \frac{1}{m \omega(m) (s_m)^{1+\beta}} \leq c \sum_{m=1}^{\infty} \frac{1}{m \omega(m) (s_m)^{1+\beta}} < \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} \left\{ \sum_{m=2^{n+1}}^{2^{n+1}} c_m^2 \right\}^{\frac{1}{2}} &\geq \sum_{n=0}^{\infty} \left\{ \sum_{m=2^{n+1}}^{2^{n+1}} \frac{1}{m (\omega(m))^2 (s_m)^{1+2\beta}} \right\}^{\frac{1}{2}} \geq \\ &\geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\omega(2^n) (s_{2^n})^{\frac{1}{2} + \beta}} = \infty. \end{aligned}$$

Therefore by Billard's theorem [1], for almost all choices of  $\varepsilon_k = \pm 1$  the series

$$\sum_{m=1}^{\infty} \varepsilon_m c_m \cos mx$$

is  $|c, \alpha|$ -nonsummable ( $\alpha > 0$ ) at almost every point  $x \in [0, 2\pi]$  though

$$\sum_{n=1}^{\infty} b_n^2 \omega(n) k(n) < \infty,$$

where  $b_n = \varepsilon_n c_n$ .

*Remark 2.* The above theorems remain also valid for  $|c, \alpha|$  methods with  $c\alpha > \frac{1}{2}$ .

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