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TWO-WEIGHTED INEQUALITIES FOR INTEGRAL OPERATORS IN LORENTZ SPACES DEFINED ON HOMOGENEOUS GROUPS

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ABSTRACT. The optimal sufficient conditions are found for weights, which guarantee the validity of two-weighted inequalities for singular integrals in the Lorentz spaces defined on homogeneous groups. In some particular case the found conditions are necessary for the corresponding inequalities to be valid. Also, the necessary and sufficient conditions are found for pairs of weights, which provide the validity of two-weighted inequalities for the generalized Hardy operator in the Lorentz spaces defined on homogeneous groups.

INTRODUCTION

In this paper, the optimal sufficient conditions are found for pairs of weights, which provide the validity of two-weighted inequalities for singular integrals in the Lorentz spaces defined on homogeneous groups. In [1–7], analogous problems were studied in the Lebesgue spaces for the Hilbert transform and singular integrals, while in [8] the sufficient conditions are found for pairs of weights, which guarantee the fulfilment of two-weighted inequalities for singular integrals in the Lorentz spaces defined on Euclidean spaces. In this paper, the necessary and sufficient conditions are also found for pairs of weights, which provide the boundedness of the generalized Hardy operators in the weighted Lorentz spaces. Analogous problems were considered in the Lebesgue spaces in [9–14] and in the Lorentz spaces in [8], [15] and [16] (see also [17]).

Finally, we would like to note that most of the results obtained in this paper are new for the classical singular integrals as well.

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Homogeneous Groups and the Spaces of Functions Defined on Them. Some Known Results

In this section, we give the notion of homogeneous groups and define the Lorentz spaces. Some known results on the Lorentz spaces and singular integrals are also presented.

Definition 1 (see [18], p. 5). A homogeneous group is a connected, simply connected nilpotent Lie group G on whose Lie algebra g a one-parametric group of extensions $\delta_t = \exp(A \ln t)$, $t > 0$, is given, where A is the diagonizable operator on g whose eigenvalues are positive.

For the homogeneous group G the mappings $\exp \circ \delta_t \circ \exp^{-1}$, $t > 0$, are the automorphisms on G which will again be denoted by δ_t.

The number $Q = trA$ is called a homogeneous dimension of the group G.

Homogeneous groups can be exemplified by an n-dimensional Euclidean space, Heisenberg groups and so on.

Onto the homogeneous group G we introduce a homogeneous norm, i.e., a continuous function $r : G \to [0, \infty)$ which is smooth on $G \setminus \{e\}$ and satisfies the following conditions:

1. $r(x) = r(x^{-1})$ for any $x \in G$;
2. $r(\delta_t x) = tr(x)$ for arbitrary $x \in G$ and $t > 0$;
3. $r(x) = 0 \Leftrightarrow x = e$;
4. there exists a constant $c_0 > 0$ such that $r(xy) \leq c_0(r(x) + r(y))$ for arbitrary x and y from G.

For $x \in G$ and $\rho > 0$ we set $B(x, \rho) = \{y \in G : r(xy^{-1}) < \rho\}$. Further, let $S(x, \rho) = \{y \in G : r(xy^{-1}) = \rho\}$. Note that $\delta_\rho B(e, 1) = B(e, \rho)$.

Onto the group G we fixed the normed Haar measure in such a manner that the measure of the unit ball $B(e, 1)$ be equal to 1. The Haar measure of any measurable set $E \subset G$ will be denoted by $|E|$, and the integral on E with respect to this measure by $\int\limits_E f(x)dx$.

It readily follows that

$$|\delta_t E| = t^Q |E|, \quad d(\delta_t x) = t^Q dx.$$

In particular, for arbitrary $x \in G$ and $\rho > 0$ we have $|B(e, \rho)| = \rho^Q$.

Definition 2. An almost everywhere positive, locally integrable function $w : G \to \mathbb{R}^1$ will be called a weight.

In what follows, we shall denote by $L^{pq}_w(G)$ the Lorentz space with weight w which is a class of all measurable functions $f : G \to \mathbb{R}^1$ for which

$$\|f\|_{L^{pq}_w(G)} = \left(q \int\limits_0^\infty \left(\int\limits_{\{x \in G\} : |f(x)| > \lambda\}} w(x)\, dx \right)^{\frac{q}{p}} \lambda^{q-1} d\lambda \right)^{\frac{1}{q}} < \infty$$

when $1 \le p \le \infty$, $1 \le q < \infty$, and

$$\|f\|_{L_w^{p\infty}(G)} = \sup_{\lambda>0} \lambda \Big(\int\limits_{\{x\in G: |f(x)|>\lambda\}} w(x)\,dx \Big)^{\frac{1}{p}} < \infty$$

when $1 \le p < \infty$.

For $p = q$, $L_w^{pq}(G)$ is the Lebesgue space which it is commonly accepted to denote by $L_w^p(G)$.

Theorem A (see [18], p. 14). *Let G be a homogeneous group, $S_G = \{x \in G : r(x) = 1\}$. There exists a unique countably additive measure σ defined on S_G such that the equality*

$$\int\limits_G f(x)\,dx = \int\limits_0^\infty t^{Q-1} \Big(\int\limits_{S_G} f(\delta_t \xi)\,d\sigma(\xi) \Big)\,dt$$

holds for any $f \in L^1(G)$.

Let $k : G \to \mathbb{R}^1$ be a measurable function such that:

(1) $|k(x)| \le \frac{c}{r(x)^Q}$ for an arbitrary element $x \ne e$;

(2) there exists a positive constant c_1 such that for arbitrary $x, y \in G$ with the condition $r(xy^{-1}) < \frac{1}{2} r(x)$ we have the inequality

$$\big|k(x) - k(y)\big| \le c_1 \omega \Big(\frac{r(xy^{-1})}{r(x)} \Big) \frac{1}{r(x)^Q},$$

where $\omega : [0,1] \to \mathbb{R}^1$ is a nondecreasing function such that $\omega(0) = 0$, $\omega(2t) \le c_2 \omega(t)$ for any $t > 0$ and $\int\limits_0^1 \frac{\omega(t)}{t}\,dt < \infty$.

It will be assumed that the kernel k together with the above-given conditions satisfy the condition: the singular integral

$$Tf(x) = \text{p.v.} \int\limits_G k(xy^{-1}) f(y)\,dy \overset{\text{def}}{=} \lim_{\varepsilon\to 0} \int\limits_{G\setminus B(x,\varepsilon)} k(xy^{-1}) f(y)\,dy$$

defines the bounded operator in the space $L^2(G)$.

The lemmas below are valid.

Lemma A ([19]). *Let $E \subset G$ be an arbitrary measurable set, w a weight function on G and f, f_1, f_2 measurable functions on G. Then we have:*

(1) $\quad \|\chi_E(\cdot)\|_{L_w^{pq}(G)} = \Big(\int\limits_E w(x)\,dx \Big)^{\frac{1}{p}};$

(2) $\quad \|f\|_{L_w^{pq_1}(G)} \le \|f\|_{L_w^{pq_2}(G)},$

for fixed p and $q_2 \le q_1$;

(3) $\|f_1 f_2\|_{L^{pq}_w(G)} \le c\|f_1\|_{L^{p_1q_1}_w(G)}\|f_2\|_{L^{p_2q_2}_w(G)}$,

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$.

Lemma B. *Let E_k be measurable sets of the homogeneous group G such that the inequality $\sum\limits_k \chi_{E_k} \le c\chi_{\underset{k}{\cup} E_k}$ with constant c is fulfilled. Then:*

(1) *for any f we have the inequality*

$$\sum_k \big\|f(\cdot)\chi_{E_k}(\cdot)\big\|^{\lambda}_{L^{rs}_w(G)} \le c_1\big\|f(\cdot)\chi_{\underset{k}{\cup} E_k}(\cdot)\big\|^{\lambda}_{L^{rs}_w(G)},$$

where the constant c_1 does not depend on f and $\max(r,s)\le\lambda$;

(2) *for any f we have the inequality*

$$\bigg\|\sum_k f(\cdot)\chi_{E_k}(\cdot)\bigg\|^{\gamma}_{L^{pq}_w(G)} \le c_2\sum_k \big\|f(\cdot)\chi_{E_k}(\cdot)\big\|^{\gamma}_{L^{pq}_w(G)},$$

where the constant c_2 does not depend on f and $0<\gamma\le\min(p,q)$.

Lemma B is proved as Lemma 2 from [8]. For the case $c=1$ analogous result was obtained in [20], [15] (see also [16]).

Definition 3. A function $v: G\to\mathbb{R}^1_+$ is called radial if there exists a function $\beta:\mathbb{R}^1_+\to\mathbb{R}^1_+$ such that the equality $v(x)=\beta(r(x))$ holds for any $x\in G$. In what follows, instead of β we shall use the notation v.

Definition 4. Let $1<p<\infty$. A weight function w belongs to $A_p(G)$ if

$$\sup\bigg(\frac{1}{|B|}\int\limits_B w(x)\,dx\bigg)\bigg(\frac{1}{|B|}\int\limits_B w^{1-p'}(x)\,dx\bigg)^{p-1}<\infty,$$

where the least upper bound is taken with respect to all balls B, $B\subset G$.

Theorem B ([21]). *Let $1<p<\infty$, $w\in A_p(G)$. Then the operator T is bounded in $L^p_w(G)$.*

Theorem C (see [22], p. 207). *Let $1<p,q<\infty$. If $w\in A_p(G)$, then the operator T is bounded in $L^{pq}_w(G)$. If the Hilbert transform acts continuously in $L^{pq}_w(\mathbb{R})$, then $w\in A_p(\mathbb{R})$.*

We shall need

Lemma C ([4], [7]). *Let $1<p<\infty$, $\rho\in A_p(G)$, $0\le c_1<c_2\le c_3<c_4<\infty$. Then there exists a positive constant c such that the inequality*

$$\int\limits_{\{x\in G: c_3t<r(x)<c_4t\}}\rho(x)\,dx \le c\int\limits_{\{x\in G: c_1t<r(x)<c_2t\}}\rho(x)\,dx$$

holds for any t, $t>0$.

1. Two-Weighted Inequalities for the Hardy Operator in the Lorentz Spaces Defined on Homogeneous Groups

In this paragraph, the necessary and sufficient conditions are found for the boundedness of the operators

$$Hf(x) = a(x) \int\limits_{r(y)\le r(x)} b(y)f(y)w(y)\,dy,$$

$$H^*f(x) = b(x) \int\limits_{r(y)\ge r(x)} a(y)f(y)w(y)\,dy$$

from $L_w^{rs}(G)$ into $L_v^{pq}(G)$, where a, b, w and v are measurable non-negative functions on G.

Theorem 1.1. *Let $r = s = 1$ or $r = s = \infty$ or $r \in (1,\infty)$ and $s \in (1,\infty)$, $p = q = 1$ or $p = q = \infty$ or $p \in (1,\infty)$ and $q \in (1,\infty)$;* $\max(r,s) \le \min(p,q)$. *For the inequality*

$$\|Hf(\cdot)\|_{L_v^{pq}(G)} \le c\|f(\cdot)\|_{L_w^{rs}(G)}, \tag{1.1}$$

where the constant c does not depend on f, to be valid it is necessary and sufficient that the condition

$$\sup_{t>0} \big\|a(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L_v^{pq}(G)} \, \big\|b(\cdot)\chi_{\{r(y)<t\}}(\cdot)\big\|_{L_w^{r's'}(G)} < \infty \tag{1.2}$$

($r' = \frac{r}{r-1}$, $s' = \frac{s}{s-1}$) be fulfilled.

Proof. Sufficiency. Assume that condition (1.2) is fulfilled. We take $f \ge 0$. If $\int\limits_G f(x)b(x)w(x)\,dx < \infty$, then it belongs to the interval $(2^m, 2^{m+1}]$ for some integer m. By virtue of Theorem A we can choose a sequence $\{x_k\}_{k=-\infty}^m$ such that

$$2^k = \int\limits_{r(y)\le x_k} b(y)f(y)w(y)\,dy = \int\limits_{x_k<r(y)\le x_{k+1}} b(y)f(y)w(y)\,dy \quad \text{for} \quad k \le m-1 \tag{1.3}$$

and

$$2^m = \int\limits_{r(y)\le x_m} b(y)f(y)w(y)\,dy. \tag{1.4}$$

Let $G_k = \{y : x_k < r(y) \le x_{k+1}\}$, $k \le m$ and $x_{m+1} = \infty$. Then the sets G_k do not intersect pairwise and, since we can assume that $\lim\limits_{k\to-\infty} x_k = 0$, we obtain

$$\bigcup_{k\le m} G_k = G \setminus \{e\}. \tag{1.5}$$

If $\int\limits_G b(y)f(y)w(y)\,dy=\infty$, then we choose a sequence $\{x_k\}_{k=-\infty}^{+\infty}$ such that (1.3) is fulfilled for any integer k (in this case $m=+\infty$). By (1.3) and (1.4) we have

$$Hf(x)\le a(x)\cdot 2^{k+1}\quad\text{for}\quad x\in G_k,\quad k\le m. \tag{1.6}$$

Choose a number σ such that $\max(r,s)\le\sigma\le\min(p,q)$. Using (1.5), Lemma B, (1.6), Hölder's inequality, we obtain

$$\begin{aligned}
\|Hf(\cdot)\|^{\sigma}_{L^{pq}_v(G)}&=\Big\|\sum_{k\le m}(Hf)(\cdot)\chi_{G_k}(\cdot)\Big\|^{\sigma}_{L^{pq}_v(G)}\le\\
&\le\sum_{k\le m}\|(Hf)(\cdot)\chi_{G_k}(\cdot)\|^{\sigma}_{L^{pq}_v(G)}\le\\
\le\sum_{k\le m}2^{\sigma(k+1)}\|a(\cdot)\chi_{G_k}(\cdot)\|^{\sigma}_{L^{pq}_v(G)}&=2^{2\sigma}\sum_{k\le m}2^{\sigma(k-1)}\|a(\cdot)\chi_{G_k}(\cdot)\|^{\sigma}_{L^{pq}_v(G)}\le\\
\le 4^{\sigma}\sum_{k\le m}\Bigg(\int\limits_{x_{k-1}<r(y)\le x_k}b(y)f(y)w(y)\,dy\Bigg)^{\sigma}&\|a(\cdot)\chi_{\{r(y)>x_k\}}(\cdot)\|^{\sigma}_{L^{pq}_v(G)}\le\\
\le 4^{\sigma}\sum_{k\le m}\|f(\cdot)\chi_{G_{k-1}}(\cdot)\|^{\sigma}_{L^{rs}_w(G)}&\|b(\cdot)\chi_{\{r(y)<x_k\}}(\cdot)\|^{\sigma}_{L^{r's'}_w(G)}\times\\
\times\|a(\cdot)\chi_{\{r(y)>x_k\}}(\cdot)\|^{\sigma}_{L^{pq}_v(G)}&\le c\|f(\cdot)\|^{\sigma}_{L^{rs}_w(G)}
\end{aligned}$$

where the constant c does not depend on f.

Necessity. Let inequality (1.1) be fulfilled. If $\|f\|_{L^{rs}_w(G)}\le 1$ and $t\in(0,\infty)$, then we have

$$\begin{aligned}
c\ge c\|f\|_{L^{rs}_w(G)}\ge\|Hf\|_{L^{pq}_v(G)}\ge\|(Hf)(\cdot)\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_v(G)}\ge\\
\ge\int\limits_{r(y)<t}b(y)f(y)w(y)\,dy\,\|a(\cdot)\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_v(G)}.
\end{aligned}$$

Taking the least upper bound with respect to all such f and t, we obtain condition (1.2). □

If we apply the dual arguments (see also [8]), then we easily obtain

Theorem 1.2. *Let the numbers r,s,p,q satisfy the conditions of Theorem 1.1. For the inequality*

$$\|H^*f(\cdot)\|_{L^{pq}_v(G)}\le c\|f(\cdot)\|_{L^{rs}_w(G)},$$

where the constant c does not depend on f, to be valid it is necessary and sufficient that the condition

$$\sup_{t>0}\big\|b(\cdot)\chi_{\{r(y)<t\}}(\cdot)\big\|_{L^{pq}_v(G)}\ \big\|a(\cdot)\chi_{\{r(y)>t\}}(\cdot)\big\|_{L^{r's'}_w(G)}<\infty$$

be fulfilled.

Analogous results for the operator $Hf(x) = a(x)\int\limits_0^x b(t)f(t)w(t)\,dt$ are given in [16]. A certain analog in $\mathbb{R}^n$ was obtained in [8].

Corollary 1.1. *Let* $1 < r,s,p,q < \infty$, $\max(r,s) \le \min(p,q)$, $\gamma < -\frac{Q}{p}$, $\beta = \gamma r + Qr(\frac{1}{r'} + \frac{1}{p})$. *Then the inequality*

$$\Big\| r(\cdot)^\gamma \int\limits_{r(y)\le r(\cdot)} f(y)\,dy \Big\|_{L^{pq}(G)} \le c\|f(\cdot)\|_{L^{rs}_{r(\cdot)^\beta}(G)}$$

holds, where the constant c *does not depend on* f.

2. Weighted Inequalities for Singular Integrals in the Lorentz Spaces Defined on Homogeneous Groups

In this paragraph, the sufficient conditions are found for pairs of weights, which provide the boundedness of a singular integral operator in the weighted Lorentz spaces defined on homogeneous groups.

First we investigate the existence of $Tf(x)$.

Lemma 1. *Let* $1 < s \le p < \infty$, $\rho \in A_p(G)$. *If the weight functions* w *and* w_1 *satisfy the conditions:*

(1) *there exists a positive increasing on* $(0,\infty)$ *function* σ *such that for almost all* $x \in G$ *we have the inequality*

$$\sigma\big(r(x)\big)\rho(x) \le bw(x)w_1^p(x),$$

where the positive constant b *does not depend on* x;

(2) $\Big\|\dfrac{1}{w(\cdot)w_1(\cdot)}\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{p's'}_w(G)} < \infty$ *for any* $t > 0$,

then for arbitrary φ *with the condition* $\|\varphi(\cdot)w_1(\cdot)\|_{L^{ps}_w(G)} < \infty$, $T\varphi(x)$ *exists almost everywhere on* G.

Proof. Fix the number α, $\alpha > 0$, and let

$$S_\alpha = \Big\{x \in G : r(x) > \frac{\alpha}{2}\Big\}.$$

Take a function φ with the condition $\|\varphi(\cdot)w_1(\cdot)\|_{L^{ps}_w(G)} < \infty$. We write ϕ as

$$\varphi(x) = \varphi_1(x) + \varphi_2(x),$$

where $\varphi_1(x) = \varphi(x)\cdot\chi_{S_\alpha}(x)$, $\varphi_2(x) = \varphi(x) - \varphi_1(x)$.

For φ_1 we obtain

$$\int_G |\varphi_1(x)|^p \rho(x)\,dx = \frac{\sigma(\frac{\alpha}{2})}{\sigma(\frac{\alpha}{2})} \int_{S_\alpha} |\varphi(x)|^p \rho(x)\,dx \leq$$

$$\leq \frac{1}{\sigma(\frac{\alpha}{2})} \int_{S_\alpha} |\varphi(x)|^p \rho(x) \sigma\big(r(x)\big)\,dx \leq \frac{b}{\sigma(\frac{\alpha}{2})} \int_{S_\alpha} |\varphi(x)|^p w_1^p(x) w(x)\,dx \leq$$

$$\leq \frac{b}{\sigma(\frac{\alpha}{2})} \|\varphi(\cdot) w_1(\cdot)\|^p_{L^p_w(G)} \leq \frac{b}{\sigma(\frac{\alpha}{2})} \|\varphi(\cdot) w_1(\cdot)\|^p_{L^{ps}_w(G)}.$$

We find by Theorem B that $T\varphi_1 \in L^p_\rho(G)$ and $T\varphi_1(x)$ exists almost everywhere on G.

Now we shall show that $T\varphi_2(x)$ converges absolutely for $r(x) > \alpha c_0$. Note that when $r(x) > \alpha c_0$ and $r(y) < \frac{\alpha}{2}$, we have $r(x) \leq c_0(r(xy^{-1}) + r(y)) \leq c_0(r(xy^{-1}) + \frac{\alpha}{2}) \leq c_0(r(xy^{-1}) + \frac{r(x)}{2c_0})$ and $\frac{\alpha}{2} < \frac{r(x)}{2c_0} \leq r(xy^{-1})$.

We obtain

$$|T\varphi_2(x)| \leq c_1 \int_{\{y: r(y) < \frac{\alpha}{2}\}} \frac{\varphi(y)}{r(xy^{-1})^Q}\,dy \leq \frac{c_2}{\alpha^Q} \int_{\{y: r(y) < \frac{\alpha}{2}\}} |\varphi(y)|\,dy =$$

$$= \frac{c_2}{\alpha^Q} \int_{\{y: r(y) < \frac{\alpha}{2}\}} \varphi(y) \frac{1}{w_1(y) w(y)} w_1(y) w(y) dy \leq$$

$$\leq \frac{c_2}{\alpha^Q} \|\varphi(\cdot) w_1(\cdot)\|_{L^{ps}_w(G)} \Big\| \chi_{\{y: r(y) < \frac{\alpha}{2}\}}(\cdot) \frac{1}{w_1(\cdot) w(\cdot)} \Big\|_{L^{p's'}_w(G)} < \infty.$$

Since we can take α arbitrarily small, $T\varphi(x)$ exists almost everywhere on G. $\square$

The next lemma is proved in a similar manner.

Lemma 2. *Let $1 < p, s < \infty$, $s \leq p$, $\rho \in A_p(G)$. If the weight functions w and w_1 satisfy the conditions:*

(1) *there exists a positive decreasing on $(0, \infty)$ function σ such that for almost all $x \in G$ we have the inequality*

$$\sigma\big(r(x)\big) \rho(x) \leq b w(x) w_1^p(x),$$

where the positive constant b does not depend on x;

(2) $\quad \Big\| \dfrac{r(\cdot)^{-Q}}{w(\cdot) w_1(\cdot)} \chi_{\{r(y) > t\}}(\cdot) \Big\|_{L^{p's'}_w(G)} < \infty$ *for any $t > 0$,*

then for arbitrary φ with the condition $\|\varphi(\cdot) w_1(\cdot)\|_{L^{ps}_w(G)} < \infty$, $T\varphi(x)$ exists almost everywhere on G.

Lemma 1 readily implies

Lemma 3. *Let $1 < p, s, < \infty$, $s \le p$. If u and u_1 are positive increasing functions on $(0, \infty)$ and*

$$\Big\| \frac{1}{u_1(r(\cdot))u(r(\cdot))} \chi_{\{r(y)<t\}}(\cdot) \Big\|_{L^{p's'}_{u(r(\cdot))}(G)} < \infty$$

for arbitrary $t > 0$, then for any φ with the condition $\|\varphi(\cdot)u_1(r(\cdot))\|_{L^{ps}_{u(r(\cdot))}(G)} < \infty$, $T\varphi(x)$ exists almost everywhere on G.

Analogously, Lemma 2 implies

Lemma 4. *Let $1 < p, s < \infty$, $s \le p$. If u and u_1 are positive decreasing functions on $(0, \infty)$ and*

$$\Big\| \frac{r(\cdot)^{-Q}}{u(r(\cdot))u_1(r(\cdot))} \chi_{\{r(y)>t\}}(\cdot) \Big\|_{L^{p's'}_{u(r(\cdot))}(G)} < \infty$$

for arbitrary $t > 0$, then for φ with the condition $\varphi(\cdot)u_1(r(\cdot)) \in L^{ps}_{u(r(\cdot))}(G) < \infty$, $T\varphi(x)$ exists almost everywhere on G.

Now we shall formulate and prove our basic theorems.

Theorem 2.1. *Let $1 < s \le p \le q < \infty$, σ be a positive increasing function on $(0, \infty)$, the function $\rho \in A_p(G)$, w a weight function on G, $v(x) = \sigma(r(x))\rho(x)$. Let the following conditions be fullfilled:*

(1) *there exists a positive constant b such that almost for all $x \in G$*

$$\sigma\big(2c_0 r(x)\big)\rho(x) \le bw(x);$$

(2) $$\|r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_v(G)} \Big\| \frac{1}{w(\cdot)} \chi_{\{r(y)<t\}}(\cdot) \Big\|_{L^{p's'}_w(G)} \le c < \infty.$$

Then there exists a positive constant c such that the inequality

$$\|Tf(\cdot)\|_{L^{pq}_v(G)} \le c\|f(\cdot)\|_{L^{ps}_w(G)} \tag{2.1}$$

holds for any $f \in L^{ps}_w(G)$.

Proof. Without loss of generality we can write the function σ as

$$\sigma(t) = \sigma(0) + \int_0^t \varphi(\tau)\,d\tau,$$

where $\sigma(0)=\lim\limits_{t\to 0}\sigma(t)$ and $\varphi(\tau)\geq 0$ on $(0,\infty)$. We have

$$\|Tf(\cdot)\|_{L^{pq}_v(G)}\leq c_1\bigg(q\int\limits_0^\infty\lambda^{q-1}\bigg(\int\limits_{\{x:|Tf(x)|>\lambda\}}\rho(x)\sigma(0)\,dx\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{1}{q}}+$$
$$+c_1\bigg(q\int\limits_0^\infty\lambda^{q-1}\bigg(\int\limits_{\{x:|Tf(x)|>\lambda\}}\rho(x)\bigg(\int\limits_0^{r(x)}\varphi(t)\,dt\bigg)dx\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{1}{q}}=$$
$$=I_1+I_2.$$

If $\sigma(0)=0$, then $I_1=0$, and if $\sigma(0)\neq 0$, then by Theorem C and Lemma A we obtain

$$I_1=c_1\sigma^{\frac{1}{p}}(0)\|Tf(\cdot)\|_{L^{pq}_\rho(G)}\leq c_2\sigma^{\frac{1}{p}}(0)\|f(\cdot)\|_{L^{pq}_\rho(G)}\leq$$
$$\leq c_2\sigma^{\frac{1}{p}}(0)\|f(\cdot)\|_{L^{ps}_\rho(G)}\leq c_3\|f(\cdot)\|_{L^{ps}_w(G)}.$$

Now we shall estimate I_2. Let $f_{1t}(x)=f(x)\cdot\chi_{\{r(x)>\frac{t}{2c_0}\}}(x)$, $f_{2t}(x)=f(x)-f_{1t}(x)$. We have

$$I_2=c_1\bigg(q\int\limits_0^\infty\lambda^{q-1}\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{\{x:r(x)>t,\,|Tf(x)|>\lambda\}}\rho(x)\,dx\bigg)dt\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{1}{q}}\leq$$
$$\leq c_4\bigg(q\int\limits_0^\infty\lambda^{q-1}\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{\{x:r(x)>t\}}\chi\Big\{x:|Tf_{1t}(x)|>\frac{\lambda}{2}\Big\}\rho(x)dx\bigg)dt\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{1}{q}}+$$
$$+c_4\bigg(q\int\limits_0^\infty\lambda^{q-1}\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_{\{x:r(x)>t\}}\chi\Big\{x:|Tf_{2t}(x)|>\frac{\lambda}{2}\Big\}\rho(x)dx\bigg)dt\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{1}{q}}=$$
$$=I_{21}+I_{22}.$$

Applying Minkowski's inequality twice ($\frac{q}{p}\geq 1$, $\frac{p}{s}\geq 1$) and Theorem C, we obtain

$$I_{21}\leq c_5\bigg(\int\limits_0^\infty\varphi(t)\bigg(\int\limits_0^\infty\lambda^{q-1}\bigg(\int\limits_{\{x:|Tf_{1t}(x)|>\lambda\}}\rho(x)\,dx\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{p}{q}}dt\bigg)^{\frac{1}{p}}\leq$$
$$\leq c_6\bigg(\int\limits_0^\infty\varphi(t)\|f_{1t}(\cdot)\|^p_{L^{pq}_\rho(G)}dt\bigg)^{\frac{1}{p}}\leq c_6\bigg(\int\limits_0^\infty\varphi(t)\|f_{1t}(\cdot)\|^p_{L^{ps}_\rho(G)}dt\bigg)^{\frac{1}{p}}\leq$$

$$\leq c_6\bigg(\int\limits_0^\infty \lambda^{s-1}\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{\{x:|f(x)|>\lambda\}} \rho(x)\chi_{\{r(y)>\frac{t}{2c_0}\}}(x)\,dx\bigg)dt\bigg)^{\frac{s}{p}}d\lambda\bigg)^{\frac{1}{s}} =$$

$$= c_6\bigg(\int\limits_0^\infty \lambda^{s-1}\bigg(\int\limits_{\{x:|f(x)|>\lambda\}} \rho(x)\bigg(\int\limits_0^{2c_0r(x)} \varphi(t)\,dt\bigg)dx\bigg)^{\frac{s}{p}}d\lambda\bigg)^{\frac{1}{s}} \leq c_7\|f\|_{L^{ps}_w(G)}.$$

Next, we shall estimate I_{22}. Note that if $r(x) > t$ and $r(y) < \frac{t}{2c_0}$, then $r(x) \leq 2c_0 r(xy^{-1})$. By Theorem 1.1 we obtain

$$I_{22} = c_4\bigg(q\int\limits_0^\infty \lambda^{q-1}\bigg(\int\limits_0^\infty \varphi(t)\bigg(\int\limits_{\{x:\frac{(2c_0)^Q}{r(x)^Q}\int\limits_{\{y:r(y)<r(x)\}} f(y)dy>\frac{\lambda}{2}\}} \rho(x)\chi_{\{r(y)>t\}}(x)\,dx\bigg)dt\bigg)^{\frac{q}{p}}d\lambda\bigg)^{\frac{1}{q}} \leq$$

$$= c_8\bigg\|\frac{1}{r(\cdot)^Q}\int\limits_{\{r(y)<r(\cdot)\}} |f(y)|\,dy\bigg\|_{L^{pq}_v(G)} \leq c_9\|f(\cdot)\|_{L^{ps}_w(G)}. \quad \square$$

If we write the function σ as

$$\sigma(t) = \sigma(+\infty) + \int\limits_t^\infty \psi(\tau)d\tau, \quad \text{where } \sigma(+\infty) = \lim_{t\to+\infty}\sigma(t), \ \ \psi(\tau)\geq 0,$$

on $(0,\infty)$, again apply Theorem C, Lemma A and Theorem 1.2, then we shall have

Theorem 2.2. *Let $1 < s \leq p \leq q < \infty$, σ be a positive decreasing function on $(0,\infty)$, w a weight function on G, $\rho \in A_p(G)$, $v(x) = \sigma(r(x))\rho(x)$. Let the following conditions be fulfilled:*

(1) *there exists a positive constant b such that the inequality*

$$\rho(x)\sigma\Big(\frac{r(x)}{2c_0}\Big) \leq bw(x)$$

holds for almost all $x \in G$;

(2) $$\sup_{t>0}\|\chi_{\{r(y)<t\}}(\cdot)\|_{L^{pq}_v(G)}\bigg\|\frac{r(\cdot)^{-Q}}{w(\cdot)}\chi_{\{r(y)>t\}}(\cdot)\bigg\|_{L^{p's'}_w(G)} < \infty.$$

Then inequality (2.1) *is valid.*

It is proved in [8] that if inequality (2.1) is fulfilled for the Hilbert transform, then the inequality $v(x) \leq b_1 w(x)$ holds almost everywhere on $\mathbb{R}^1$.

Corollary 2.1. *Let $1 < s \le p \le q < \infty$, σ_1 and σ_2 be positive increasing functions on $(0,\infty)$, $\rho \in A_p(G)$, $v(x) = \sigma_2(r(x))\rho(x)$, $w(x) = \sigma_1(r(x))\rho(x)$. If the conditions*

(1) *there exists a positive constant b such that the inequality*

$$\sigma_2(2c_0 t) \le b\sigma_1(t)$$

holds for any $t > 0$;

(2) $$\sup_{t>0} \|r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_v(G)} \Big\| \frac{1}{w(\cdot)}\,\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{p's'}_w(G)} < \infty$$

are fulfilled, then inequality (2.1) is valid.

Corollary 2.2. *Let $1 < s \le p \le q < \infty$, σ_1 and σ_2 be positive decreasing functions on $(0,\infty)$, $\rho \in A_p(G)$, $v(x) = \sigma_2(r(x))\rho(x)$, $w(x) = \sigma_1(r(x))\rho(x)$. If the conditions*

(1) $\sigma_2\Big(\dfrac{t}{2c_0}\Big) \le b\sigma_1(t)$ *for any $t > 0$;*

(2) $$\sup_{t>0} \|\chi_{\{r(y)<t\}}(\cdot)\|_{L^{pq}_v(G)} \Big\| \frac{r(\cdot)^{-Q}}{w(\cdot)}\,\chi_{\{r(y)>t\}}(\cdot)\Big\|_{L^{p's'}_w(G)} < \infty$$

are fulfilled, then inequality (2.1) is valid.

Theorem 2.3. *Let $1 < p \le q < \infty$, σ_1 and σ_2 be positive increasing functions on $(0,\infty)$, $\rho \in A_p(G)$, $v(x) = \sigma_2(r(x))\rho(x)$, $w(x) = \sigma_1(r(x))\rho(x)$. If v and w satisfy the condition*

$$\sup_{t>0} \|r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_v(G)} \Big\| \frac{1}{w(\cdot)}\,\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{p'}_w(G)} < \infty$$

then the inequality

$$\|Tf(\cdot)\|_{L^{pq}_v(G)} \le c\|f(\cdot)\|_{L^p_w(G)}$$

holds, where the positive constant c does not depend on f.

Proof. By Corollary 2.1. it is sufficient to show that the inequality

$$\sigma_2(2c_0 t) \le b\sigma_1(t)$$

holds for any $t > 0$.

Using Hölder's inequality and Lemma C ($\rho^{1-p'} \in A_{p'}(G)$), we obtain

$$\frac{\sigma_2(2c_0 t)}{\sigma_1(t)} \le$$

$$\le c_1\,\frac{\sigma_2(2c_0 t)}{\sigma_1(t)}\,t^{-Qp}\Bigg(\int\limits_{\{x:2c_0t<r(x)<4c_0t\}} \rho(x)\,dx\Bigg)\Bigg(\int\limits_{\{x:2c_0t<r(x)<4c_0t\}} \rho^{1-p'}(x)\,dx\Bigg)^{p-1} \le$$

$$\leq c_2\,\frac{\sigma_2(2c_0t)}{\sigma_1(t)}\,t^{-Qp}\Bigg(\int\limits_{\{x:2c_0t<r(x)<4c_0t\}}\rho(x)\,dx\Bigg)\Bigg(\int\limits_{\{x:r(x)<t\}}\rho^{1-p'}(x)\,dx\Bigg)^{p-1}\leq$$

$$\leq c_2t^{-Qp}\Bigg(\int\limits_{\{x:2c_0t<r(x)<4c_0t\}}v(x)\,dx\Bigg)\Bigg(\int\limits_{\{x:r(x)<t\}}w^{1-p'}(x)\,dx\Bigg)^{p-1}=$$

$$=c_2t^{-Qp}\|\chi_{\{x:2c_0t<r(x)<4c_0t\}}(\cdot)\|^p_{L^{pq}_v(G)}\Big\|\frac{1}{w(\cdot)}\,\chi_{\{r(y)<t\}}(\cdot)\Big\|^p_{L^{p'}_w(G)}\leq$$

$$\leq c_3\|r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\|^p_{L^{pq}_v(G)}\Big\|\frac{1}{w(\cdot)}\,\chi_{\{r(y)<t\}}(\cdot)\Big\|^p_{L^{p'}_w(G)}<\infty.\quad\square$$

Theorem 2.4. *Let $1<s\leq p\leq q<\infty$; σ_1, σ_2, u_1 and u_2 be weight functions on G, $\rho\in A_p(G)$, $v=\sigma_2\rho$, $w=\sigma_1\rho$. Let the following three conditions be fulfilled:*

(1) *there exists a positive constant b such that for any $t>0$ we have the inequality*

$$\sup_{F_t}\sigma_2^{\frac1p}(x)\sup_{F_t}u_2(x)\leq b\inf_{F_t}\sigma_1^{\frac1p}(x)\inf_{F_t}u_2(x),$$

where $F_t=\{x\in G:\frac{t}{c_0}<r(x)<8c_0t\}$;

(2) $\displaystyle\sup_{t>0}\|u_2(\cdot)r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_v(G)}\Big\|\frac{1}{u_1(\cdot)w(\cdot)}\chi_{\{r(y)<t\}}(\cdot)\Big\|_{L^{p's'}_w(G)}<\infty;$

(3) $\displaystyle\sup_{t>0}\|u_2(\cdot)\chi_{\{r(y)<t\}}(\cdot)\|_{L^{pq}_v(G)}\Big\|\frac{1}{u_1(\cdot)w(\cdot)}r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\Big\|_{L^{p's'}_w(G)}<\infty.$

Then the inequality

$$\|u_2(\cdot)Tf(\cdot)\|_{L^{pq}_v(G)}\leq c\|u_1(\cdot)f(\cdot)\|_{L^{ps}_w(G)}\tag{2.2}$$

is valid, where the positive constant c does not depend on f.

Proof. Let $E_k=\{x\in G:2^k<r(x)\leq 2^{k+1}\}$, $G_{k1}=\{x\in G:r(x)\leq\frac{2^{k-1}}{c_0}\}$, $G_{k2}=\{x\in G:\frac{2^{k-1}}{c_0}<r(x)\leq c_02^{k+2}\}$, $G_{k3}=\{x\in G:r(x)>c_02^{k+2}\}$. We estimate the left-hand side of inequality (2.2) as follows:

$$\|u_2(\cdot)Tf(\cdot)\|^p_{L^{pq}_v(G)}=\Big\|\sum_{k\in\mathbb{Z}}Tf(\cdot)u_2(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L^{pq}_v(G)}\leq$$

$$\leq c_1\Big\|\sum_{k\in\mathbb{Z}}u_2(\cdot)T(f\cdot\chi_{G_{k1}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L^{pq}_v(G)}+$$

$$+c_1\Big\|\sum_{k\in\mathbb{Z}}u_2(\cdot)T(f\cdot\chi_{G_{k2}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L^{pq}_v(G)}+$$

$$+c_1\Big\|\sum_{k\in\mathbb{Z}}u_2(\cdot)T(f\cdot\chi_{G_{k3}})(\cdot)\chi_{E_k}(\cdot)\Big\|^p_{L^{pq}_v(G)}=c_1(S_1^p+S_2^p+S_3^p).$$

Next, we shall estimate S_1^p. Note that for $x\in E_k$ and $y\in G_{k1}$ we have $r(y)\le\frac{2^{k-1}}{c_0}=\frac{2^k}{2c_0}\le\frac{r(x)}{2c_0}<r(x)$. Moreover, $r(xy^{-1})\ge\frac{r(x)}{2c_0}$ and we obtain

$$|T(f\cdot\chi_{G_{k1}})(x)|=\Big|\int\limits_G k(xy^{-1})f(y)\chi_{G_{k1}}(y)\,dy\Big|\le$$
$$\le c_2\int\limits_G\frac{|f(y)|\chi_{G_{k1}}(y)}{r(xy^{-1})^Q}\,dy\le c_3\frac{1}{r(x)^Q}\int\limits_{r(y)<r(x)}|f(y)|\,dy$$

for any $x\in E_k$. By Theorem 1.1 we find that

$$S_1^p\le c_3^p\Big\|u_2(\cdot)r(\cdot)^{-Q}\Big(\int\limits_{\{r(y)\le r(x)\}}f(y)\,dy\Big)\Big\|^p_{L^{pq}_v(G)}\le c_4\|u_1(\cdot)f(\cdot)\|^p_{L^{ps}_w(G)}.$$

Let us estimate S_3^p. As is easy to verify, for $x\in E_k$ and $y\in G_{k3}$ we have $r(y)\ge r(x)$ and $r(xy^{-1})\ge\frac{r(y)}{2c_0}$. For $x\in E_k$ we obtain

$$|T(f\cdot\chi_{G_{k3}})(x)|\le c_5\int\limits_{\{r(y)\ge r(x)\}}\frac{|f(y)|}{r(y)^Q}\,dy.$$

By Theorem 1.2 we find that

$$S_3^p\le c_5^p\Big\|u_2(\cdot)\int\limits_{\{r(y)\ge r(x)\}}\frac{f(y)}{r(y)^Q}\,dy\Big\|^p_{L^{pq}_v(G)}\le c_6\|u_1(\cdot)f(\cdot)\|^p_{L^{ps}_w(G)}.$$

Now we shall estimate S_2^p. By Lemma B (second part) we obtain

$$S_2^p\le\sum_{k\in\mathbb{Z}}\|u_2(\cdot)T(f\cdot\chi_{G_{k2}})(\cdot)\chi_{E_k}(\cdot)\|^p_{L^{pq}_v(G)}=\sum_k S_{k2}^p.$$

Introducing the notation $u_{2k}=\sup\limits_{x\in E_k}u_2(x)$, $\sigma_{2k}=\sup\limits_{x\in E_k}\sigma_2(x)$, $u_{1k}=\inf\limits_{x\in E_k}u_1(x)$, $\sigma_{1k}=\inf\limits_{x\in E_k}\sigma_1(x)$, by Lemma A and Theorem C we obtain

$$S_{k2}\le u_{2k}\sigma_{2k}^{\frac1p}\|T(f\cdot\chi_{G_{k2}})(\cdot)\|_{L^{pq}_\rho(G)}\le c_7u_{2k}\sigma_{2k}^{\frac1p}\|f(\cdot)\chi_{G_{k2}}(\cdot)\|_{L^{pq}_\rho(G)}\le$$
$$\le c_7u_{2k}\sigma_{2k}^{\frac1p}\|f(\cdot)\chi_{G_{k2}}(\cdot)\|_{L^{ps}_\rho(G)}\le c_8u_{1k}\sigma_{1k}^{\frac1p}\|f(\cdot)\chi_{G_{k2}}(\cdot)\|_{L^{ps}_\rho(G)}\le$$
$$\le c_9\Big[\sum_j 2^{js}\Big(\int\limits_{G_{k2}\cap\{|f|>2^j\}}u_{1k}^p\sigma_{1k}\rho(x)\,dx\Big)^{\frac sp}\Big]^{\frac1s}\le$$

$$\leq c_9 \Bigg[\sum_j (u_{1k} 2^j)^s \Bigg(\int\limits_{G_{k2} \cap \{u_1 f > u_{1k} 2^j\}} w(x)\,dx \Bigg)^{\frac{s}{p}} \Bigg]^{\frac{1}{s}} \leq c_{10} \|u_1(\cdot) f(\cdot) \chi_{G_{k2}}(\cdot)\|_{L^{ps}_w(G)}.$$

By Lemma B (first part) we have

$$S_2^p \leq c_{11} \|u_1(\cdot) f(\cdot)\|^p_{L^{ps}_w(G)}. \quad \square$$

In [8] (see also [1], [4]) it is shown that for the Hilbert transform conditions (2) and (3) of Theorem 2.4 are necessary for inequality (2.2) to be fulfilled.

Remark 1. One can easily verify that Theorem 2.4 remains in force when condition (1) is replaced by the condition

$$(1') \qquad \sup_{\frac{r(x)}{4c_0} < r(y) \leq 4c_0 r(x)} \big(\sigma_2(y) u_2^p(y)\big) \leq b_1 \sigma_1(x) u_1^p(x) \text{ for a.a. } x \in G.$$

Now let us consider the case with radial weight functions.

Corollary 2.3. *Let $1 < s \leq p \leq q < \infty$; u_1, u_2 and v be positive increasing functions on $(0,\infty)$ which satisfy the condition*

$$\sup_{t>0} \|u_2(\cdot) r(\cdot)^{-Q} \chi_{\{r(y)>t\}}(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \Big\| \frac{1}{u_1(r(\cdot))}\, \chi_{\{r(y)<t\}}(\cdot) \Big\|_{L^{p's'}(G)} = \\ = \sup_{t>0} B(t) < \infty.$$

Then the inequality

$$\|Tf(\cdot) u_2(r(\cdot))\|_{L^{pq}_{v(r(\cdot))}(G)} \leq c \|f(\cdot) u_1(r(\cdot))\|_{L^{ps}(G)} \tag{2.3}$$

holds, where the positive constant c does not depend on f.

Proof. First we shall show that the inequality

$$u_2(8c_0 t) v^{\frac{1}{p}}(8c_0 t) \leq b u_1\Big(\frac{t}{c_0}\Big)$$

holds, where the positive constant b does not depend on t. Indeed, using Lemma A and Theorem A, we obtain

$$B(t) \geq \|u_2(r(\cdot)) r(\cdot)^{-Q} \chi_{\{t<r(y)<2t\}}(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \times \\ \times \Big\| \frac{1}{u_1(r(\cdot))}\, \chi_{\{r(y)<\frac{t}{8c_0^2}\}} \Big\|_{L^{p's'}(G)} \geq c_1 t^{-Q} u_2(t) \Bigg(\int\limits_t^{2t} v(\tau) \tau^{Q-1} d\tau \Bigg)^{\frac{1}{p}} \times \\ \times \frac{1}{u_1(\frac{t}{8c_0^2})} \Big| \Big\{ y : r(y) < \frac{t}{8c_0^2} \Big\} \Big|^{\frac{1}{p'}} \geq c_2 u_2(t) v^{\frac{1}{p}}(t)\, \frac{1}{u_1(\frac{t}{8c_0^2})}.$$

Moreover,

$$B_1(t) = \|u_2(r(\cdot)) \chi_{\{r(y)<t\}}(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)} \times$$

$$\times\Big\|\frac{1}{u_1(r(\cdot))}\,r(\cdot)^{-Q}\chi_{\{r(y)>t\}}\Big\|_{L^{p's'}(G)}\le$$
$$\le c_3u_2(t)v^{\frac{1}{p}}(t)t^{\frac{Q}{p}}\,\frac{1}{u_1(\frac{t}{8c_0^2})}\,\|r(\cdot)^{-Q}\chi_{\{r(y)>\frac{t}{8c_0^2}\}}(\cdot)\|_{L^{p's'}(G)}.$$

It is easy to verify that

$$\|r(\cdot)^{-Q}\chi_{\{r(y)>\frac{t}{8c_0^2}\}}(\cdot)\|_{L^{p's'}(G)}\le c_4t^{-\frac{Q}{p}},$$

where c_4 does not depend on $t>0$. We have $\sup\limits_{t>0}B_1(t)\le c_5$. Using Theorem 2.4, we obtain inequality (2.3). □

An analogous reasoning is used to prove

Corollary 2.4. *Let $1<s\le p\le q<\infty$; u_1, u_2 and v be positive decreasing functions on $(0,\infty)$ and*

$$\sup_{t>0}\|u_2(r(\cdot))\chi_{\{r(y)<t\}}(\cdot)\|_{L^{pq}_{v(r(\cdot))}(G)}\Big\|\frac{1}{u_1(r(\cdot))}\,r(\cdot)^{-Q}\chi_{\{r(y)>t\}}(\cdot)\Big\|_{L^{p's'}(G)}<\infty.$$

Then inequality (2.3) *holds.*

Now we shall give examples illustrating pairs of weights.

Example 1. Let $1<p<q<\infty$, $v(t)=t^{p-1}$, $w(t)=t^{p-1}\ln^{\gamma}\frac{2\pi}{t}$, where $p-1<\gamma<p$ and $\gamma=\frac{p}{q}+p-1$. Then the pair (v,w) satisfies the condition

$$\sup_{0<t<\pi}\||\cdot|^{-1}\chi_{\{t<|y|<\pi\}}(\cdot)\|_{L^{pq}_{v(|\cdot|)}}\Big\|\frac{1}{w^{\frac{1}{p}}(|\cdot|)}\,\chi_{\{|y|<t\}}(\cdot)\Big\|_{L^{p'}}<\infty$$

and therefore the inequality

$$\|\widetilde{f}(\cdot)\|_{L^{pq}_{v(|\cdot|)}}\le c\|f(\cdot)\|_{L^{p}_{w(|\cdot|)}}$$

holds, where $\widetilde{f}$ is the conjugate function and the constant c does not depend on f.

Example 2. Let $1<s\le p<\infty$, $\frac{p}{s'}\le\gamma<p$, $s=\frac{p}{p+1-\gamma}$ $(\gamma=\frac{p}{s'}+1)$. Then the inequality

$$\|\widetilde{f}(\cdot)|\cdot|^{\frac{p-1}{p}}\|_{L^p}\le c\Big\|f(\cdot)|\cdot|^{\frac{p-1}{p}}\ln^{\frac{\gamma}{p}}\frac{2\pi}{|\cdot|}\Big\|_{L^{ps}}$$

holds, where the positive constant c does not depend on f.

Example 3. Let $1 < s \leq p \leq q < \infty$, $\gamma = p(\frac{1}{q} + 1 - \frac{1}{s})$. Then the inequality

$$\|\widetilde{f}(\cdot)|\cdot|^{\frac{p-1}{p}}\|_{L^{pq}} \leq c\Big\|f(\cdot)|\cdot|^{\frac{p-1}{p}} \ln^{\frac{\gamma}{p}} \frac{2\pi}{|\cdot|}\Big\|_{L^{ps}} \tag{2.4}$$

holds, where the constant $c > 0$ does not depend on f. Inequality (2.4) remains in force for $\gamma > p(\frac{1}{q} + 1 - \frac{1}{s})$ too.

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