

## ON PERIODIC SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

I. KIGURADZE AND B. PUŽA

ABSTRACT. Sufficient conditions are established for the existence and uniqueness of an  $\omega$ -periodic solution of the functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t),$$

where  $f$  is a continuous operator acting from the space of  $n$ -dimensional  $\omega$ -periodic continuous vector functions into the space of  $n$ -dimensional  $\omega$ -periodic and summable on  $[0, \omega]$  vector functions.

### 1. STATEMENT OF THE PROBLEM AND BASIC NOTATION

Let  $n$  be a natural number,  $\omega > 0$ , and  $f : C_\omega(R^n) \rightarrow L_\omega(R^n)$  be a continuous operator. Consider the vector functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t). \tag{1.1}$$

A vector function  $x : R \rightarrow R^n$  is called an  $\omega$ -periodic solution of equation (1.1) if it is absolutely continuous, satisfies (1.1) almost everywhere on  $R$  and

$$x(t + \omega) = x(t) \quad \text{for } t \in R.$$

In the second section of this paper, using the principle of a priori boundedness we establish new sufficient conditions for the existence and uniqueness of an  $\omega$ -periodic solution of equation (1.1). In the third section we give corollaries of the main results for the vector differential equation with deviating arguments

$$\frac{dx(t)}{dt} = f_0(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))), \tag{1.2}$$

---

1991 *Mathematics Subject Classification.* 34C25, 34K15.

*Key words and phrases.* Functional differential equation,  $\omega$ -periodic solution, principle of a priori boundedness.

where  $f_0 : R \times R^{(m+1)n} \rightarrow R^n$  satisfies the local Carathéodory conditions and is  $\omega$ -periodic in the first argument, i.e., satisfies the equality

$$f_0(t + \omega, x_0, x_1, \dots, x_m) = f_0(t, x_0, x_1, \dots, x_m) \quad (1.3)$$

for almost all  $t \in R$  and for all  $x_k \in R^n$  ( $k = 0, 1, \dots, m$ ). As for  $\tau_k : R \rightarrow R$  ( $k = 1, \dots, m$ ), they are measurable and such that

$$(\tau_k(t + \omega) - \tau_k(t))/\omega \quad (k = 1, \dots, m) \text{ are integer numbers.} \quad (1.4)$$

The above-mentioned propositions strengthen the earlier results on periodic solutions of systems of ordinary differential equations and functional differential equations of types (1.1) and (1.2) (see [1–23] and the references cited therein).

Throughout this paper, use will be made of the following notation:

$R^n$  is the space of all  $n$ -dimensional column vectors  $x = (x_i)_{i=1}^n$  with the elements  $x_i \in R$  ( $i = 1, \dots, n$ ) and the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

$R^{n \times n}$  is the space of all  $n \times n$ -matrices  $X = (x_{ik})_{i,k=1}^n$  with the elements  $x_{ik} \in R$  ( $i, k = 1, \dots, n$ ) and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|.$$

$$R_+^n = \{(x_i)_{i=1}^n \in R^n : x_i \geq 0 \ (i = 1, \dots, n)\}.$$

$$R_+^{n \times n} = \{(x_{ik})_{i,k=1}^n \in R^{n \times n} : x_{ik} \geq 0 \ (i, k = 1, \dots, n)\}.$$

If  $x, y \in R^n$  and  $X, Y \in R^{n \times n}$ , then

$$x \leq y \iff y - x \in R_+^n, \quad X \leq Y \iff Y - X \in R_+^{n \times n}.$$

$x \cdot y$  is the scalar product of the vectors  $x$  and  $y \in R^n$ .

If  $x = (x_i)_{i=1}^n \in R^n$  and  $X = (x_{ik})_{i,k=1}^n \in R^{n \times n}$ , then

$$|x| = (|x_i|)_{i=1}^n, \quad |X| = (|x_{ik}|)_{i,k=1}^n, \\ \operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n.$$

$\det(X)$  is the determinant of the matrix  $X$ .

$X^{-1}$  is the matrix inverse to  $X$ .

$r(X)$  is the spectral radius of the matrix  $X$ .  $E$  is the unit matrix.

$C([0, \omega]; R^n)$  is the space of all continuous vector functions  $x : [0, \omega] \rightarrow R^n$  with the norm

$$\|x\|_C = \max \{ \|x(t)\| : 0 \leq t \leq \omega \}.$$

$C_\omega(R^n)$  with  $\omega > 0$  is the space of all continuous  $\omega$ -periodic vector functions  $x : R \rightarrow R^n$  with the norm

$$\|x\|_{C_\omega} = \max \{ \|x(t)\| : 0 \leq t \leq \omega \};$$

if  $x = (x_i)_{i=1}^n \in C_\omega(R^n)$ , then

$$\|x\|_{C_\omega} = (\|x_i\|_{C_\omega})_{i=1}^n.$$

$L([0, \omega]; R^n)$  is the space of all vector functions  $x : R \rightarrow R^n$  with summable on  $[0, \omega]$  elements and with the norm

$$\|x\|_L = \int_0^\omega \|x(t)\| dt.$$

$L_\omega(R^n)$  is the space of all  $\omega$ -periodic vector functions  $x : R \rightarrow R^n$  with summable on  $[0, \omega]$  elements and with the norm

$$\|x\|_{L_\omega} = \int_0^\omega \|x(t)\| dt.$$

$$L_\omega(R_+) = \{x \in L_\omega(R) : x(t) \geq 0 \text{ for } t \in R\};$$

$$L_\omega(R_-) = \{x \in L_\omega(R) : x(t) \leq 0 \text{ for } t \in R\}.$$

$L_\omega(R^{n \times n})$  is the space of all matrix functions  $X : R \rightarrow R^{n \times n}$  with elements from  $L_\omega(R)$ .

If  $Z : R \rightarrow R^{n \times n}$  is an  $\omega$ -periodic continuous matrix function with columns  $z_1, \dots, z_n$ , and  $g : C_\omega(R^n) \rightarrow L_\omega(R^n)$  is a linear operator, then by  $g(Z)$  we understand the matrix function with columns  $g(z_1), \dots, g(z_n)$ .

## 2. PERIODIC SOLUTIONS OF EQUATION (1.1)

Throughout this section,  $f : C_\omega(R^n) \rightarrow L_\omega(R^n)$  is assumed to be a continuous operator such that

$$f^*(\cdot, \rho) \in L_\omega(R_+) \text{ for } \rho \in ]0, +\infty[,$$

where

$$f^*(t, \rho) = \sup \{ \|f(x)(t)\| : x \in C_\omega(R^n), \|x\|_{C_\omega} \leq \rho \}.$$

We introduce

**Definition 2.1.** Let  $\beta$  be a positive number. We say that an operator  $p : C_\omega(R^n) \times C_\omega(R^n) \rightarrow L_\omega(R^n)$  belongs to the class  $V_\omega^n(\beta)$  if it is continuous and satisfies the following three conditions:

(i)  $p(x, \cdot) : C_\omega(R^n) \rightarrow L_\omega(R^n)$  is a linear operator for any arbitrarily fixed  $x \in C_\omega(R^n)$ ;

(ii) there exists a nondecreasing in the second argument function  $\alpha : R \times R_+ \rightarrow R_+$  such that  $\alpha(\cdot, \rho) \in L_\omega(R)$  for  $\rho \in ]0, +\infty[$ , and for any  $x, y \in C_\omega(R)$  and for almost all  $t \in R$  the inequality

$$\|p(x, y)(t)\| \leq \alpha(t, \|x\|_{C_\omega}) \|y\|_{C_\omega}$$

holds;

(iii) for any  $x \in C_\omega(R^n)$  and  $q \in L_\omega(R^n)$ , an arbitrary  $\omega$ -periodic solution  $y$  of the differential equation

$$\frac{dy(t)}{dt} = p(x, y)(t) + q(t) \quad (2.1)$$

admits the estimate

$$\|y\|_{C_\omega} \leq \beta \|q\|_{L_\omega}. \quad (2.2)$$

**Definition 2.2.** We say that an operator  $p : C_\omega(R^n) \times C_\omega(R^n) \rightarrow L_\omega(R^n)$  belongs to the set  $V_\omega^n$  if there exists  $\beta > 0$  such that  $p \in V_\omega^n(\beta)$ .

**Theorem 2.1.** *Let there exist a positive number  $\rho_0$  and an operator  $p \in V_\omega^n$  such that for any  $\lambda \in ]0, 1[$  an arbitrary  $\omega$ -periodic solution of the differential equation*

$$\frac{dx(t)}{dt} = (1 - \lambda)p(x, x)(t) + \lambda f(x)(t) \quad (2.3)$$

*admits the estimate*

$$\|x\|_{C_\omega} \leq \rho_0. \quad (2.4)$$

*Then equation (1.1) has at least one  $\omega$ -periodic solution.*

*Proof.* For arbitrary  $x \in C([0, \omega]; R^n)$ , we denote by  $v_\omega(x)$  the vector function defined by the equality

$$v_\omega(x)(t) = x(t - j\omega) + \frac{t - j\omega}{\omega} [x(0) - x(\omega)] \quad \text{for } j\omega \leq t < (j+1)\omega \quad (2.5)$$

$$(j = 0, 1, -1, 2, -2, \dots),$$

and for any  $x$  and  $y \in C([0, \omega]; R^n)$  we set

$$\begin{aligned} \tilde{p}(x, y)(t) &= p(v_\omega(x), v_\omega(y))(t), \quad l(x, y) = y(\omega) - y(0), \\ \tilde{f}(x)(t) &= f(v_\omega(x))(t). \end{aligned} \quad (2.6)$$

Obviously,  $v_\omega : C([0, \omega]; R^n) \rightarrow C_\omega(R^n)$  is a linear bounded operator, while  $\tilde{f} : C([0, \omega]; R^n) \rightarrow L([0, \omega]; R^n)$  and  $\tilde{p} : C([0, \omega]; R^n) \times C([0, \omega]; R^n) \rightarrow L([0, \omega]; R^n)$  are continuous operators. Moreover, the restrictions on  $[0, \omega]$  of  $\omega$ -periodic solutions of equations (1.1) and (2.3) are respectively solutions of the differential equations

$$\frac{dx(t)}{dt} = \tilde{f}(x)(t) \quad (2.7)$$

and

$$\frac{dx(t)}{dt} = (1 - \lambda)\tilde{p}(x, x)(t) + \lambda\tilde{f}(x)(t) \quad (2.8)$$

with the periodic boundary condition

$$x(\omega) = x(0), \quad (2.9)$$

and vice versa the periodic extension of an arbitrary solution of problem (2.7), (2.9) (problem (2.8), (2.9)) is an  $\omega$ -periodic solution of equation (1.1) (equation (2.3)). Consequently, for any  $\lambda \in ]0, 1[$ , an arbitrary solution of problem (2.8), (2.9) admits estimate (2.4).

On the other hand, it follows from the condition  $p \in V_\omega^n$  and equalities (2.6) that the pair of operators  $(\tilde{p}, l)$  is compatible in the sense of Definition 1 from [14].

Thus we have shown that for problem (2.7), (2.9), all the conditions of Theorem 1 from [14] are fulfilled, which guarantees the solvability of this problem. However, according to the above-said, the existence of an  $\omega$ -periodic solution of equation (1.1) follows from the solvability of problem (2.7), (2.9).  $\square$

**Corollary 2.1.** *Let there exist  $\beta > 0$  and  $p \in V_\omega^n(\beta)$  such that for any  $x \in C_\omega(R^n)$  almost everywhere on  $R$  the inequality*

$$\|f(x)(t) - p(x, x)(t)\| \leq \gamma(t, \|x\|_{C_\omega}) \quad (2.10)$$

*holds, where  $\gamma(\cdot, \rho) \in L_\omega(R_+)$  for  $0 < \rho < +\infty$ , and*

$$\limsup_{\rho \rightarrow +\infty} \frac{1}{\rho} \int_a^b \gamma(t, \rho) dt < \frac{1}{\beta}. \quad (2.11)$$

*Then equation (1.1) has at least one  $\omega$ -periodic solution.*

*Proof.* By (2.11) there exists  $\rho_0 > 0$  such that

$$\beta \int_0^\omega \gamma(t, \rho) dt < 1 \quad \text{for } \rho \geq \rho_0. \quad (2.12)$$

Let  $x$  be an  $\omega$ -periodic solution of (2.3) for some  $\lambda \in ]0, 1[$ . Assume

$$\delta(t) = f(x)(t) - p(x, x)(t).$$

Then

$$\frac{dx(t)}{dt} = p(x, x)(t) + \lambda\delta(t)$$

and, as follows from (2.10), the vector function  $\delta$  satisfies almost everywhere on  $R$  the inequality

$$\|\delta(t)\| \leq \gamma(t, \|x\|_{C_\omega}),$$

whence, owing to  $p \in V_\omega^n(\beta)$ , we have

$$\|x\|_{C_\omega} \leq \beta \int_0^\omega \|\delta(t)\| dt \leq \beta \int_0^\omega \gamma(t, \|x\|_{C_\omega}) dt.$$

From this inequality, by virtue of (2.12), follows estimate (2.4).

If now we take into account Theorem 2.1 the validity of the corollary will become obvious.  $\square$

**Corollary 2.2.** *Let for any  $x \in C_\omega(R^n)$ , inequality (2.10) be fulfilled almost everywhere on  $R$ , where  $\gamma(\cdot, \rho) \in L_\omega(R_+)$  for  $0 < \rho < +\infty$ , and  $p : C_\omega(R^n) \times C_\omega(R^n) \rightarrow L_\omega(R^n)$  is a continuous operator such that  $p(x, \cdot) : C_\omega(R^n) \rightarrow L_\omega(R^n)$  is linear and  $\int_0^\omega p(x, E)(s) ds$  is a nonsingular matrix for an arbitrarily fixed  $x \in C_\omega(R^n)$ . Let, moreover, there exist matrices  $A$  and  $B \in R_+^{n \times n}$  such that*

$$r(A + BA^2) < 1, \quad (2.13)$$

$$\int_0^\omega |p(x, y)(s)| ds \leq A|y|_{C_\omega}, \quad \left| \left[ \int_0^\omega p(x, E)(s) ds \right]^{-1} \right| \leq B \quad (2.14)$$

for any  $x$  and  $y \in C_\omega(R^n)$  and the function  $\gamma$  satisfies condition (2.11), where  $\beta = \|(E - A - BA^2)^{-1}(E + BA)\|$ . Then equation (1.1) has at least one  $\omega$ -periodic solution.

*Proof.* By virtue of Corollary 2.1, to prove Corollary 2.2 it suffices to establish that for any  $x \in C_\omega(R^n)$  and  $q \in L_\omega(R^n)$  an arbitrary  $\omega$ -periodic solution  $y$  of equation (2.1) admits estimate (2.2).

By (2.5), from (2.1) we have

$$y(t) = y(0) + p^1(x, y)(t) + q_0(t), \quad (2.15)$$

where

$$p^1(x, y)(t) = \int_0^t p(x, v_\omega(y))(s) ds, \quad q_0(t) = \int_0^t q(s) ds. \quad (2.16)$$

Therefore

$$y(t) = y(0) + p^1(x, E)(t)y(0) + p^1(x, p^1(x, y))(t) + p^1(x, q_0)(t),$$

whence because of the  $\omega$ -periodicity of  $y$  and the nonsingularity of the matrix

$$p^1(x, E)(\omega) = \int_0^\omega p(x, E)(s) ds$$

we obtain

$$y(0) = - \left[ \int_0^\omega p(x, E)(s) ds \right]^{-1} [p^1(x, p^1(x, y))(\omega) + p^1(x, q_0)(\omega)].$$

According to (2.14) and (2.16), the latter representation results in

$$\begin{aligned} |y(0)| &\leq B(A|v_\omega(p^1(x, y))|_{C_\omega} + A|v_\omega(q_0)|_{C_\omega}) \leq \\ &\leq B(A|p^1(x, y)|_{C_\omega} + A|q_0|_{C_\omega}) \leq BA^2|y|_{C_\omega} + BA|q|_{L_\omega}. \end{aligned}$$

Taking this estimate into account, from (2.15) we find that

$$|y|_{C_\omega} \leq BA^2|y|_{C_\omega} + BA|q|_{L_\omega} + A|y|_{C_\omega} + |q|_{L_\omega}$$

and

$$(E - A - BA^2)|y|_{C_\omega} \leq (E + BA)|q|_{L_\omega}.$$

Hence by (2.13) we have

$$|y|_{C_\omega} \leq (E - A - BA^2)^{-1}(E + BA)|q|_{L_\omega}.$$

Thus estimate (2.2) is valid.  $\square$

Corollary 2.2 deals with the case where

$$\sup \left\{ \frac{1}{1 + \|x\|_{C_\omega}} \int_0^\omega \|f(x)(t)\| dt : x \in C_\omega(R^n) \right\} < +\infty,$$

whereas Corollary 2.1 covers the class of equations of type (1.1) for which the last condition is violated. As an example, consider the integro-differential equation

$$\frac{dx(t)}{dt} = p_1(t, x(t)) \int_0^\omega p_2(s, x(s))x(s) d\varphi(s) + p_0(t, x(t)), \quad (2.17)$$

where the functions  $p_i : R \times R \rightarrow R$  ( $i = 0, 1$ ) satisfy the local Carathéodory conditions and are  $\omega$ -periodic in the first argument,  $p_2 : [0, \omega] \times R \rightarrow R$  is continuous and  $\varphi : [0, \omega] \rightarrow R$  is nondecreasing.

**Corollary 2.3.** *Let on  $[0, \omega] \times R$  the inequalities*

$$\sigma_i p_i(t, x) \geq \delta_i(t) \quad (i = 1, 2) \quad (2.18)$$

be fulfilled, where  $\sigma_i \in \{-1, 1\}$  ( $i = 1, 2$ ),  $\delta_1 : [0, \omega] \rightarrow R_+$  is a summable function and  $\delta_2 : [0, \omega] \rightarrow R_+$  is a continuous function such that

$$\delta = \left( \int_0^\omega \delta_1(s) ds \right) \left( \int_0^\omega \delta_2(s) d\varphi(s) \right) > 0. \quad (2.19)$$

Let further

$$\limsup_{\rho \rightarrow +\infty} \left( \frac{1}{\rho} \int_0^\omega \gamma(t, \rho) dt \right) < \frac{\delta}{1 + 3\delta},$$

where

$$\gamma(t, \rho) = \max \{ |p_0(t, x)| : |x| \leq \rho \}.$$

Then equation (2.17) has at least one  $\omega$ -periodic solution.

*Proof.* Suppose

$$p(x, y)(t) = p_1(t, x(t)) \int_0^\omega p_2(s, x(s)) y(s) d\varphi(s).$$

By virtue of Corollary 2.1, to prove Corollary 2.3 it suffices to establish that  $p \in V_\omega^1(\frac{1+3\delta}{\delta})$ .

Let  $x \in C_\omega(R)$ ,  $q \in L_\omega(R)$  and let  $y$  be an arbitrary  $\omega$ -periodic solution of the equation

$$\frac{dy(t)}{dt} = p_1(t, x(t)) \int_0^\omega p_2(s, x(s)) y(s) d\varphi(s) + q(t).$$

Suppose

$$\int_0^\omega p_2(s, x(s)) y(s) d\varphi(s) = c. \quad (2.20)$$

Then

$$y(t) = y(0) + c \int_0^t p_1(s, x(s)) ds + \int_0^t q(s) ds. \quad (2.21)$$



Therefore

$$c \int_0^{\omega} p_1(s, x(s)) ds + \int_0^{\omega} q(s) ds = 0$$

and consequently

$$c = - \left( \int_0^{\omega} p_1(s, x(s)) ds \right)^{-1} \left( \int_0^{\omega} q(s) ds \right). \quad (2.22)$$

If we substitute (2.21) in (2.20) and calculate  $y(0)$ , then we obtain

$$\begin{aligned} y(0) &= \\ &= c \left( \int_0^{\omega} p_2(s, x(s)) d\varphi(s) \right)^{-1} \left[ 1 - \int_0^{\omega} p_2(s, x(s)) \left( \int_0^s p_1(\xi, x(\xi)) d\xi \right) d\varphi(s) \right] - \\ &\quad - \left( \int_0^{\omega} p_2(s, x(s)) d\varphi(s) \right)^{-1} \int_0^{\omega} p_2(s, x(s)) \left( \int_0^s q(\xi) d\xi \right) d\varphi(s). \end{aligned} \quad (2.23)$$

Introduce the function

$$\eta(t) = \left( \int_0^{\omega} p_1(s, x(s)) ds \right)^{-1} \int_0^t p_1(s, x(s)) ds. \quad (2.24)$$

Then from (2.21) and (2.22) we get

$$y(t) = y(0) + \int_0^t (1 - \eta(t)) q(s) ds - \eta(t) \int_t^{\omega} q(s) ds.$$

On the other hand, taking into account (2.18) and (2.19), from (2.22)–(2.24) we find

$$|y(0)| \leq \left( \frac{1}{\delta} + 2 \right) \|q\|_{L_\omega}, \quad 0 \leq \eta(t) \leq 1 \quad \text{for } 0 \leq t \leq \omega.$$

Therefore

$$\|y\|_{L_\omega} \leq |y(0)| + \|q\|_{L_\omega} \leq \frac{1 + 3\delta}{\delta} \|q\|_{L_\omega},$$

which because of the arbitrariness of  $x \in C_\omega(R)$  and  $q \in L_\omega(R)$  results in  $p \in V_\omega^1(\frac{1+3\delta}{\delta})$ .  $\square$

**Theorem 2.2.** *Let for any  $x \in C_\omega(R^n)$  almost everywhere on  $R$  the inequality*

$$f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) \leq p_0(t)\|x(t)\| + \gamma(t, \|x\|_{C_\omega}) \quad (2.25)$$

*be fulfilled, where  $\sigma \in \{-1, 1\}$ ,  $p_0 \in L_\omega(R)$ ,  $\gamma(\cdot, \rho) \in L_\omega(R_+)$  for  $0 < \rho < +\infty$ ,*

$$\int_0^\omega p_0(s) ds < 0 \quad (2.26)$$

*and*

$$\begin{aligned} & \limsup_{\rho \rightarrow +\infty} \frac{1}{\rho} \left( \int_t^{t+\omega} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right) \gamma(s, \rho) ds \right) < \\ & < \left| \exp \left( -\sigma \int_0^\omega p_0(\xi) d\xi \right) - 1 \right| \text{ uniformly with respect to } t \in [0, \omega]. \end{aligned} \quad (2.27)$$

*Then equation (1.1) has at least one  $\omega$ -periodic solution.*

To prove this theorem, it is necessary to establish an a priori estimate of nonnegative  $\omega$ -periodic solutions of the differential inequality

$$\sigma u'(t) \leq p_0(t)u(t) + \gamma(t, \|u\|_{C_\omega}), \quad (2.28)$$

where  $\sigma \in \{-1, 1\}$ ,  $p_0 \in L_\omega(R)$ , and  $\gamma(\cdot, \rho) \in L_\omega(R_+)$  for  $0 < \rho < +\infty$ .

Note that by an  $\omega$ -periodic solution of inequality (2.28) we mean an absolutely continuous  $\omega$ -periodic function  $u : R \rightarrow R$  satisfying this inequality almost everywhere on  $R$ .

**Lemma 2.1.** *Let inequality (2.26) be fulfilled, and let there exist a nonnegative constant  $\rho_0$  such that*

$$\begin{aligned} & \int_t^{t+\omega} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right) \gamma(s, \rho) ds < \\ & < \left| \exp \left( -\sigma \int_0^\omega p_0(s) ds \right) - 1 \right| \rho \text{ for } 0 \leq t \leq \omega, \rho > \rho_0. \end{aligned} \quad (2.29)$$

*Then an arbitrary nonnegative  $\omega$ -periodic solution  $u$  of (2.28) admits the estimate*

$$\|u\|_{C_\omega} \leq \rho_0. \quad (2.30)$$

*Proof.* Let  $u$  be an arbitrary  $\omega$ -periodic solution of the differential inequality (2.28). Suppose

$$q(t) = u'(t) - \sigma p_0(t)u(t).$$

Then by Theorem 6.4 from [11] we find

$$u(t) = \left[ \exp \left( -\sigma \int_0^\omega p_0(s) ds \right) - 1 \right]^{-1} \int_t^{t+\omega} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right) q(s) ds. \quad (2.31)$$

On the other hand, owing to (2.28), the inequality

$$\sigma q(t) \leq \gamma(t, \|x\|_{C_\omega})$$

holds almost everywhere on  $R$ . If along with this inequality we take into consideration inequality (2.26), then from (2.31) we obtain

$$u(t) \leq \left| \exp \left( -\sigma \int_0^\omega p_0(s) ds \right) - 1 \right|^{-1} \times \\ \times \int_t^{t+\omega} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right) \gamma(s, \|x\|_{C_\omega}) ds \quad \text{for } 0 \leq t \leq \omega. \quad (2.32)$$

Suppose now that estimate (2.30) is not valid. Then there exists a point  $t_0 \in [0, \omega]$  such that

$$\|u\|_{C_\omega} = u(t_0) > \rho_0.$$

Taking into account this fact and condition (2.29), from (2.32) we obtain the contradiction

$$\|u\|_{C_\omega} < \|u\|_{C_\omega},$$

which proves the lemma.  $\square$

*Proof of Theorem 2.2.* First of all it should be noted that by condition (2.27), there exists a positive number  $\rho_0$  such that inequality (2.29) is fulfilled.

For any  $x$  and  $y \in C_\omega(R^n)$  suppose

$$p(x, y)(t) = \sigma p_0(t)y(t).$$

Then by Theorem 6.4 from [11], inequality (2.26) guarantees the condition

$$p \in V_\omega^n.$$

According to this fact and Theorem 2.1, to prove Theorem 2.2 it suffices to establish that for any  $\lambda \in ]0, 1[$  an arbitrary  $\omega$ -periodic solution of the differential equation

$$\frac{dx(t)}{dt} = (1 - \lambda)\sigma p_0(t)x(t) + \lambda f(x)(t) \quad (2.33)$$

admits estimate (2.4).

Indeed, let  $x$  be such a solution. Suppose

$$u(t) = \|x(t)\|.$$

Then by (2.25) from (2.33) we find

$$\begin{aligned} \sigma u'(t) &= x'(t) \cdot \operatorname{sgn}(\sigma x(t)) = \\ &= (1 - \lambda)p_0(t)\|x(t)\| + \lambda f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) \leq \\ &\leq p_0(t)\|x(t)\| + \gamma(t, \|x\|_{C_\omega}) = p_0(t)u(t) + \gamma(t, \|u\|_{C_\omega}). \end{aligned}$$

Consequently  $u$  is a nonnegative  $\omega$ -periodic solution of the differential inequality (2.28). This function by Lemma 2.1 admits estimate (2.30). Therefore  $x$  admits estimate (2.4).  $\square$

**Theorem 2.3.** *Let for any  $x$  and  $y \in C_\omega(\mathbb{R}^n)$ , almost everywhere on  $\mathbb{R}$  the condition*

$$\begin{aligned} [f(x)(t) - f(y)(t)] \cdot \operatorname{sgn}[\sigma(x(t) - y(t))] &\leq \\ &\leq p_0(t)\|x(t) - y(t)\| + \gamma_0(t)\|x - y\|_{C_\omega} \end{aligned} \quad (2.34)$$

be fulfilled, where  $\sigma \in \{-1, 1\}$ , the function  $p_0 \in L_\omega(\mathbb{R})$  satisfies inequality (2.26) and the function  $\gamma_0 \in L_\omega(\mathbb{R}_+)$  satisfies the inequality

$$\begin{aligned} &\int_t^{t+\omega} \exp\left(\sigma \int_s^t p_0(\xi) d\xi\right) \gamma_0(s) ds < \\ &< \left| \exp\left(-\sigma \int_0^\omega p_0(\xi) d\xi\right) - 1 \right| \quad \text{for } 0 \leq t \leq \omega. \end{aligned} \quad (2.35)$$

Then equation (1.1) has one and only one  $\omega$ -periodic solution.

*Proof.* From (2.34) and (2.35) we arrive at conditions (2.25) and (2.27), where

$$\gamma(t, \rho) = \gamma_0(t)\rho + \|f(t, 0, \dots, 0)\|.$$

Therefore by Theorem 2.2, equation (1.1) has at least one  $\omega$ -periodic solution  $x$ .

To complete the proof of the theorem, it remains to show that an arbitrary  $\omega$ -periodic solution  $y$  of equation (1.1) coincides with  $x$ . Suppose

$$u(t) = \|x(t) - y(t)\|.$$

Then by (2.34),  $u$  is a nonnegative  $\omega$ -periodic solution of the differential inequality (2.28), where

$$\gamma(t, \rho) = \gamma_0(t)\rho.$$

On the other hand, owing to (2.35), the function  $\gamma$  satisfies inequality (2.29), where  $\rho_0 = 0$ , whence by Lemma 2.1 it follows that  $u(t) \equiv 0$ . Consequently  $x(t) \equiv y(t)$ .  $\square$

**Example 2.1.** Consider the integro-differential equation

$$\frac{dx(t)}{dt} = \sigma p_0(t)x(t) + \sigma_0 \left[ |p_0(t)| \int_{-\infty}^{+\infty} p_1(s)|x(s)| ds + p_2(t) \right], \quad (2.36)$$

where  $\sigma$  and  $\sigma_0 \in \{-1, 1\}$ ,  $p_1 : R \rightarrow R_+$  is a summable function, while  $p_0 \in L_\omega(R_-)$  and  $p_2 \in L_\omega(R_+)$  are functions different from zero on a set of positive measure. Because of the restrictions imposed on  $p_0$  and  $p_2$ ,

$$\int_t^{t+\omega} |g(t, s)| |p_0(s)| ds = 1 \quad (2.37)$$

and

$$\int_t^{t+\omega} |g(t, s)| p_2(s) ds \geq \delta, \quad (2.38)$$

where

$$g(t, s) = \left[ \exp \left( -\sigma \int_0^\omega p_0(\xi) d\xi \right) - 1 \right]^{-1} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right),$$

and  $\delta$  is a positive constant. On the other hand, the operator

$$f(x)(t) = \sigma p_0(t)x(t) + \sigma_0 \left[ |p_0(t)| \int_{-\infty}^{+\infty} p_1(s)|x(s)| ds + p_2(t) \right]$$

satisfies condition (2.34), where

$$\gamma_0(t) = |p_0(t)| \int_{-\infty}^{+\infty} p_1(s) ds.$$

This and equality (2.37) imply that if

$$\int_{-\infty}^{+\infty} p_1(s) ds < 1, \quad (2.39)$$

then inequality (2.35) is fulfilled. In that case, by Theorem 2.3 equation (2.36) has a unique  $\omega$ -periodic solution.

Show that if

$$\int_{-\infty}^{+\infty} p_1(s) ds \geq 1, \quad (2.40)$$

then equation (2.36) has no  $\omega$ -periodic solution. Assume to the contrary that it has a solution  $x$ . Then

$$x(t) = \sigma_0 \int_t^{t+\omega} g(t, s) \left[ |p_0(s)| \int_{-\infty}^{+\infty} p_1(\xi) |x(\xi)| d\xi + p_2(s) \right] ds.$$

Hence, taking into account (2.37), (2.38) and (2.40), we find

$$\mu \geq \mu \int_{-\infty}^{+\infty} p_1(\xi) d\xi + \delta \geq \mu + \delta,$$

where  $\mu = \min\{|x(t)| : t \in R\}$ . The obtained contradiction shows that equation (2.36) has no  $\omega$ -periodic solution when condition (2.40) is fulfilled.

The example under consideration shows that in Theorem 2.2 (Theorem 2.3) it is impossible to replace the strict inequality (2.27) (the strict inequality (2.35)) by the nonstrict one.

**Theorem 2.4.** *Let there exist  $\rho_0 \in ]0, +\infty[$ ,  $\delta \in ]0, 1[$  and  $\sigma \in \{-1, 1\}$  such that for any  $x \in C_\omega(R^n)$  satisfying*

$$\|x\|_{C_\omega} > \rho_0, \quad (2.41)$$

*almost everywhere on the set*

$$\{t \in R : \|x(t)\| > (1 - \delta)\|x\|_{C_\omega}\} \quad (2.42)$$

*the inequality*

$$f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) \leq 0 \quad (2.43)$$

*is fulfilled. Then equation (1.1) has at least one  $\omega$ -periodic solution.*

*Proof.* By Theorem 2.1, to prove Theorem 2.4 it suffices to establish that for any  $\lambda \in ]0, 1[$ , an arbitrary  $\omega$ -periodic solution of the differential equation

$$\frac{dx(t)}{dt} = -\sigma(1 - \lambda)x(t) + \lambda f(x)(t) \quad (2.44)$$

admits estimate (2.4).

Assume to the contrary that for some  $\lambda \in ]0, 1[$  equation (2.44) has an  $\omega$ -periodic solution  $x$  satisfying (2.41). Then there exist  $t_0 \in ]0, +\infty[$ ,  $t_* \in ]0, t_0[$  and  $t^* \in ]t_0, +\infty[$  such that

$$\|x(t_0)\| = \|x\|_{C_\omega}, \quad \|x(t)\| > (1 - \delta)\|x\|_{C_\omega} \quad \text{for } t_* \leq t \leq t^*.$$

Hence it is clear that  $[t_*, t^*]$  is included in (2.42). Therefore inequality (2.43) is fulfilled almost everywhere on  $[t_*, t^*]$ . Consequently

$$\begin{aligned} \sigma \frac{d\|x(t)\|}{dt} &= -(1 - \lambda)\|x(t)\| + \lambda f(x)(t) \cdot \text{sgn}(\sigma x(t)) < 0 \\ &\text{for almost all } t \in [t_*, t^*]. \end{aligned}$$

Hence for  $\sigma = 1$  ( $\sigma = -1$ ) it follows that

$$\|x(t_*)\| > \|x(t_0)\| \quad (\|x(t^*)\| > \|x(t_0)\|).$$

But this is impossible because  $\|x(t_0)\| = \|x\|_{C_\omega}$ . The obtained contradiction proves the theorem.  $\square$

As an example, consider the nonlinear differential system

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -\sigma l_0(t)|x_i(t)|^\lambda \text{sgn } x_i(t) + \sigma f_{1i}(t, x_1(t), \dots, x_n(t)) + \\ &+ f_{2i}(t, l_1(x_1)(t), \dots, l_n(x_n)(t)) \quad (i = 1, \dots, n), \end{aligned} \quad (2.45)$$

where  $\sigma \in \{-1, 1\}$ ,  $\lambda \in ]0, +\infty[$ ,  $l_0 \in L_\omega(R_+)$ ,  $f_{1i}$  and  $f_{2i} : R \times R^n \rightarrow R$  ( $i = 1, \dots, n$ ) are  $\omega$ -periodic in the first argument functions satisfying the local Carathéodory conditions, and  $l_i : C_\omega(R) \rightarrow C_\omega(R)$  ( $i = 1, \dots, n$ ) are linear bounded operators with norms  $\|l_1\|, \dots, \|l_n\|$ .

Set

$$\mu(\lambda) = \begin{cases} \lambda - 1 & \text{for } \lambda > 1 \\ 0 & \text{for } \lambda \leq 1 \end{cases}.$$

Theorem 2.4 implies

**Corollary 2.4.** *Let on  $R \times R^n$  the inequalities*

$$\sum_{i=1}^n f_{1i}(t, x_1, \dots, x_n) \text{sgn } x_i \leq l_0(t) \left[ \eta_1 \left( \sum_{i=1}^n |x_i| \right)^\lambda + \eta_0 \right], \quad (2.46)$$

$$\sum_{i=1}^n |f_{2i}(t, x_1, \dots, x_n)| \leq l_0(t) \left[ \eta_2 \left( \sum_{i=1}^n |x_i| \right)^\lambda + \eta_0 \right] \quad (2.47)$$

be fulfilled, where  $\eta_i$  ( $i = 0, 1, 2$ ) are positive constants such that

$$\eta_1 + \|l_i\|^\lambda \eta_2 < n^{-\mu(\lambda)} \quad (i = 1, \dots, n). \quad (2.48)$$

Then system (2.45) has at least one  $\omega$ -periodic solution.

*Proof.* If we assume  $x(t) = (x_i(t))_{i=1}^n$ ,

$$f_i(x)(t) = -\sigma l_0(t)|x_i(t)|^\lambda \operatorname{sgn} x_i(t) + \sigma f_{1i}(t, x_1(t), \dots, x_n(t)) + f_{2i}(t, l_1(x_1)(t), \dots, l_n(x_n)(t)) \quad (i = 1, \dots, n)$$

and

$$f(x)(t) = (f_i(x)(t))_{i=1}^n,$$

then system (2.45) will take form (1.1). On the other hand, taking into account (2.46) and (2.47), the operator  $f$  satisfies the condition

$$\begin{aligned} f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) &\leq \\ &\leq -l_0(t) \sum_{i=1}^n |x_i(t)|^\lambda + l_0(t) [(\eta_1 + \|l_{i_0}\|^\lambda \eta_2) \|x\|_{C_\omega}^\lambda + 2\eta_0], \end{aligned}$$

where  $\|l_{i_0}\| = \max\{\|l_1\|, \dots, \|l_n\|\}$ . Hence in view of the inequality

$$\|x\|^\lambda \leq n^{\mu(\lambda)} \sum_{i=1}^n |x_i|^\lambda$$

we obtain

$$\begin{aligned} f(x)(t) \cdot \operatorname{sgn}(\sigma x(t)) &\leq \\ &\leq -l_0(t) [n^{-\mu(\lambda)} \|x(t)\|^\lambda - (\eta_1 + \|l_{i_0}\|^\lambda \eta_2) \|x\|_{C_\omega}^\lambda - 2\eta_0]. \end{aligned} \quad (2.49)$$

By virtue of (2.48), there exists  $\delta \in ]0, 1[$  such that

$$\varepsilon = (1 - \delta)^\lambda n^{-\mu(\lambda)} - \eta_1 - \|l_{i_0}\|^\lambda \eta_2 > 0.$$

Set

$$\rho_0 = (2\eta_0/\varepsilon)^{\frac{1}{\lambda}}.$$

Let  $x \in C_\omega(R^n)$  be an arbitrary vector function satisfying (2.41). Then by (2.49) inequality (2.43) holds almost everywhere on set (2.42). Therefore all the conditions of Theorem 2.4 are fulfilled, which guarantees the existence of at least one  $\omega$ -periodic solution of (2.45).  $\square$

### 3. PERIODIC SOLUTIONS OF EQUATION (1.2)

Throughout this section  $f_0 : R \times R^{(m+1)n} \rightarrow R^n$  is assumed to be a vector function satisfying the local Carathéodory conditions and also condition (1.3), while  $\tau_k : R \rightarrow R$  ( $k = 1, \dots, m$ ) are assumed to be measurable functions satisfying condition (1.4).

For any  $x \in C_\omega(R^n)$  we assume that

$$f(x)(t) = f_0(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))).$$

Then the operator  $f : C_\omega(R^n) \rightarrow L_\omega(R^n)$  is continuous. Therefore from Corollary 2.2 and Theorems 2.2–2.4 we obtain the following propositions.



**Corollary 3.1.** *Let the inequality*

$$\begin{aligned} \left\| f_0(t, x_0, x_1, \dots, x_m) - \sum_{k=0}^m \mathcal{P}_k(t, x_0, x_1, \dots, x_m) x_k \right\| &\leq \\ &\leq \gamma(t, \|x_0\|, \|x_1\|, \dots, \|x_m\|) \end{aligned}$$

be fulfilled on  $R \times R^{(m+1)n}$ , where  $\mathcal{P}_k : R \times R^{(m+1)n} \rightarrow R^{n \times n}$  ( $k = 0, 1, \dots, m$ ) are  $\omega$ -periodic in the first argument matrix functions satisfying the local Carathéodory conditions, and  $\gamma : R \times R_+^{m+1} \rightarrow R_+$  is nondecreasing in the last  $n$  arguments and  $\omega$ -periodic in the first argument. Let, moreover, there exist matrices  $A$  and  $B \in R_+^{n \times n}$  such that

$$\begin{aligned} r(A + BA^2) &< 1, \\ \limsup_{\rho \rightarrow +\infty} \frac{1}{\rho} \int_0^\omega \gamma(s, \rho, \dots, \rho) ds &< \|(E - A - BA^2)^{-1}(E + BA)\| \end{aligned}$$

and for any  $x \in C_\omega(R^n)$  the matrix

$$\sum_{k=0}^m \int_0^\omega \mathcal{P}_k(s, x(s), x(\tau_1(s)), \dots, x(\tau_m(s))) ds$$

is nondegenerate,

$$\sum_{k=0}^m \int_0^\omega |\mathcal{P}_k(s, x(s), x(\tau_1(s)), \dots, x(\tau_m(s)))| ds \leq A,$$

and

$$\left| \left[ \sum_{k=0}^m \int_0^\omega \mathcal{P}_k(s, x(s), x(\tau_1(s)), \dots, x(\tau_m(s))) ds \right]^{-1} \right| \leq B.$$

Then equation (1.2) has at least one  $\omega$ -periodic solution.

**Corollary 3.2.** *Let the inequality*

$$f_0(t, x_0, x_1, \dots, x_m) \cdot \operatorname{sgn}(\sigma x_0) \leq p_0(t) \|x_0\| + \gamma(t, \|x_0\|, \dots, \|x_m\|)$$

be fulfilled on  $R \times R^{(m+1)n}$ , where  $\sigma \in \{-1, 1\}$ ,  $p_0 \in L_\omega(R)$ ,  $\int_0^\omega p_0(s) ds < 0$ ,

$\gamma(\cdot, \rho, \dots, \rho) \in L_\omega(R_+)$  for  $0 < \rho < +\infty$ , and

$$\begin{aligned} & \limsup_{\rho \rightarrow +\infty} \left( \frac{1}{\rho} \int_t^{t+\omega} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right) \gamma(s, \rho, \dots, \rho) ds \right) < \\ & < \left| \exp \left( -\sigma \int_0^\omega p_0(\xi) d\xi \right) - 1 \right| \text{ uniformly with respect to } t \in [0, \omega]. \end{aligned}$$

Then equation (1.2) has at least one  $\omega$ -periodic solution.

**Corollary 3.3.** *Let the condition*

$$\begin{aligned} & [f_0(t, x_0, x_1, \dots, x_m) - f_0(t, y_0, y_1, \dots, y_m)] \cdot \operatorname{sgn}(\sigma(x_0 - y_0)) \leq \\ & \leq p_0(t) \|x_0 - y_0\| + \sum_{k=1}^m \gamma_k(t) \|x_k - y_k\| \end{aligned}$$

be fulfilled on  $R \times R^{(m+1)n}$ , where  $\sigma \in \{-1, 1\}$ ,  $p_0 \in L_\omega(R)$ ,  $\int_0^\omega p_0(s) ds < 0$ ,  $\gamma_k \in L_\omega(R_+)$  ( $k = 1, \dots, m$ ) and

$$\begin{aligned} & \sum_{k=1}^m \int_t^{t+\omega} \exp \left( \sigma \int_s^t p_0(\xi) d\xi \right) \gamma_k(s) ds < \\ & < \left| \exp \left( -\sigma \int_0^\omega p_0(\xi) d\xi \right) - 1 \right| \text{ for } 0 \leq t \leq \omega. \end{aligned}$$

Then equation (1.2) has one and only one  $\omega$ -periodic solution.

**Corollary 3.4.** *Let there exist  $\rho \in ]0, +\infty[$ ,  $\delta \in ]0, 1[$  and  $\sigma \in \{-1, 1\}$  such that on the set*

$$\begin{aligned} & \{(t, x_0, x_1, \dots, x_m) \in R \times R^{(m+1)n} : \|x_0\| \geq \rho, \\ & (1 - \delta) \|x_k\| \leq \|x_0\| \quad (k = 1, \dots, m)\} \end{aligned}$$

the inequality

$$f_0(t, x_0, x_1, \dots, x_m) \cdot \operatorname{sgn}(\sigma x_0) \leq 0$$

is fulfilled. Then equation (1.2) has at least one  $\omega$ -periodic solution.

## ACKNOWLEDGEMENT

This work was supported by Grant 201/96/0410 of the Grant Agency of the Czech Republic (Prague) and by Grant 619/1996 of the Development Fund of Czech Universities.

## REFERENCES

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, Introduction to the theory of functional differential equations. (Russian) *Nauka, Moscow*, 1991.
2. S. R. Bernfeld and V. Lakshmikantham, An introduction to nonlinear boundary value problems. *Academic Press, Inc., New York and London*, 1974.
3. J. Cronin, Periodic solutions of some nonlinear differential equations. *J. Differential Equations* **3**(1967), 31–46.
4. Sh. Gelashvili and I. Kiguradze, On multi-point boundary value problems for systems of functional differential and difference equations. *Mem. Differential Equations Math. Phys.* **5**(1995), 1–113.
5. A. Halanay, Optimal control of periodic solutions. *Rev. Roumaine Math. Pures Appl.* **19**(1974), 3–16.
6. J. K. Hale, Periodic and almost periodic solutions of functional-differential equations. *Arch. Rational Mech. Anal.* **15**(1964), 289–304.
7. P. Hartman, Ordinary differential equations. *John Wiley, New York*, 1964.
8. G. S. Jones, Asymptotic fixed point theorems and periodic systems of functional-differential equations. *Contr. Diff. Eqn.* **2**(1963), 385–405.
9. I. T. Kiguradze, On periodic solutions of systems of non-autonomous ordinary differential equations. (Russian) *Mat. Zametki* **39**(1986), No. 4, 562–575.
10. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. *J. Soviet Math.* **43**(1988), No. 2, 2259–2339.
11. I. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations. I. (Russian) *Metsniereba, Tbilisi*, 1997.
12. I. Kiguradze, On periodic solutions of first order nonlinear differential equations with deviating arguments. *Mem. Differential Equations Math. Phys.* **10**(1997), 134–137.
13. I. Kiguradze and B. Půža, On periodic solutions of systems of linear functional differential equations. *Arch. Math.* **33**(1997), No. 3, 197–212.
14. I. Kiguradze and B. Půža, On boundary value problems for functional differential equations. *Mem. Differential Equations Math. Phys.* **12**(1997), 106–113.

15. M. A. Krasnosel'skij, An alternative principle for establishing the existence of periodic solutions of differential equations with a lagging argument. (Russian) *Dokl. Akad. Nauk SSSR* **152**(1963), No. 4, 801–805.
16. M. A. Krasnosel'skij, The theory of periodic solutions of non-autonomous differential equations. *Math. Surveys* **21**(1966), 53–74.
17. M. A. Krasnosel'skij, The operator of translation along the trajectories of differential equations. (Russian) *Nauka, Moscow*, 1966.
18. M. A. Krasnosel'skij and A. I. Perov, On existence principle for bounded, periodic and almost periodic solutions of systems of ordinary differential equations. (Russian) *Dokl. Akad. Nauk SSSR* **123**(1958), 235–238.
19. J. Mawhin, Periodic solutions of nonlinear functional differential equations. *J. Differential Equations* **10**(1971), 240–261.
20. T. A. Osechkina, Criterion of unique solvability of periodic boundary value problem for functional differential equation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **10**(1994), No. 1, 48–52.
21. B. Půža, On periodic solutions of linear differential systems with deviating arguments. (Russian) *Differentsial'nye Uravneniya* **31**(1995), No. 11, 1937–1938.
22. K. Schmitt, Periodic solutions of nonlinear differential systems. *J. Math. Anal. and Appl.* **40**(1972), 174–182.
23. S. Sedziwy, Periodic solutions of a system of nonlinear differential equations. *Proc. Amer. Math. Soc.* **48**(1975), 328–336.

(Received 12.09.1997)

Authors' addresses:

I. Kiguradze  
A. Razmadze Mathematical Institute  
Georgian Academy of Sciences  
1, M. Aleksidze St., Tbilisi 380093  
Georgia

B. Půža  
Department of Mathematics  
Masaryk University  
Janáčkovo nám. 2a, 66295 Brno  
Czech Republic