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TENSOR PRODUCTS OF NON-ARCHIMEDEAN WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. It is shown that the completion of the tensor product of two non-Archimedean weighted spaces of continuous functions is topologically isomorphic to another weighted space. Several applications of this result are given.

1. Introduction

Weighted spaces of continuous functions were introduced in the complex case by L. Nachbin in [1], and in the vector case by J. Prolla in [2]. Many other authors have continued the investigation of such spaces. W. H. Summers has shown in [3] that if X and Y are locally compact topological spaces and U, V Nachbin families on X, Y, respectively, then $CU_0(X) \otimes CV_0(Y)$ is topologically isomorphic to a dense subspace of $CW_0(X \times Y)$, where $W = U \times V = \{u \times v : u \in U, v \in V\}$ and $(u \times v)(x, y) = u(x)v(y)$.

The p-adic weighted spaces of continuous functions were introduced by J. P. Q. Carneiro in [4]. Several of the properties of these spaces were studied by the authors in [5] and [6]. In this paper we show that if X, Y are Hausdorff topological spaces, not necessarily locally compact, U, V Nachbin families on X, Y respectively and E a non-Archimedean polar locally convex space, then $CU_0(X) \otimes CV_0(Y, E)$ is topologically isomorphic to a dense subspace of $CW_0(X \times Y, E)$, where $W = U \times V$. We give several applications of this result. We also show that on the space $C_b(X, E)$ of all bounded continuous E-valued functions on X, the strict topology defined in [7] is the weighted topology which corresponds to a certain Nachbin family on X.

2. Preliminaries

Throughout this paper, $\mathbf{K}$ will stand for a complete non-Archimedean valued field whose valuation is nontrivial. By a seminorm, on a vector

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space E over $\mathbf{K}$, we mean a non-Archimedean seminorm. Let E be a locally convex space over $\mathbf{K}$. The collection of all continuous seminorms on E will be denoted by $cs(E)$. The algebraic and the topological duals of E will be denoted by E^* and $E^{'}$, respectively. For a subset B of E, B^0 denotes its polar subset of $E^{'}$. A seminorm p on E is called polar if

$$p = \sup\{|f| : f \in E^*,\ |f| \leq p\},$$

where $|f|$ is defined by $|f|(x) = |f(x)|$. The space E is called polar if its topology is generated by a family of polar seminorms. If E, F are locally convex spaces over $\mathbf{K}$, then $E \otimes F$ denotes the projective tensor product of these spaces. By $E\widehat{\otimes}F$ we denote the completion of $E \otimes F$. Also, by $p \otimes q$ we denote the tensor product of the seminorms p and q. For all unexplained terms concerning non-Archimedean spaces we refer to [8].

Next we recall the definition of non-Archimedean weighted spaces. Let X be a Hausdorff topological space and E a locally convex space. The space of all continuous E-valued functions on X is denoted by $C(X, E)$. By $C_b(X, E)$ and $C_0(X, E)$ we denote the spaces of all members of $C(X, E)$ which are bounded on X or vanish at infinity on X, respectively. In case $E = \mathbf{K}$, we write $C(X), C_b(X)$ and $C_0(X)$ instead of $C(X, \mathbf{K})$, $C_b(X, \mathbf{K})$ and $C_0(X, \mathbf{K})$.

A Nachbin family on X is a family V of non-negative upper-semicontinuous functions on X such that:

(1) For all $v_1, v_2 \in V$ and any $a > 0$ there exists $v \in V$ with $v \geq av_1, av_2$ (pointwise) on X.

(2) For every $x \in X$ there exists $v \in V$ with $v(x) > 0$.

Let now $p \in cs(E)$ and $v \in V$. For an E-valued function f on X, we define

$$q_{v,p}(f) = \|f\|_{v,p} = \sup\{v(x)p(f(x)) : x \in X\}.$$

In case f is $\mathbf{K}$-valued, we define

$$q_v(f) = \|f\|_v = \sup\{v(x)|f(x)| : x \in X\}.$$

Also, for an $\mathbf{R}$-valued or $\mathbf{K}$-valued function f on X, we define

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

The weighted space $CV(X, E)$ is defined to be the space of all f in $C(X, E)$ such that $q_{v,p}(f) < \infty$ for all $v \in V$ and all $p \in cs(E)$. Note that $q_{v,p}$ is a non-Archimedean seminorm on $CV(X, E)$. We will denote by $CV_0(X, E)$ the subspace of $CV(X, E)$ consisting of all f such that the function $x \mapsto v(x)p(f(x))$ vanishes at infinity on X for each $v \in V$ and each $p \in cs(E)$. On $CV(X, E)$ and on $CV_0(X, E)$ we will consider the weighted topology τ_ν generated by the seminorms $q_{v,p}, v \in V, p \in cs(E)$. When $E = \mathbf{K}$, we will simply write $CV(X)$ and $CV_0(X)$ instead of $CV(X, \mathbf{K})$ and $CV_0(X, \mathbf{K})$.

3. On the Strict Topology

For a locally compact zero dimensional topological space X and a non-Archimedean normed space E, J. Prolla has defined, in [9], the strict topology β on $C_b(X,E)$ as the topology defined by the seminorms

$$f \mapsto \|\phi f\| = \sup\{\|\phi(x)f(x)\| : x \in X\},$$

where $\phi \in C_0(X)$. For an arbitrary topological space X and a locally convex space E, the strict topology β_0 on $C_b(X,E)$ was defined in [7]. This is the topology generated by the seminorms

$$f \mapsto \|\phi f\|_p = \sup\{|\phi(x)|p(f(x)) : x \in X\}$$

where $p \in cs(E)$ and ϕ belongs to the family $B_0(X)$ of all bounded $\mathbf{K}$-valued functions f on X which vanish at infinity. As shown in [7], $\beta_0 = \beta$ when X is locally compact zero-dimensional. In this section we will show that β_0 is a weighted topology.

Let X be a Hausdorff topological space and let $B_{0u}(X)$ denote the family of all $\phi \in B_0(X)$ for which $|\phi|$ is upper-semicontinuous.

Lemma 3.1.
(1) *If $V = |B_{0u}(X)| = \{|\phi| : \phi \in B_{0u}(X)\}$, then V is a Nachbin family on X.*
(2) *For each $\phi \in B_0(X)$ there exists $\psi \in B_{0u}(X)$ such that $|\phi| \leq |\psi|$.*

Proof. (1) If $\phi_1, \phi_2 \in B_{0u}(X)$ and if ϕ is defined on X by

$$\phi(x) = \begin{cases} \phi_1(x) + \phi_2(x) & \text{if } |\phi_1(x)| \neq |\phi_2(x)| \\ \phi_1(x) & \text{otherwise,} \end{cases}$$

then $|\phi| = \max\{|\phi_1|, |\phi_2|\}$ and $\phi \in B_{0u}(X)$. It follows now easily that V is a Nachbin family on X.

(2) Let $\phi \in B_0(X)$ and choose $\lambda \in \mathbf{K}$, $0 < |\lambda| < 1$. Without loss of generality we may assume that $\|\phi\| < |\lambda|$. There exists an increasing sequence (D_n) of compact subsets of X such that $\{x \in X : |\phi(x)| > |\lambda|^n\} \subseteq D_n$. Let ϕ_n denote the $\mathbf{K}$-characteristic function of D_n. For each $x \in X$, the series $\sum_{n=1}^{\infty} \lambda^n \phi_n(x)$ converges in $\mathbf{K}$. Define ψ on X by

$$\psi(x) = \sum_{n=1}^{\infty} \lambda^n \phi_n(x).$$

If $x \in D_n \setminus D_{n-1}$, then $|\psi(x)| = |\lambda|^n$. Given $\epsilon > 0$, choose n such that $|\lambda|^n < \epsilon$. Now $\{x \in X : |\psi(x)| > \epsilon\} \subseteq D_n$ and so $\psi \in B_0(X)$. Also, for each $\epsilon > 0$, the set $A = \{x : |\psi(x)| < \epsilon\}$ is open. Indeed, if $|\lambda| < \epsilon$, then $A = X$. Assume $\epsilon \leq |\lambda|$ and let κ be such that $|\lambda|^{\kappa+1} < \epsilon \leq |\lambda|^{\kappa}$. If $x_0 \in A$, then $x_0 \notin D_\kappa$. Also, for $x \notin D_\kappa$, we have $|\psi(x)| \leq |\lambda|^{\kappa+1} < \epsilon$ and so $x \in A$. Thus $A = X \setminus D_\kappa$, which shows that A is open. Finally, $|\lambda\phi| \leq |\psi|$. Indeed, let $\phi(x) \neq 0$. If $x \in D_1$, then $|\psi(x)| = |\lambda| \geq |\lambda\phi(x)|$. If $x \in D_{n+1} \setminus D_n$, then $|\phi(x)| \leq |\lambda|^n$ and so $|\psi(x)| = |\lambda|^{n+1} \geq |\lambda\phi(x)|$. $\square$

Theorem 3.2. *If V is as in the preceding Lemma, then*

$$CV(X,E) = CV_0(X,E) = C_b(X,E) \quad (\textit{algebraically})$$

and the weighted topology on $CV(X,E)$ coincides with the strict topology β_0 on $C_b(X,E)$.

Proof. It is clear that $C_b(X,E) \subseteq CV_0(X,E)$. On the other hand, assume that some $f \in CV(X,E)$ is not bounded. Then, for $|\lambda| > 1$, there exist $p \in cs(E)$ and a sequence (x_n) of distinct elements of X such that $p(f(x_n)) > |\lambda|^{2n}$ for all n. Let ϕ_n be the $\mathbf{K}$-characteristic function of the set $\{x_1, \ldots, x_n\}$. As in the proof of the preceding Lemma, we get that the function $\phi = \sum_{n=1}^{\infty} \lambda^{-n}\phi_n$ is in $B_{0u}(X)$ and $|\phi(x_n)| = |\sum_{\kappa \geq n} \lambda^{-\kappa}\phi_\kappa(x_n)| = |\lambda|^{-n}$. Thus $\sup_n |\phi(x_n)|p(f(x_n)) = \infty$ contradicts the fact that $f \in CV(X,E)$. This proves the first part. The second part follows from (2) of the preceding Lemma. $\square$

4. Tensor Products of Weighted Spaces

Let X, Y be Hausdorff topological spaces and let U, V be Nachbin families on X, Y respectively. Set $W = U \times V = \{u \times v : u \in U,\ v \in V\}$ where $u \times v$ is defined on $X \times Y$ by $(u \times v)(x,y) = u(x)v(y)$. It is easy to see that W is a Nachbin family on $X \times Y$. In the complex case, Summers has shown in [3] that, for locally compact X, Y, $CU_0(X) \otimes CV_0(Y)$ is topologically isomorphic to a dense subspace of $CW_0(X \times Y)$. The following is an analogous result in our case. Note that we do not assume that X, Y are locally compact.

Theorem 4.1. *Let U, V, W be as above and let E be a Hausdorff locally convex space over $\mathbf{K}$. Then:*

(1) *$CU_0(X) \otimes CV_0(Y,E)$ is topologically isomorphic to a subspace G of $CW_0(X \times Y, E)$;*

(2) *if X is zero-dimensional and E a polar space, then G is a dense subspace of $CW_0(X \times Y, E)$.*

Proof. (1) Let

$$B : CU_0(X) \times CV_0(Y,E) \mapsto CW_0(X \times Y, E),$$
$$B(\phi, f) = \phi \times f, \quad (\phi \times f)(x,y) = \phi(x)f(y).$$

Then B is bilinear. Let

$$T = \tilde{B} : CU_0(X) \otimes CV_0(Y,E) \mapsto CW_0(X \times Y, E)$$

be the corresponding linear map. Then T is one-to-one. Indeed, assume that for some $h = \sum_1^n \phi_\kappa \otimes f_\kappa$ we have $T(h) = 0$. We claim that $h = 0$. We prove it by induction on n. This is clearly true if $n = 1$. Assume that it is true for $n-1$. If some $\phi_\kappa \neq 0$, say $\phi_n \neq 0$, then f_n is a linear combination of $f_1, \ldots, f_{n-1}$, i.e., $f_n = \sum_{\kappa=1}^{n-1} \lambda_\kappa f_\kappa$. Thus

$$0 = \sum_1^n \phi_\kappa \times f_\kappa = \sum_1^{n-1} \phi_\kappa \times f_\kappa + \sum_1^{n-1} \lambda_\kappa(\phi_n \times f_\kappa) = \sum_1^{n-1} (\phi_\kappa + \lambda_\kappa \phi_n) \times f_\kappa.$$

By our inductive hypothesis, we have

$$\begin{aligned} 0 = \sum_1^{n-1} (\phi_\kappa + \lambda_\kappa \phi_n) \otimes f_\kappa &= \sum_1^{n-1} \phi_\kappa \otimes f_\kappa + \sum_1^{n-1} \lambda_\kappa \phi_n \otimes f_\kappa = \\ = \sum_1^{n-1} \phi_\kappa \otimes f_\kappa + \phi_n \otimes \left(\sum_1^{n-1} \lambda_\kappa f_\kappa \right) &= \sum_1^n \phi_\kappa \otimes f_\kappa. \end{aligned}$$

This proves that T is one-to-one. Also, if $M = CU_0(X) \otimes CV_0(Y, E)$ and $G = T(M)$, then T is a topological isomorphism from M onto G. Indeed, let $h \in M, u \in U, v \in V, w = u \times v, p \in cs(E)$. For any representation $h = \sum_1^n \phi_\kappa \otimes f_\kappa$ of h we have

$$\begin{aligned} \|Th\|_{w,p} &= \sup_{x,y} u(x) v(y)\, p\left(\sum_1^n \phi_\kappa(x) f_\kappa(y) \right) \leq \\ \leq \max_\kappa \left[\left(\sup_x u(x) |\phi_\kappa(x)| \right) \cdot \left(\sup_y v(y) p(f_\kappa(y)) \right) \right] &= \max_\kappa \|\phi_\kappa\|_u \|f_\kappa\|_{v,p}. \end{aligned}$$

Thus $\|Th\|_{w,p} \leq (\|\cdot\|_u \otimes \|\cdot\|_{v,p})(h)$. On the other hand, given $0 < t < 1$, there exists a representation $h = \sum_{\kappa=1}^m \phi_\kappa \otimes f_\kappa$ of h such that $\{f_1, \ldots, f_m\}$ is t-orthogonal with respect to the seminorm $\|\cdot\|_{v,p}$. Now, for any $x \in X$,

$$\left\| \sum_{\kappa=1}^m \phi_\kappa(x) f_\kappa \right\|_{v,p} \geq t \max_\kappa |\phi_\kappa(x)| \|f_\kappa\|_{v,p}$$

and so

$$\begin{aligned} \|Th\|_{w,p} = \sup_x \left[\left\| \sum_1^m \phi_\kappa(x) f_\kappa \right\|_{v,p} \right] u(x) &\geq t \max_\kappa \sup_x |\phi_\kappa(x)| \|f_\kappa\|_{v,p} u(x) = \\ = t \max_\kappa \|\phi_\kappa\|_u \|f_\kappa\|_{v,p} &\geq t\, (\|\cdot\|_u \otimes \|\cdot\|_{v,p})(h). \end{aligned}$$

It follows that $\|Th\|_{w,p} = (\|\cdot\|_u \otimes \|\cdot\|_{v,p})(h)$ and so $T : M \mapsto G$ is a topological isomorphism.

(2) Assume that E is polar and X zero-dimensional.

Let $f \in CW_0(X \times Y, E), u \in U, v \in V, w = u \times v, \epsilon > 0$ and $p \in cs(E)$, where p is polar. The set $D = \{(x, y) : u(x) v(y)\, p(f(x, y)) \geq \epsilon\}$ is compact in $X \times Y$. If D_1, D_2 are the projections of D on X, Y respectively, then $D \subseteq D_1 \times D_2$. Let $d > \sup_{x \in D_1} u(x)$, $\sup_{y \in D_2} v(y)$.

The set $\Omega = \{x \in X : u(x) < d\}$ is open in X and contains D_1. Since X is zero-dimensional, there exists a clopen subset A of X with $D_1 \subseteq A \subseteq \Omega$. For each $x \in D_1$ there exists $y \in Y$ with $(x, y) \in D$ and so $u(x) > 0$. Also, for $x_0 \in X$, the map $y \mapsto f(x_0, y)$ is in $CV_0(Y, E)$. Indeed, there exists $u_1 \in U$ with $u_1(x_0) \neq 0$. Let $v_1 \in V$, $\epsilon_1 > 0$ and $q \in cs(E)$. We want to show that the set $B = \{y \in Y : v_1(y)\, q(f(x_0, y)) \geq \epsilon_1\}$ is compact. The set $B_1 = \{(x, y) : u_1(x) v_1(y) q(f(x, y)) \geq \epsilon_1 u_1(x_0)\}$ is compact. If $y \in B$, then $(x_0, y) \in B_1$ and so B is contained in the projection of B_1 in Y. Since B is closed, it follows that B is compact. This proves that the map $y \mapsto f(x_0, y)$ is in $CV_0(Y, E)$.

Also, for each $y_0 \in Y$ and each $x' \in E'$, the function $x \mapsto x'(f(x, y_0))$ is in $CU_0(X)$. Indeed, the seminorm $q(x) = |x'(x)|$ is continuous on E. Choose $v_1 \in V$ with $v_1(y_0) \neq 0$. For $u_1 \in U$, let $H = \{x : u_1(x) q(f(x, y_0)) \geq \epsilon_1\}$. Then, H is contained in the projection on X of the compact set $B_2 = \{(x, y) : u_1(x) v_1(y)\, q(f(x, y)) \geq \epsilon_1 v_1(y_0)\}$ and so H is compact, which proves that the function $x \mapsto x'(f(x, y_0))$ is in $CU_0(X)$.

Let now $x \in D_1$. There exists $y_0 \in Y$ with $(x, y_0) \in D$. Since $p(f(x, y_0)) > 0$ and p is polar, there exists $x' \in E'$ with $x'(f(x, y_0)) \neq 0$. Since the function $z \mapsto x'(f(z, y_0))$ is in $CU_0(X)$, it is clear that there exists $\phi_x \in CU_0(X)$ with $\phi_x(x) = 1$. By the compactness of D_2, there exists a clopen neighborhood A_x and $0 < \epsilon_x < 1$, with

$$d^2 \cdot \epsilon_x \cdot \sup_{y \in D_2} p(f(x, y)) < \epsilon,$$

such that

$$A_x \subseteq A \cap \{z : |\phi_x(z) - 1| < \epsilon_x\} \cap \{z : u(z) < 2u(x)\}$$

and $p(f(z, y) - f(x, y)) < \epsilon / d^2$ for all $z \in A_x$ and all $y \in D_2$. In view of the compactness of D_1, there are $x_1, \ldots, x_m$ in D_1 such that $D_1 \subseteq \bigcup_1^m A_{x_i}$.

Let $A_1 = A_{x_1}$, $\quad A_{\kappa+1} = A_{x_{\kappa+1}} \setminus \big(\bigcup_1^{\kappa} A_{x_i}\big) \quad$ for $\kappa = 1, \ldots, m-1$.

Set $\phi_\kappa = \phi_{x_\kappa} \cdot \mathcal{X}_{A_\kappa}$, $f_\kappa = f(x_\kappa, \cdot) \in CV_0(Y, E)$, where $\mathcal{X}_{A_\kappa}$ is the **K**-characteristic function of A_κ. Then $h = \sum_1^m \phi_\kappa \times f_\kappa \in G$. Moreover, for all $x \in X$ and $y \in Y$, we have

$$u(x) v(y)\, p(f(x, y) - h(x, y)) \leq 2\epsilon. \tag{$*$}$$

To show $(*)$ we consider three possible cases.

Case I: $x \notin \bigcup_1^m A_\kappa$.

In this case, we have $h(x, y) = 0$, $(x, y) \notin D$ and $u(x) v(y)\, p(f(x, y)) < \epsilon$.

Case II: $x \in A_\kappa$ and $y \in D_2$.

Then

$$f(x,y)-h(x,y)=f(x,y)-\phi_\kappa(x)f_\kappa(y)=$$
$$=[f(x,y)-f(x_\kappa,y)]\,+\,f(x_\kappa,y)(1-\phi_\kappa(x)).$$

Since

$$u(x)v(y)\,p\,(f(x,y)-f(x_\kappa,y))<d^2\cdot\epsilon/d^2=\epsilon$$

and

$$u(x)v(y)|1-\phi_\kappa(x)|\,p(f(x_\kappa,y))\le d^2\cdot\epsilon_{x_\kappa}\cdot p(f(x_\kappa,y))<\epsilon,$$

we have that $(*)$ holds.

Case III: $x\in A_\kappa$, $y\notin D_2$.

In this case we have that $(x,y)\notin D$ and so $u(x)v(y)p(f(x,y))<\epsilon$. Also, since $x\in A_\kappa\subseteq A_{x_\kappa}$, we have $\phi_\kappa(x)=\phi_{x_\kappa}(x)$ and $|\phi_{x_\kappa}(x)-1|<1$, which implies that $|\phi_{x_\kappa}(x)|=1$. Thus

$$u(x)v(y)|\phi_\kappa(x)|p(f(x_\kappa,y))\le 2u(x_\kappa)v(y)p(f(x_\kappa,y))<2\epsilon,$$

since $(x_\kappa,y)\notin D$. Thus $(*)$ holds in all cases and so $\|f-h\|_{w,p}\le 2\epsilon$. $\square$

Remark 4.2. Looking at the proof of (2) in the preceding Theorem, we see that instead of the hypothesis that E is polar we may just assume that $E^{'}$ separates the points of E, i.e., for each $s\neq 0$ in E there exists $x^{'}\in E^{'}$ with $x^{'}(s)\neq 0$. Of course polar spaces have this property.

Taking as V the family of all constant positive functions on X, we get that $CV_0(X,E)$ coincides with $C_0(X,E)$ with the topology τ_u of uniform convergence.

Lemma 4.3. *Considering on both $C_0(X,E)$ and $C_0(X,\hat{E})$ the topology τ_u of uniform convergence, we have that $C_0(X,\hat{E})$ is the completion of $C_0(X,E)$.*

Proof. It is easy to see that $C_0(X,\hat{E})$ is complete. Let $f\in C_0(X,\hat{E})$ and $p\in cs(E)$. We will denote also by p the unique continuous extension of p to all of $\hat{E}$.

The set $Z=\{x\in X:\,p(f(x))\ge 1\}$ is clopen and compact. There are $x_1,\ldots,x_n$ in Z such that the sets

$$Z_\kappa=\{x\in X:\,p(f(x)-f(x_\kappa))_{\le}1\}\,,\quad \kappa=1,\ldots,n,$$

are pairwise disjoint and cover Z. For each κ, choose $s_\kappa\in E$ with $p(s_\kappa-f(x_\kappa))<1$. Set

$$h=\sum_1^n \mathcal{X}_{A_\kappa}s_\kappa\,\in\,C_0(X,E),$$

where $A_\kappa=Z_\kappa\cap Z$. Note that the sets $A_1,\ldots,A_n$ are clopen and compact and their union is Z. Since $\|f-h\|_p\le 1$, the result follows. $\square$

Combining Theorem 1 with Lemma 2, we get as a corollary the following

Theorem 4.4. *Let X, Y be Hausdorff topological spaces and E a Hausdorff locally convex space. Then:*

(1) $C_0(X) \otimes C_0(Y, E)$ *is topologically isomorphic to a subspace of* $C_0(X \times Y, E)$;

(2) *if X is zero-dimensional and E' separates the points of E (e.g. if E is polar), then*

$$C_0(X) \hat{\otimes} C_0(Y, E) \cong C_0(X \times Y, \hat{E}).$$

Lemma 4.5. *Let X, Y be Hausdorff topological spaces, $U = |B_{0u}(X)|$, $V = |B_{0u}(Y)|$, $W = U \times V$, $W_1 = |B_{0u}(X \times Y)|$. Then, the Nachbin families W and W_1 are equivalent.*

Proof. Clearly, $W \subseteq W_1$. On the other hand, let $\phi \in B_{0u}(X \times Y)$ and $\lambda \in \mathbf{K}$, $\mu \in \mathbf{K}$ with $|\mu| > 1$, $|\lambda| \geq |\mu|^2$. Without loss of generality, we may assume that $\|\phi\| < |\lambda|^{-1}$. For each positive integer n, the set

$$D_n = \{(x, y) : |\phi(x, y)| \geq |\lambda|^{-n}\}$$

is compact. Let A_n, B_n be the projections of D_n on X, Y, respectively. Set

$$\phi_1 = \sum_{n=1}^{\infty} \mu^{-n} \mathcal{X}_{A_n} \quad , \quad \phi_2 = \sum_{n=1}^{\infty} \mu^{-n} \mathcal{X}_{B_n} .$$

Since (A_n), (B_n) are increasing sequences of compact sets, we get (as in the proof of Lemma 1) that $\phi_1 \in B_{0u}(X)$ and $\phi_2 \in B_{0u}(Y)$. Moreover, $|\phi| \leq |\lambda|(|\phi_1| \times |\phi_2|)$. Indeed, let $(x_0, y_0) \in X \times Y$ with $\phi(x_0, y_0) \neq 0$, and let n be the smallest of all integers κ with $(x_0, y_0) \in D_\kappa$. If m is the smallest integer κ with $x_0 \in A_\kappa$, then $m \leq n$ and $|\phi_1(x_0)| = |\mu|^{-m} \geq |\mu|^{-n}$. Similarly, $|\phi_2(y_0)| \geq |\mu|^{-n}$ and so

$$|\phi_1(x_0)\phi_2(y_0)| \geq |\mu|^{-2n} \geq |\lambda|^{-n}.$$

Since $(x_0, y_0) \notin D_{n-1}$, we have that

$$|\phi(x_0, y_0)| < |\lambda|^{-(n-1)} \leq |\lambda||\phi_1(x_0)\phi_2(y_0)|.$$

This clearly completes the proof. □

Combining the preceding Lemma with Theorems 3.2 and 4.1, we get

Theorem 4.6. *Let X, Y be Hausdorff topological spaces and E a Hausdorff locally convex space. Then:*

(1) $(C_b(X), \beta_0) \otimes (C_b(Y, E), \beta_0)$ *is topologically isomorphic to a subspace G of* $(C_b(X \times Y, E), \beta_0) = M$.

(2) *If X is zero-dimensional and E' separates the points of E, then G is a dense subspace of M.*

Let X, Y be Hausdorff topological spaces, U the Nachbin family of all positive multiples of the $\mathbf{R}$-characteristic functions of the compact subsets of X, $V = |B_{0u}(Y)|$ and $W = U \times V$. Let $f \in E^{X\times Y}$ be such that the restriction $f|_D$ to each compact subset D of $X \times Y$ is continuous.

Consider the following properties of f:

(1) For each compact subset D_1 of X, the restriction of f to $D_1 \times Y$ is bounded.

(2) For any $u \in U$, $v \in V$, $w = u \times v$, $p \in cs(E)$, the function $w \cdot (p \circ f)$ vanishes at infinity on $X \times Y$.

(3) $\|f\|_{w,p} < \infty$ for any $w = u \times v \in W$ and any $p \in cs(E)$.

Then (1), (2), (3) are equivalent. Indeed, it is easy to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To prove that (3) $\Rightarrow$ (1), assume that there exist a compact subset D_1 of X and $p \in cs(E)$ such that

$$\sup\{p(f(x,y)) : x \in D_1,\, y \in Y\} = \infty.$$

Let $|\lambda| > 1$ and choose a sequence (x_n) in D_1 and a sequence (y_n) of distinct elements of Y such that $p(f(x_n, y_n)) > |\lambda|^{2n}$. Let w_n be the $\mathbf{K}$-characteristic function of $\{y_1, \ldots, y_n\}$ and consider the function $\phi = \sum_{n=1}^{\infty} \lambda^{-n} w_n$. Then $v = |\phi| \in V$. If u is the $\mathbf{R}$-characteristic function of D_1, then $w = u \times v \in W$ and

$$u(x_n)v(y_n)p(f(x_n, y_n)) = |\lambda|^{-n} p(f(x_n, y_n)) \geq |\lambda|^n$$

and so $\|f\|_{w,p} = \infty$, a contradiction. Thus (1),(2),(3) are equivalent.

Let now U, V, W be as above and denote by $CW_\kappa(X \times Y, E)$ the vector space of all $f \in E^{X\times Y}$ such that:

(a) $f|_{D\times Y}$ is continuous for each compact subset D of X.

(b) $\|f\|_{w,p} < \infty$ for each $w \in W$ and each $p \in cs(E)$.

If we consider on $CW_\kappa(X \times Y, E)$ the weighted topology τ_w generated by the seminorms $\|\cdot\|_{w,p}$, $w \in W$, $p \in cs(E)$, we have

Theorem 4.7. *Let X, Y be zero-dimensional Hausdorff topological spaces and E a Hausdorff locally convex space. If τ_c is the topology of compact convergence, then:*

(1) *the map*

$$\omega : (C(X), \tau_c) \otimes (C_b(Y, E), \beta_0) \mapsto CW_\kappa(X \times Y, E), \quad f \otimes g \mapsto f \times g,$$

is a topological isomorphism onto a dense subspace G of $CW_\kappa(X \times Y, E)$;

(2) *if Y is locally compact, then*

$$(C(X), \tau_c)\hat{\otimes}(C_b(Y, E), \beta_0) \cong CW_\kappa(X \times Y, \hat{E}).$$

Proof. The mapping ω is a topological isomorphism onto G by Theorem 4.1, since $CW_0(X \times Y, E)$ is a topological subspace of $CW_\kappa(X \times Y, E)$. To prove that G is dense, let $f \in CW_\kappa(X \times Y, E)$, $w = u \times v \in W$ and $p \in cs(E)$.

We may assume that u is the $\mathbf{R}$-characteristic function of a compact subset D_1 of X. Given $\epsilon > 0$, let $D = \{(x,y) : x \in D_1,\, v(y)p(f(x,y)) \geq \epsilon\}$. If D_2 is the projection of D on Y, then D_2 is compact, since D is compact, and $D \subseteq D_1 \times D_2$. The restriction h of f to $D_1 \times D_2$ is continuous. Let $\epsilon_2 > 0$ with $\epsilon_2 \|v\| < \epsilon$. There are $(x_\kappa, y_\kappa) \in D_1 \times D_2$, $\kappa = 1, \ldots, n$, such that the sets $A_\kappa = \{s \in E : p(s - f(x_\kappa, y_\kappa)) \leq \epsilon_2\}$ are pairwise disjoint and cover $h(D_1 \times D_2)$. Set $B_\kappa = h^{-1}(A_\kappa)$. Clearly, B_κ is compact and $D_1 \times D_2 = \bigcup_\kappa B_\kappa$.

It is easy to see that if $C, C_1, \ldots, C_n$ are clopen in X and $F, F_1, \ldots, F_n$ clopen in Y, then the set

$$C \times F \setminus \left(\bigcup_{\kappa=1}^{n} C_\kappa \times F_\kappa \right)$$

is a finite disjoint union of sets of the form $Z_1 \times Z_2$, with Z_1 clopen in X and Z_2 clopen in Y.

There are pairwise disjoint sets $O_1 \ldots, O_n$ in $X \times Y$ with $B_\kappa \subseteq O_\kappa$. For $(x,y) \in B_\kappa$ there are clopen neighbourhoods M_x, D_y of x, y respectively such that $M_x \times D_y \subseteq O_\kappa$ and $p(f(x,y) - f(a,b)) \leq \epsilon_2$ for all $a \in M_x \cap D_1$ and $b \in D_y$. In view of the compactness of B_κ, there are clopen sets $A_{\kappa 1}, \ldots, A_{\kappa m_\kappa}$ in X and clopen sets $D_{\kappa 1}, \ldots, D_{\kappa m_\kappa}$ in Y such that the sets $A_{\kappa j} \times D_{\kappa j}$, $j = 1, \ldots, m_\kappa$, are pairwise disjoint, cover B_κ, are contained in O_κ and $p(f(x,y) - f(a,b)) \leq \epsilon_2$ if (x,y) and (a,b) are in $(A_{\kappa j} \cap D_1) \times D_{\kappa j}$.

Choose $(x_{\kappa j}, y_{\kappa j}) \in (A_{\kappa j} \cap D_1) \times D_{\kappa j}$ and set

$$g = \sum_{\kappa=1}^{n} \left(\sum_{j=1}^{m_\kappa} \mathcal{X}_{C_{\kappa j}} \times \left(\mathcal{X}_{F_{\kappa j}} f(x_\kappa, y_\kappa) \right) \right)$$

is in G. Moreover, $\|f - g\|_{w,p} \leq \epsilon$. Indeed, let $x \in D_1$, $y \in Y$.

Case I: $(x,y) \in A_{\kappa j} \times B_{\kappa j}$.

Then $g(x,y) = f(x_\kappa, y_\kappa)$ and so $p(f(x,y) - g(x,y)) \leq \epsilon_2$, which implies that

$$v(y)p(f(x,y) - g(x,y)) \leq \|v\|\epsilon_2 < \epsilon.$$

Case II: $(x,y) \notin \bigcup_{\kappa,j} A_{\kappa j} \times B_{\kappa j}$.

Then $g(x,y) = 0$ and $(x,y) \notin D$ and so

$$w(x,y)p(f(x,y) - g(x,y)) \leq v(y)p(f(x,y)) < \epsilon.$$

This proves the first part of the theorem.

(2) To prove the second part, we show first that $CW_\kappa(X \times Y, \hat{E})$ is complete. To this end, let (f_α) be a Cauchy net in $CW_\kappa(X \times Y, \hat{E})$.

Given $(x_0, y_0) \in X \times Y$, there exist $u \in U$, $v \in V$ with $u(x_0) > 0$, $v(y_0) > 0$. Using this, we get that the net $(f_\alpha(x_0, y_0))$ is Cauchy and hence convergent in $\hat{E}$. Define

$$f : X \times Y \mapsto \hat{E}, \quad f(x,y) = \lim f_\alpha(x,y).$$

(i) For each compact subset D_1 of X, $f|_{D_1 \times Y}$ is continuous. Indeed, let $x_0 \in D_1$ and $y_0 \in Y$. There exists a compact clopen neighbourhood W of $y_0 \in Y$.

If u, v are the **R**-characteristic functions of D_1, W, respectively, then $w = u \times v \in W$ and

$$\|f_\alpha - f_\beta\|_{w,p} = \sup\{p(f_\alpha(x,y) - f_\beta(x,y)) : x \in D_1,\, y \in W\}.$$

It follows that $f_\alpha \to f$ uniformly on $D_1 \times W$. Since $D_1 \times W$ is open in $D_1 \times Y$ and $(x_0, y_0) \in D_1 \times W$ it follows that f is continuous at (x_0, y_0) on $D_1 \times Y$.

(ii) If $w = u \times v \in W$, then $\|f\|_{w,p} < \infty$ for each $p \in cs(E)$. Indeed, there exists α_0 such that $\|f_{\alpha_0} - f_\alpha\|_{w,p} \le 1$, for all $\alpha \succeq \alpha_0$, which implies that

$$\|f_{\alpha_0} - f\|_{w,p} \le 1 \text{ and so } \|f\|_{w,p} \le \max\{1, \|f_{\alpha_0}\|_{w,p}\} < \infty.$$

It follows from the above that $f \in CW_\kappa(X \times Y, \hat{E})$ and $f_\alpha \to f$ in the topology τ_w. To finish the proof, it suffices to show that $CW_\kappa(X \times Y, E)$ is dense in $CW_\kappa(X \times Y, \hat{E})$. So, let $f \in CW_\kappa(X \times Y, \hat{E})$, $w = u \times v \in W$ and $p \in cs(E)$. As in the proof of the first part, there are clopen subsets $A_1, \ldots, A_n$ of X, clopen subsets $B_1, \ldots, B_n$ of Y and (x_κ, y_κ) in $X \times Y$ such that the sets $A_\kappa \times B_\kappa$, $\kappa = 1, \ldots, n$, are pairwise disjoint and $\|f - g\|_{w,p} \le 1$, where

$$g = \sum_{\kappa=1}^{n} \mathcal{X}_{A_\kappa} \times (\mathcal{X}_{B_\kappa} f(x_\kappa, y_\kappa)).$$

Since w is bounded, we have that $\|w\| = d < \infty$. For each κ, choose $s_\kappa \in E$ such that $p(s_\kappa - f(x_\kappa, y_\kappa)) < 1/d$. Now

$$h = \sum_{\kappa=1}^{n} \mathcal{X}_{A_\kappa} \times (\mathcal{X}_{B_\kappa} s_\kappa) \in G.$$

If $(x,y) \in A_\kappa \times B_\kappa$, then $g(x,y) = f(x_\kappa, y_\kappa)$, $h(x,y) = s_\kappa$, and so

$$w(x,y)p(f(x,y) - h(x,y)) \le$$
$$\le \max\{w(x,y)p(f(x,y) - g(x,y)), w(x,y)p(f(x_\kappa, y_\kappa) - s_\kappa)\} \le 1.$$

Thus $\|f - h\|_{w,p} \le 1$ and the result clearly follows. $\square$

Let $C_{\kappa,0}(X \times Y, E)$ denote the space of all E-valued functions f on $X \times Y$ such that $f|_{D_1 \times Y} \in C_0(D_1 \times Y, E)$ for each compact subset D_1 of X. If we consider on $C_{\kappa,0}(X \times Y, E)$ the locally convex topology generated by the seminorms $\|f\|_{D_1,p} = \sup\{p(f(x,y) : x \in D_1,\ y \in Y\}$, where $p \in cs(E)$ and D_1 is a compact subset of X, then we have

Theorem 4.8. *Let X, Y be zero-dimensional Hausdorff topological spaces, where Y is locally compact, and let E be a Hausdorff complete locally convex space. Then*

$$(C(X), \tau_c) \hat{\otimes} (C_0(Y, E), \tau_u) \cong C_{\kappa,0}(X \times Y, E).$$

Proof. The proof is analogous to the one of the preceding theorem, using an additional fact that the clopen compact subsets of Y form the base for the open subsets of Y. $\square$

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