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A RADIAL DERIVATIVE WITH BOUNDARY VALUES OF THE SPHERICAL POISSON INTEGRAL *

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Abstract. A formula of a radial derivative $\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi)$ is obtained with the aid of derivatives with respect to θ and to ϕ of the functions closely connected with the spherical Poisson integral $\mathcal{U}_f(r,\theta,\phi)$ and the boundary values are determined for $\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi)$. The boundary values are also found for partial derivatives with respect to the Cartesian coordinates $\frac{\partial}{\partial x}U_F$, $\frac{\partial}{\partial y}U_F$ and $\frac{\partial}{\partial z}U_F$.

Introduction

0.1. Let the function $F(X,Y,Z)$ be summable on the two-dimensional unit sphere σ with center at the origin. The Poisson integral corresponding to this function F will be denoted by $\mathcal{U}_F(x,y,z)$, where (x,y,z) is a point in the open unit ball $\mathbb{B}$ bounded by σ. This integral written in terms of spherical ccordinates will be denoted by $\mathcal{U}_f(r,\theta,\phi)$.

We shall represent the spehrical kernel of the Poisson integral P_r by the series (see, for instance, [2], pp. 335 and 143)

$$P_r(\theta,\phi;\theta',\phi') = 1 + \sum_{n=1}^{\infty}(2n+1)r^n P_n(\cos\theta)P_n(\cos\theta') +$$
$$+2\sum_{n=1}^{\infty}(2n+1)r^n\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\phi-\phi'). \quad (0.1)$$

0.2. In the case of the unit circle, the radial derivative of the Poisson integral $v_\lambda(r,\phi)$ for a function $\lambda \in L(0,2\pi)$ and the derivative with respect

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* The results of this paper were announced in the author's paper [1]. The terms used there are slightly different from those of this paper.

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to ϕ of the conjugate Poisson integral $\widetilde{v}_\lambda(r,\phi)$ are connected through the equality

$$\frac{\partial}{\partial r}\,v_\lambda(r,\phi)=\frac{1}{r}\,\frac{\partial}{\partial\phi}\,\widetilde{v}_\lambda(r,\phi).$$

As is well known, the above equality is based on the property of the function $v_\lambda(r,\phi)+i\widetilde{v}_r(r,\phi)$ to be analytic with respect to the complex variable $re^{i\phi}$, $0\le r<1$, $0\le\phi\le 2\pi$. The equality is immediately obtained from the representations of $v_\lambda(r,\phi)$ and $\widetilde{v}_\lambda(r,\phi)$ as the series

$$v_\lambda(r,\phi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n(a_n\cos n\phi+b_n\sin n\phi),$$

$$\widetilde{v}_\lambda(r,\phi)=\sum_{n=1}^{\infty}r^n(a_n\sin n\phi-b_n\cos n\phi)$$

since the exponent of r coincides with multiplicity of the polar angle ϕ. Moreover, if the conjugate function $\widetilde{\lambda}$ is summable on $(0,2\pi)$, then the equality $\widetilde{v}_\lambda(r,\phi)=v_{\widetilde{\lambda}}(r,\phi)$ holds by virtue of Smirnov's theorem ([3], p. 263; [4], p. 583). Therefore by the well-known Fatou's theorem ([3], p. 100) the radial derivative $\frac{\partial}{\partial r}\,v_r(r,\phi)$ will have an angular limit at the point $(1,\phi_0)$ if the conjugate function $\widetilde{\lambda}$ with $\widetilde{\lambda}(\phi)\in L(0,2\pi)$ has the finite derivative $(\widetilde{\lambda})'(\phi_0)$ at the point ϕ_0.

It should be noted in the first place that there exists no analogue of the theory of analytic functions of a complex variable in a three-dimensional real space. In the second place, the exponent of r in series (0.2) differs from the multiplicity of the polar angle ϕ. Hence to obtain a formula for $\frac{\partial}{\partial r}\,\mathcal{U}_f(r,\theta,\phi)$ we have to overcome some difficulties. That this is so is seen from the fact that in the final result the products $\cos m\phi\cdot P_{nm}(\cos\theta)$ and $\sin m\phi\cdot P_{nm}(\cos\theta)$ must remain unchanged after all transformations if we want to remain in the class of harmonic functions; such an intention is dictated by the problem itself. At the same time, the trigonometric system $(\cos m\phi,\sin m\phi)$ has some properties which the system $(P_{nm}(\cos(\theta))$ does not possess. For instance, a derivative of the trigonometric system is the same system to within a constant multiplier with respect to ϕ. The system $(P_{nm}(\cos(\theta))$ does not possess the latter property! Furthermore, the trigonometric system is bounded, while the system $(P_{nm}(\cos(\theta))$ is unbounded! These and other differences (for instance, the 2π-periodicity with respect to ϕ and the non-π-periodicity with respect to θ) between these two systems make series (0.2) have different properties with respect to θ and to ϕ (a similar observation is obviously true for series (0.3)). The same reason accounts for different preconditions with respect to θ and to ϕ, which are imposed on the boundary function $f(\theta,\phi)$ from equalities (0.2) and (0.3). Nevertheless the system of spherical functions $(P_n(\cos(\theta),\cos m\phi\cdot$

$P_{nm}(\cos\theta), \sin m\phi \cdot P_{nm}(\cos\theta))$ is successfully used to solve many important spatial problems of theoretical and mathematical physics as well as of mechanics. Theoretical geodesics which studies the external gravitational field of the Earth and other planets makes active use of the method of Laplace series.

0.3. In what follows we shall use the representation of the spherical Poisson integral as a series (see, e.g., [5], p. 444)

$$\mathcal{U}_f(r,\theta,\phi) = a_{00} + \sum_{n=1}^{\infty} a_{n0} r^n P_n(\cos\theta) + \\ + \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}(\cos\theta), \tag{0.2}$$

coinciding with the Abel–Poisson mean values (A-mean values) of the following Fourier–Laplace series $S[f]$ for the function $f(\theta,\phi) \in L(R)$ (see, e.g., [5], p. 444):

$$S[f] = a_{00} + \sum_{n=1}^{\infty} a_{n0} P_n(\cos\theta) + \\ + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}(\cos\theta). \tag{0.3}$$

The Legendre polynomials $P_n(x)$ and associated Legendre functions $P_{nm}(x)$ figuring in these equalities are defined on $[-1,1]$ by the following equalities:

$$P_n(x) = \frac{1}{n!2^n} \cdot \frac{d^n}{dx^n}(x^2-1)^n, \quad n = 0,1,2,\dots, \tag{0.4}$$

$$P_{nm}(x) = (1-x^2)^{\frac{1}{2}m} \frac{d^m}{dx^m} P_n(x) = \tag{0.5}$$

$$= \frac{(1-x^2)^{\frac{1}{2}m}}{n!2^n} \frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n, \tag{0.6}$$

$$1 \le m \le n, \quad n = 1,2,\dots,$$

where $(1-x^2)^{\frac{1}{2}}$ is a non-negative value of the square root. Note that we have $P_n^0(x) = P_n(x)$ and $P_{nm}(x) = 0$ for $m > n$.

0.4. The formula for $\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi)$ will contain the functions $\mathcal{U}_f^*$ and $\widetilde{\mathcal{U}}_f^*$ which are allied with $\mathcal{U}_f$ and defined as follows (see [6], §1). To the spherical

Poisson integral $\mathcal{U}_f(r,\theta,\phi)$, $f\in L(R)$, given in the form of series (0.2) there correspond the following functions allied with respect to θ and to (θ,ϕ):

$$\mathcal{U}_f^*(r,\theta,\phi)=\sum_{n=1}^{\infty}a_{n0}\lambda_{n0}r^nP_n(\cos\theta)+$$
$$+\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta) \tag{0.7}$$

and

$$\widetilde{\mathcal{U}}_f^*(r,\theta,\phi)=\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)P_{nm}(\cos\theta), \tag{0.8}$$

where the numbers λ_{nm} are given by the equalities

$$\lambda_{nm}=\frac{1}{n+m}+\frac{1}{n-m+1}\quad\text{for}\quad 0\le m\le n,\ \ n=1,2,\dots. \tag{0.9}$$

Analogously, to the spherical Poisson kernel $P_r(\theta,\phi;\theta',\phi')$ there correspond the following kernels allied with respect to θ and (θ,ϕ) (see [6], §2):

$$P_r^*(\theta,\phi;\theta',\phi')=\sum_{n=1}^{\infty}(2n+1)\lambda_{n0}r^nP_n(\cos\theta)P_n(\cos\theta')+$$
$$+2\sum_{n=1}^{\infty}(2n+1)r^n\sum_{m=1}^{n}\lambda_{nm}\frac{(n-m)!}{(n+m)!}\times$$
$$\times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\phi-\phi'), \tag{0.10}$$
$$\widetilde{P}_r^*(\theta,\phi;\theta',\phi')=$$
$$+2\sum_{n=1}^{\infty}(2n+1)r^n\sum_{m=1}^{n}\lambda_{nm}\frac{(n-m)!}{(n+m)!}\times$$
$$\times P_{nm}(\cos\theta)P_{nm}(\cos\theta')\sin m(\phi-\phi'). \tag{0.11}$$

Finally, for each function $f\in L^2(R)$ there are functions $f^*(\theta,\phi)\in L^2(R)$ and $\widetilde{f}^*(\theta,\phi)\in L^2(R)$ allied with respect to θ and to (θ,ϕ) respectively, such that (see [6], §4)

$$\mathcal{U}_f^*(r,\theta,\phi)=\mathcal{U}_{f^*}(r,\theta,\phi),\quad \widetilde{\mathcal{U}}_f^*(r,\theta,\phi)=\mathcal{U}_{\widetilde{f}^*}(r,\theta,\phi). \tag{0.12}$$

§ 1. A Radial Derivative of the Spherical Poisson Integral and Kernel

Our aim now is to derive a formula for the radial derivative $\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi)$ of the spherical Poisson integral $\mathcal{U}_f(r,\theta,\phi)$ for $f \in L(R)$, and also a formula for $\frac{\partial}{\partial r} P_r$. At the end of the paper a formula for $\frac{\partial^2}{\partial r^2}\mathcal{U}_f(r,\theta,\phi)$ will be given too.

1.1. For the derivative $\frac{\partial}{\partial r}\mathcal{U}_f$ we have

Theorem 1.1. *For every function $f(\theta,\phi) \in L(R)$ the equality*

$$2r\,\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi) + \mathcal{U}_f(r,\theta,\phi) - a_{00} = \\ = \frac{\partial^2}{\partial\theta^2}\mathcal{U}_f^*(r,\theta,\phi) + \cot\theta\,\frac{\partial}{\partial\theta}\mathcal{U}_f^*(r,\theta,\phi) + \cot^2\theta\,\frac{\partial^2}{\partial\phi^2}\mathcal{U}_f^*(r,\theta,\phi) - \\ - \frac{\partial}{\partial\phi}\widetilde{\mathcal{U}}_f^*(r,\theta,\phi) \tag{1.1}$$

holds, where the allied harmonic functions $\mathcal{U}_f^*$ and $\widetilde{\mathcal{U}}_f^*$ in the ball $\mathbb{B}$ are defined by equalities (0.7) *and* (0.8).

By introducing the operator

$$T_{\theta,\phi} = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial}{\partial\theta}\Big) + \cot^2\theta\,\frac{\partial^2}{\partial\phi^2} = \\ = \frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \cot^2\theta\,\frac{\partial^2}{\partial\phi^2}\ , \tag{1.2}$$

equality (1.1) *can be rewritten as*

$$2r\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi) + \mathcal{U}_f(r,\theta,\phi) - a_{00} = \\ = T_{\theta,\phi}\mathcal{U}_f^*(r,\theta,\phi) - \frac{\partial}{\partial\phi}\widetilde{\mathcal{U}}_f^*(r,\theta,\phi). \tag{1.3}$$

Proof. From equality (0.2) we obtain

$$r\frac{\partial}{\partial r}\mathcal{U}_f(r,\theta,\phi) = \sum_{n=1}^{\infty} nr^n a_{n0}P_n(\cos\theta) + \\ + \sum_{n=1}^{\infty} nr^n \sum_{m=1}^{n}\big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big)P_{nm}(\cos\theta). \tag{1.4}$$

Our next aim is to find convenient expressions for $nP_n(\cos\theta)$ and $nP_{nm}(\cos\theta)$. If in equality (0.5) we replace $P_n(x)$ by its value from the equality

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad P_{-1}(x) = 0$$

(see,for instance, [2], p. 33, equality (34), or [7], p. 228, equality (7.8.2)), then we obtain the relations

$$\begin{aligned}(2n+1)P_{nm}(x) &= (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} \big[P'_{n+1}(x) - P'_{n-1}(x)\big] = \\ &= (1-x^2)^{-\frac{1}{2}} \Big[(1-x^2)^{\frac{m+1}{2}} \frac{d^{m+1}}{dx^{m+1}} P_{n+1}(x) - \\ &\quad - (1-x^2)^{\frac{m+1}{2}} \frac{d^{m+1}}{dx^{m+1}} P_{n-1}(x)\Big] = \\ &= (1-x^2)^{-\frac{1}{2}} \big[P_{n+1,m+1}(x) - P_{n-1,m+1}(x)\big].\end{aligned}$$

Hence

$$\begin{gathered}(2n+1)P_{nm}(x) = \\ = (1-x^2)^{-\frac{1}{2}} \big[P_{n+1,m+1}(x) - P_{n-1,m+1}(x)\big], \quad -1 < x < 1.\end{gathered} \tag{1.5}$$

Further, the differentiation of equality (0.5) gives

$$\begin{aligned}\frac{d}{dx} P_{nm}(x) &= -\frac{mx}{1-x^2} (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x) + \\ &\quad + \frac{(1-x^2)^{\frac{m+1}{2}}}{(1-x^2)^{\frac{1}{2}}} \cdot \frac{d^{m+1}}{dx^{m+1}} P_n(x) = \\ &= -\frac{mx}{1-x^2} P_{nm}(x) + \frac{1}{(1-x^2)^{\frac{1}{2}}} P_{n,m+1}(x).\end{aligned}$$

Thus

$$\frac{d}{dx} P_{nm}(x) = \frac{1}{(1-x^2)^{\frac{1}{2}}} P_{n,m+1}(x) - \frac{mx}{1-x^2} P_{nm}(x) \tag{1.6}$$

from which we have

$$P_{n,m+1}(x) = \frac{mx}{(1-x^2)^{\frac{1}{2}}} P_{nm}(x) + (1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_{nm}(x). \tag{1.7}$$

Using the latter equality to define successively $P_{n+1,m+1}(x)$, $P_{n-1,m+1}$ and substituting them into (1.5), we obtain the equality

$$\begin{aligned}(2n+1)P_{nm}(x) &= \frac{d}{dx} \big[P_{n+1,m}(x) - P_{n-1,m}(x)\big] + \\ &\quad + \frac{mx}{1-x^2} \big[P_{n+1,m}(x) - P_{n-1,m}(x)\big]\end{aligned} \tag{1.8}$$

which for $x = \cos\theta$ and $0 < \theta < \pi$ can be rewritten as

$$(2n+1)P_{nm}(\cos\theta) = \frac{1}{\sin\theta}\,\frac{d}{d\theta}\,\big[P_{n-1,m}(\cos\theta) - P_{n+1,m}(\cos\theta)\big] + \\ +m\,\frac{\cos\theta}{\sin^2\theta}\,\big[P_{n+1,m}(\cos\theta) - P_{n-1,m}(\cos\theta)\big]. \qquad (1.9)$$

Hence we find

$$nP_{nm}(\cos\theta) = \frac{1}{2\sin\theta}\,\frac{d}{d\theta}\,\big[P_{n-1,m}(\cos\theta) - P_{n+1,m}(\cos\theta)\big] + \\ +\,m\,\frac{\cos\theta}{2\sin^2\theta}\,\big[P_{n+1,m}(\cos\theta) - P_{n-1,m}(\cos\theta)\big] - \\ -\,\frac{1}{2}\,P_{nm}(\cos\theta). \qquad (1.10)$$

For $m = 0$ the latter equality implies the equality

$$nP_n(\cos\theta) = \frac{1}{2\sin\theta}\,\frac{d}{d\theta}\,\big[P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\big] - \frac{1}{2}\,P_n(\cos\theta). \qquad (1.11)$$

By the substitution of the expressions we have found for $nP_n(\cos\theta)$ and $nP_{nm}(\cos\theta)$ into (1.4) we obtain the following equality, using in doing so the right of termwise differentiation of the sries (see the arguments from [6], §1):

$$2r\,\frac{\partial}{\partial r}\,\mathcal{U}_f(r,\theta,\phi) = \\ = \frac{1}{\sin\theta}\,\frac{d}{d\theta}\sum_{n=1}^{\infty} a_{n0}r^n\big[P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\big] - \sum_{n=1}^{\infty} a_{n0}r^n P_n(\cos\theta) + \\ +\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) \times \\ \times\big[P_{n-1,m}(\cos\theta) - P_{n+1,m}(\cos\theta)\big] + \\ +\frac{\cos\theta}{\sin^2\theta}\sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} m\big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) \times \\ \times\big[P_{n+1,m}(\cos\theta) - P_{n-1,m}(\cos\theta)\big] - \\ -\sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big)P_{nm}(\cos\theta).$$

Hence

$$2r\,\frac{\partial}{\partial r}\,\mathcal{U}_f(r,\theta,\phi) + \mathcal{U}_f(r,\theta,\phi) - a_{00} =$$

$$
\begin{aligned}
&= \frac{1}{\sin\theta}\,\frac{d}{d\theta}\sum_{n=1}^{\infty} a_{n0}r^n\big[P_{n-1}(\cos\theta)-P_{n+1}(\cos\theta)\big]+\\
&+\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sum_{n=1}^{\infty} r^n\sum_{m=1}^{n}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)\times\\
&\qquad\times\big[P_{n-1,m}(\cos\theta)-P_{n+1,m}(\cos\theta)\big]+\\
&+\frac{\cos\theta}{\sin^2\theta}\,\frac{\partial}{\partial\phi}\sum_{n=1}^{\infty} r^n\sum_{m=1}^{n}\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)\times\\
&\qquad\times\big[P_{n+1,m}(\cos\theta)-P_{n-1,m}(\cos\theta)\big]\equiv\\
&\qquad\equiv\frac{1}{\sin\theta}\,A+\frac{1}{\sin\theta}\,B+\frac{\cos\theta}{\sin^2\theta}\,C. \qquad (1.12)
\end{aligned}
$$

Now we are to find expressions for the differences contained within the square brackets from equality (1.12).

The well-known equality ([2], p. 107, equality (41)[1] with replaced $\cos\theta = x$, $\cot\theta = x(1-x^2)^{-\frac{1}{2}}$; [7], p. 241, equality (7.12.9), or [8], p. 99, equality $(46')$ with the same replacement)

$$
\begin{aligned}
(-1)^{m+2}P_{n,m+2}(x)+(-1)^{m+1}\frac{2(m+1)x}{(1-x^2)^{\frac{1}{2}}}\,P_{n,m+1}(x)+\\
+(-1)^m(n-m)(n+m+1)P_{nm}(x)=0 \qquad (1.13)
\end{aligned}
$$

implies, after reducing by $(-1)^m$ and replacing m by $(m-1)$, the equality

$$
P_{n,m+1}(x)=\frac{2mx}{(1-x^2)^{\frac{1}{2}}}\,P_{nm}(x)+(n+m)(n-m+1)P_{n,m-1}(x). \qquad (1.14)
$$

By (1.7) and (1.14) we have the equality

$$
\begin{aligned}
(1-x^2)\,\frac{d}{dx}\,P_{nm}(x)=mxP_{nm}(x)+\\
+(n+m)(n-m+1)(1-x^2)^{\frac{1}{2}}P_{n,m-1}(x), \quad -1<x<1. \qquad (1.15)
\end{aligned}
$$

Next, if in equality (1.5) m is replaced by $(m-1)$, then we obtain the equality

$$
\begin{aligned}
P_{n+1,m}(x)-P_{n-1,m}(x)=(2n+1)(1-x^2)^{\frac{1}{2}}P_{n,m-1}(x), \qquad (1.16)\\
1\le m\le n, \quad -1<x<1.
\end{aligned}
$$

[1]Here the Legendre functions are defined together with multipliers $(-1)^k$. This fact is taken into account in equality (1.13), i.e., in (1.13) $P_{nk}(x)$ are defined by equality (0.5).

Now, by equalities (1.15) and (1.16) we find the product $(1-x^2)^{\frac{1}{2}} P_{n,m-1}(x)$ and equate them to each other. This results in the equality

$$P_{n+1,m}(x) - P_{n-1,m}(x) = \frac{(2n+1)(1-x^2)}{(n+m)(n-m+1)} \frac{d}{dx} P_{nm}(x) - \\ - \frac{m(2n+1)x}{(n+m)(n-m+1)} P_{nm}(x), \quad -1 < x < 1. \tag{1.17}$$

Thus

$$P_{n-1,m}(\cos\theta) - P_{n+1,m}(\cos\theta) = \frac{(2n+1)\sin\theta}{(n+m)(n-m+1)} \frac{d}{d\theta} P_{nm}(\cos\theta) + \\ + \frac{m(2n+1)\cos\theta}{(n+m)(n-m+1)} P_{nm}(\cos\theta), \quad 0 < \theta < \pi. \tag{1.18}$$

For $m = 0$ equality (1.18) gives rise to the equality

$$P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta) = \frac{(2n+1)\sin\theta}{n(n+1)} \frac{d}{d\theta} P_n(\cos\theta), \\ 0 < \theta < \pi. \tag{1.19}$$

Now, taking into account equalities (1.18) and (1.19) we define A, B and C appearing in (1.12). In doing so, we make use of the numbers λ_{nm} from equality (0.9).

$$A = \frac{d}{d\theta}\Big(\sin\theta \frac{d}{d\theta} \sum_{n=1}^{\infty} \lambda_{n0} a_{n0} r^n P_n(\cos\theta)\Big) = \\ = \cos\theta \frac{d}{d\theta} \sum_{n=1}^{\infty} \lambda_{n0} a_{n0} r^n P_n(\cos\theta) + \\ + \sin\theta \frac{d^2}{d\theta^2} \sum_{n=1}^{\infty} \lambda_{n0} a_{n0} r^n P_n(\cos\theta). \tag{1.20}$$

Further,

$$B = \frac{\partial}{\partial\theta}\Big[\sin\theta \frac{\partial}{\partial\theta} \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \lambda_{nm} \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}(\cos\theta)\Big] + \\ + \frac{\partial}{\partial\theta}\Big[\cos\theta \sum_{n=1}^{\infty} r^n \sum_{m=1}^{n} \lambda_{nm} m \big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) P_{nm}(\cos\theta)\Big].$$

After performing the above-indicated differentiations and taking into account the equality

$$m\big(a_{nm}\cos m\phi + b_{nm}\sin m\phi\big) = \frac{\partial}{\partial\phi}\big(a_{nm}\sin m\phi - b_{nm}\cos m\phi\big),$$

we have

$$B=$$

$$=\cos\theta\,\frac{\partial}{\partial\theta}\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta)+$$

$$+\sin\theta\,\frac{\partial^2}{\partial\theta^2}\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta)-$$

$$-\sin\theta\,\frac{\partial}{\partial\phi}\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)P_{nm}(\cos\theta)+$$

$$+\cos\theta\,\frac{\partial^2}{\partial\theta\partial\phi}\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)P_{nm}(\cos\theta).\quad(1.21)$$

In a similar manner we obtain

$$C=-\sin\theta\,\frac{\partial^2}{\partial\theta\partial\phi}\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\times$$

$$\times\big(a_{nm}\sin m\phi-b_{nm}\cos m\phi\big)P_{nm}(\cos\theta)+$$

$$+\cos\theta\,\frac{\partial^2}{\partial\phi^2}\sum_{n=1}^{\infty}r^n\sum_{m=1}^{n}\lambda_{nm}\big(a_{nm}\cos m\phi+b_{nm}\sin m\phi\big)P_{nm}(\cos\theta).\quad(1.22)$$

The substitution of the found values of A, B and C into equality (1.12) shows that the one-dimensional series with $\cot\theta\,\frac{d}{d\theta}$ and the double-series with $\cot\theta\,\frac{\partial}{\partial\theta}$ are summable. The situation for series with $\frac{\partial^2}{\partial\theta^2}$ is similar. In the obtained series we have $0\le m\le n$. The sum of the series with $\cot\theta\,\frac{\partial^2}{\partial\theta\partial\phi}$ and $(-\cot\theta\,\frac{\partial^2}{\partial\theta\partial\phi})$ is equal to zero. Further, in the series with $\cot^2\theta\,\frac{\partial^2}{\partial\phi^2}$, where $1\le m\le n$, we add $\lambda_{n0}a_{n0}r^nP_n(\cos\theta)$ which does not depend on ϕ. This will not affect the result, since the derivative has been taken with respect to ϕ. Now m will vary from zero to n inclusive.

Summarizing the above arguments, we obtain equality (1.1) with the functions $\mathcal{U}_f^*(r,\theta,\phi)$ and $\widetilde{\mathcal{U}}_f^*(r,\theta,\phi)$ from equalities (0.7) and (0.8). It should be noted that formula (1.1) contains no mixed derivative $\frac{\partial^2}{\partial\theta\partial\phi}$ whose presence would make the situation much more difficult on the whole. □

1.2. By the method of proving Theorem 1.1 we can obtain a formula for the radial derivative $\frac{\partial}{\partial r}P_r$ of the Poisson kernel P_r.

Theorem 1.2. *The equalities*

$$2r\,\frac{\partial}{\partial r}P_r+P_r-1=$$

$$\frac{\partial^2}{\partial\theta^2} P_r^* + \cot\theta \frac{\partial}{\partial\theta} P_r^* + \cot^2\theta \frac{\partial^2}{\partial\phi^2} P_r^* - \frac{\partial}{\partial\phi} \widetilde{P}_r^*, \tag{1.23}$$

$$2r \frac{\partial}{\partial r} P_r + P_r - 1 = T_{\theta,\phi} P_r^* - \frac{\partial}{\partial\phi} \widetilde{P}_r^* \tag{1.24}$$

hold, where the allied harmonic kernels P_r^* *and* $\widetilde{P}_r^*$ *in the ball* $\mathbb{B}$ *are defined by equalities* (0.10) *and* (0.11)*, and the operator* $T_{\theta,\phi}$ *by equality* (1.2)*.*

Proof. If we itroduce the values $A_{00} = 1$, $A_{n0} = (2n+1)P_n(\cos\theta')$, $A_{nm} = 2(2n+1)\frac{(n-m)!}{(n+m)!}P_{nm}(\cos\theta')\cos m\phi'$, $B_{nm} = 2(2n+1)\frac{(n-m)!}{(n+m)!}P_{nm}(\cos\theta')\sin m\phi'$, and use the equality $\cos m(\phi - \phi') = \cos m\phi \cos m\phi' + \sin m\phi \sin m\phi'$, then series (0.1) will take the form of series (0.2) with the coefficients we have just introduced. From the resulting series we can obtain equality (1.23). □

§ 2. Boundary values of the radial derivative $\frac{\partial}{\partial r}\mathcal{U}_f$

The formula obtained in §1 for $\frac{\partial}{\partial r}\mathcal{U}_f$ makes it possible to find the boundary values of $\frac{\partial}{\partial r}\mathcal{U}_f$ with the aid of the author's previous results. By equality (1.1) it is clear that the fact that $\frac{\partial}{\partial r}\mathcal{U}_f$ has an angular limit (limit) imposes restrictions of differential nature on the functions f^* and $\widetilde{f}^*$.

2.1. The corresponding result for $f \in L^2(R)$ consists in the following statement:

Theorem 2.1. *Let the function* $f \in L^2(R)$*. If the function* f *is continuous at the point* (θ_0, ϕ_0) *with* $0 < \theta_0 < \pi$*, the function* $\widetilde{f}^*$ *has a continuous partial derivative at* (θ_0, ϕ_0)*, and the function* f^* *is twice differentiable*[2] (*twice continuously differentiable*) *at the point* (θ_0, ϕ_0)*, then the radial derivative* $\frac{\partial}{\partial r}\mathcal{U}_f(r, \theta, \phi)$ *has an angular limit* (*has a limit*) *at the point* $(1, \theta_0, \phi_0)$ *which is equal to the value of*

$$\frac{1}{2}\Big[a_{00} - f(\theta_0,\phi_0) + \frac{\partial^2 f^*}{\partial\theta^2}(\theta_0,\phi_0) + \cot\theta_0 \frac{\partial f^*}{\partial\theta}(\theta_0,\phi_0) + \cot^2\theta_0 \frac{\partial^2 f^*}{\partial\phi^2}(\theta_0,\phi_0) - \frac{\partial \widetilde{f}^*}{\partial\phi}(\theta_0,\phi_0)\Big] \tag{2.1}$$

or, which is the same,

$$\frac{1}{2}\Big[a_{00} - f(\theta_0,\phi_0) + T_{\theta,\phi} f^*(\theta_0,\phi_0) - \frac{\partial \widetilde{f}^*}{\partial\phi}(\theta_0,\phi_0)\Big]. \tag{2.2}$$

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[2]The function $\lambda(x,y)$ is called twice differentiable at the point (x_0,y_0) if in a neighborhood of (x_0,y_0) there exist partial derivatives $\lambda'_x(x,y)$, $\lambda'_y(x,y)$ and they are differentiable at (x_0,y_0). Moreover, if $\lambda'_x(x,y)$ and $\lambda'_y(x,y)$ are continuously differentiable at the point (x_0,y_0), then the function $\lambda(x,y)$ is called twice continuously differentiable at (x_0,y_0).

Proof. The fact that the function f is continuous at the point (θ_0,ϕ_0) implies that by the well known theorem (see, for instance, [9]. p. 209; [10], p. 243) the Poisson integral $\mathcal{U}_f(r,\theta,\phi)$ has a limit at (θ_0,ϕ_0) (though instead of the continuity it is sufficient that f have a finite limit at (θ_0,ϕ_0)). Since $f\in L^2(R)$, the functions $\mathcal{U}_f^*$ and $\widetilde{\mathcal{U}}_f^*$ in equality (1.1) can be replaced by $\mathcal{U}_{f^*}$ and $\mathcal{U}_{\widetilde{f}^*}$, respectively, in accordance with equalities (0.12). The continuity of the partial derivative $\frac{\partial}{\partial\phi}\widetilde{f}^*$ at the point (θ_0,ϕ_0) implies that $\frac{\partial}{\partial\phi}\mathcal{U}_{\widetilde{f}^*}$ (see [11], Theorem 8.2) has a limit at the point $(1,\theta_0,\phi_0)$ which is equal to the value of $\frac{\partial\widetilde{f}^*}{\partial\phi}(\theta_0,\phi_0)$. Further, since the function f^* is twice differentiable (twice continuously differentiable) at the point (θ_0,ϕ_0), all second order derivatives of $\mathcal{U}_{f^*}$ have angular limits at the point $(1,\theta_0,\phi_0)$ (limits) which are equal to the values of the corresponding derivatives of f^* at (θ_0,ϕ_0) (see [12], Theorems 3.6 and 5.5). □

§ 3. Boundary values of derivatives with respect to Cartesian derivatives of the Poisson integral

Applying the well known rules, derivatives with respect to the Cartesian coordinates of $U_F(x,,y,z)$ can be expressed by derivatives with respect to spherical coordinates (for $-1<z<1$, $0<\theta<\pi$, $0<r<1$) as follows (see, for instance, [13], p.107):

$$\frac{\partial}{\partial x}U_F=\sin\theta\cos\phi\,\frac{\partial}{\partial r}\mathcal{U}_f+\frac{1}{r}\,\cos\theta\cos\phi\,\frac{\partial}{\partial\theta}\mathcal{U}_f-\\ -\frac{1}{r}\,\frac{\sin\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}\mathcal{U}_f, \tag{3.1}$$

$$\frac{\partial}{\partial y}U_F=\sin\theta\sin\phi\,\frac{\partial}{\partial r}\mathcal{U}_f+\frac{1}{r}\,\cos\theta\sin\phi\,\frac{\partial}{\partial\theta}\mathcal{U}_f+\\ +\frac{1}{r}\,\frac{\cos\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}\mathcal{U}_f, \tag{3.2}$$

$$\frac{\partial}{\partial z}U_F=\cos\theta\,\frac{\partial}{\partial r}\mathcal{U}_f-\frac{1}{r}\,\sin\theta\,\frac{\partial}{\partial\theta}\mathcal{U}_f. \tag{3.3}$$

The derivatives figuring in the right-hand sides of equalities (3.1)–(3.3) have angular limits (limits) at the point $(1,\theta_0,\phi_0)$ with $0<\theta_0<\pi$, if $f(\theta,\phi)$ is differentiable (continuously differentiable) at the point (θ_0,ϕ_0) (see [11], Theorems 6.1 and 8.3). On the other hand, for $\frac{\partial}{\partial r}\mathcal{U}_f$ to have an angular limit (limit), we had to impose certain restrictions on f^* and $\widetilde{f}^*$. Hence the following statements are valid.

Theorem 3.1. *Let the function $f \in L^2(R)$ be differentiable (continuously differentiable) at the point (θ_0, ϕ_0) with $0 < \theta_0 < \pi$. If the function $\widetilde{f}^*$ has a partial derivative $\frac{\partial}{\partial\phi}\widetilde{f}^*$ continuous at (θ_0, ϕ_0) and the function f^* is twice differentiable (twice continuously differentiable) at the point (θ_0, ϕ_0), then the derivatives of* (3.1)–(3.3) *have angular limits (have limits) at $(1, \theta_0, \phi_0)$ which are calculated by equalities* (3.1)–(3.3), *in whose right-hand sides (θ, ϕ) is replaced by (θ_0, ϕ_0), $\frac{\partial}{\partial\theta}\mathcal{U}_f$ and $\frac{\partial}{\partial\phi}\mathcal{U}_f$ by $\frac{\partial f}{\partial\theta}(\theta_0, \phi_0)$ and $\frac{\partial f}{\partial\phi}(\theta_0, \phi_0)$, respectively, and, finally $\frac{\partial}{\partial r}\mathcal{U}_f$ is replaced by value* (2.1) *or* (2.2).

Theorem 3.2. *Let for the function $f \in L(R)$ there exist summable allied functions f^* and $\widetilde{f}^*$ on R such that $\mathcal{U}_f^* = \mathcal{U}_{f^*}$ and $\widetilde{\mathcal{U}}_f^* = \mathcal{U}_{\widetilde{f}^*}$. If the function f is differentiable (continuously differentiable) at the point (θ_0, ϕ_0) with $0 < \theta_0 < \pi$, the function $\widetilde{f}^*$ has a partial derivative $\frac{\partial}{\partial\phi}\widetilde{f}^*$ continuous at (θ_0, ϕ_0) and the function f^* is twice differentiable (twice continuously differentiable) at the point (θ_0, ϕ_0), then the derivatives of* (3.1)–(3.3) *have angular limits (have limits) at the point $(1, \theta_0, \phi_0)$ which are calculated by equalities* (3.1)–(3.3) *in whose right-hand sides (θ, ϕ) is replaced by (θ_0, ϕ_0), $\frac{\partial}{\partial\theta}\mathcal{U}_f$ and $\frac{\partial}{\partial\phi}\mathcal{U}_f$ by $\frac{\partial f}{\partial\theta}(\theta_0, \phi_0)$ and $\frac{\partial f}{\partial\phi}(\theta_0, \phi_0)$ respectively and, finally $\frac{\partial}{\partial r}\mathcal{U}_f$ is replaced by value* (2.1) *or* (2.2).

Remark 3.1. Since the spherical Poisson integral is a harmonic function in the open unit ball, it satisfies the Laplace equation

$$2r\,\frac{\partial\mathcal{U}_f}{\partial r} + r^2\,\frac{\partial^2\mathcal{U}_f}{\partial r^2} + \frac{\partial^2\mathcal{U}_f}{\partial\theta^2} + \cot\theta\,\frac{\partial\mathcal{U}_f}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2\mathcal{U}_f}{\partial\phi^2} = 0. \tag{3.4}$$

If the first term here is replaced by its value from formula (1.3), then for $r^2\,\frac{\partial^2\mathcal{U}_f}{\partial r^2}$ we shall have

$$r^2\,\frac{\partial^2\mathcal{U}_f}{\partial r^2} = -a_{00} + \mathcal{U}_f - T_{\theta,\phi}\mathcal{U}_f^* + \\ + \frac{\partial}{\partial\phi}\widetilde{\mathcal{U}}_f^* - \frac{\partial^2\mathcal{U}_f}{\partial\theta^2} - \cot\theta\,\frac{\partial\mathcal{U}_f}{\partial\theta} - \frac{1}{\sin^2\theta}\,\frac{\partial^2\mathcal{U}_f}{\partial\phi^2}\,. \tag{3.5}$$

To find the boundary values for $\frac{\partial^2\mathcal{U}_f}{\partial r^2}$ it is sufficient to use in an appropriate manner the same facts as we used in finding the boundary values for $\frac{\partial\mathcal{U}_f}{\partial r}$.

In a similar manner we obtain the formula

$$r^2\frac{\partial^2 P_r}{\partial r^2} = -\,1 + P_r - T_{\theta,\phi}P_r^* + \frac{\partial\widetilde{P}_r^*}{\partial\phi} - \frac{\partial^2 P_r}{\partial\theta^2} - \\ - \cot\theta\,\frac{\partial P_r}{\partial\theta} - \frac{1}{\sin^2\theta}\,\frac{\partial^2 P_r}{\partial\phi^2}\,. \tag{3.6}$$

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