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ON THE NUMBER OF REPRESENTATIONS OF INTEGERS BY QUADRATIC FORMS IN TWELVE VARIABLES

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ABSTRACT. A way of finding exact explicit formulas for the number of representations of positive integers by quadratic forms in 12 variables with integral coefficients is suggested.

INTRODUCTION

In [1] we derived a formula for the sum of singular series corresponding to the number of representations of positive integers by a positive primitive quadratic form

$$f = a_1(x_1^2+x_2^2)+a_2(x_3^2+x_4^2)+a_3(x_5^2+x_6^2)+ \\ +a_4(x_7^2+x_8^2)+a_5(x_9^2+x_{10}^2)+a_6(x_{11}^2+x_{12}^2) \tag{1}$$

with integral coefficients $a_1, a_2, \dots, a_6$.

In [2, 3], several classes of entire modular forms of weight 6 are constructed for the congruence group $\Gamma_0(4N)$ with Fourier coefficients in a simple arithmetical sense, which allows one sometimes to obtain explicit exact formulas for the number of representations of positive integers by quadratic forms of type (1).

In the present paper we have obtained such formulas for the forms

$$\begin{aligned} f_1 &= x_1^2+\cdots+x_{10}^2+2(x_{11}^2+x_{12}^2), \\ f_2 &= x_1^2+\cdots+x_6^2+2(x_7^2+\cdots+x_{12}^2), \\ f_3 &= x_1^2+x_2^2+2(x_3^2+\cdots+x_{12}^2), \\ f_4 &= x_1^2+\cdots+x_8^2+2(x_9^2+x_{10}^2)+4(x_{11}^2+x_{12}^2). \end{aligned} \tag{2}$$

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Namely, we prove the following formulas:

$$\text{(I)}\quad r(n;f_1)=4\sigma_5(u)-8\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2,x_1>0,x_2>0}} x_1^4-3x_1^2x_2^2 \quad\text{if}\quad n\equiv 1 \pmod 2,$$

$$=128\sigma_5(u)+56\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=2n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\cdots,x_4>0}}\Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4$$

$$\text{if}\quad n\equiv 2 \pmod 4,$$

$$=\frac{72}{31}\,(2^{5(\alpha-1)}55+7)\sigma_5(u)\quad\text{if}\quad n\equiv 0 \pmod 4;$$

$$\text{(II)}\quad r(n;f_2)=\sigma_5(u)-6\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2,x_1>0,x_2>0}} x_1^4-3x_1^2x_2^2-$$

$$-\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=4n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\cdots,x_4>0}}\Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4$$

$$\text{if}\quad n\equiv 1 \pmod 2,$$

$$=32\sigma_5(u)+40\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=2n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\cdots,x_4>0}}\Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4$$

$$\text{if}\quad n\equiv 2 \pmod 4,$$

$$=\frac{24}{31}\,(2^{5(\alpha-1)}41+21)\sigma_5(u)\quad\text{if}\quad n\equiv 0 \pmod 4;$$

$$\text{(III)}\quad r(n;f_3)=\frac{1}{4}\,\sigma_5(u)-2\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2,x_1>0,x_2>0}} x_1^4-3x_1^2x_2^2-$$

$$-\frac{1}{4}\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=4n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\cdots,x_4>0}}\Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4$$

$$\text{if}\quad n\equiv 1 \pmod 2,$$

$$=8\sigma_5(u)+16\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=2n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\cdots,x_4>0}}\Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4$$

$$\text{if}\quad n\equiv 2 \pmod 4,$$

$$=\frac{24}{31}\,(2^{5\alpha-4}5+21)\sigma_5(u)\quad\text{if}\quad n\equiv 0 \pmod 4;$$

(IV) $$r(n;f_4) = \sigma_5(u) + 15 \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=4n \\ 2\nmid x_1,\ldots,2\nmid x_4, x_1>0,\cdots,x_4>0}} \Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4 \quad \text{if } n\equiv 1 \pmod 4,$$

$$= \sigma_5(u) + 15 \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=4n \\ 2\nmid x_1,\ldots,2\nmid x_4, x_1>0,\cdots,x_4>0}} \Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4 -$$

$$- 256 \sum_{\substack{2(x_1^2+x_2^2)+x_3^2+x_4^2=2n \\ 2\nmid x_1,\ldots,2\nmid x_4, x_1>0,\cdots,x_4>0}} x_1^4 - 3x_1^2x_2^2 +$$

$$+ 8 \sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n \\ 2\nmid x_1, 2\nmid x_2, x_1>0, x_2>0}} x_1^4 - 3x_1^2x_2^2 \quad \text{if } n\equiv 3 \pmod 4,$$

$$= 32\sigma_5(u) - 56 \sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=n \\ 2\nmid x_1, 2\nmid x_2, x_1>0, x_2>0}} x_1^4 - 3x_1^2x_2^2 -$$

$$- 28 \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=2n \\ 2\nmid x_1,\ldots,2\nmid x_4, x_1>0,\cdots,x_4>0}} \Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4 \quad \text{if } n\equiv 2 \pmod 4,$$

$$= 1024\sigma_5(u) + 568 \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=n \\ 2\nmid x_1,\ldots,2\nmid x_4, x_1>0,\cdots,x_4>0}} \Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4 \quad \text{if } n\equiv 4 \pmod 8,$$

$$= \frac{8}{21}\,(2^{5(\alpha-2)}3967 + 63)\sigma_5(u) \quad \text{if } n\equiv 0 \pmod 8.$$

1. Some Known Results

1.1. In the present paper N, a, k, n, q, r denote positive integers; b, u, v are odd positive integers; p is a prime number; c, g, h, j, m, α, β, γ, δ are integers; i is the imaginary unit; z, τ are complex variables ($\operatorname{Im}\tau > 0$); $e(z) = \exp 2\pi iz$; $Q = e(\tau)$; $(\frac{h}{u})$ is the generalized Jacobi symbol; $\sum_{h \bmod q}$ and $\sum'_{h \bmod q}$ denote sums in which h runs through the complete and the reduced residue system modulo q, respectively; $\sigma_5(u)$ is the sum of the fifth powers of positive divisors of u.

Let

$$\vartheta_{gh}(z|\tau;c,N)=$$
$$=\sum_{m\equiv c \pmod N}(-1)^{h(m-c)/N}e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big)e\Big(\Big(m+\frac{g}{2}\Big)z\Big); \quad (1.1)$$

hence

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}(z|\tau;c,N)=(\pi i)^n\sum_{m\equiv c \pmod N}(-1)^{h(m-c)/N}(2m+g)^n\times$$
$$\times e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big)e\Big(\Big(m+\frac{g}{2}\Big)z\Big). \quad (1.2)$$

Let

$$\vartheta_{gh}(\tau;c,N)=\vartheta_{gh}(0|\tau;c,N),\ \ \vartheta^{(n)}_{gh}(\tau;c,N)=\frac{\partial^n}{\partial z^n}\vartheta_{gh}(z|\tau;c,N)\Big|_{z=0}. \quad (1.3)$$

We have (see, e.g., [4], formulas (2.3), (2.5) on page 112)

$$\begin{aligned}\vartheta_{g+2j,h}(\tau;c,N)&=\vartheta_{gh}(\tau;c+j,N),\\ \vartheta^{(n)}_{g+2j,h}(\tau;c,N)&=\vartheta^{(n)}_{gh}(\tau;c+j,N);\end{aligned} \quad (1.4)$$

$$\begin{aligned}\vartheta_{gh}(\tau;c+N_j,N)&=(-1)^{hj}\vartheta_{gh}(\tau;c,N),\\ \vartheta^{(n)}_{gh}(\tau;c+N_j)&=(-1)^{hj}\vartheta_{gh}(\tau;c,N).\end{aligned} \quad (1.5)$$

In particular, according to (1.3), it follows from (1.1) that

$$\vartheta_{gh}(\tau;0,N)=\sum_{m=-\infty}^{\infty}(-1)^{hm}Q^{(2Nm+g)^2/8N}, \quad (1.6)$$

$$\vartheta^{(n)}_{gh}(\tau;0,N)=(\pi i)^n\sum_{m=-\infty}^{\infty}(-1)^{hm}(2Nm+g)^nQ^{(2Nm+g)^2/8N}. \quad (1.7)$$

Expressions (1.6) and (1.7) yield

$$\vartheta_{-g,h}(\tau;0,N)=\vartheta_{gh}(\tau;0,N),\ \ \vartheta^{(n)}_{-g,h}(\tau;0,N)=(-1)^n\vartheta_{gh}(\tau;0,N). \quad (1.8)$$

Throughout this paper, $\Delta=\prod\limits_{k=1}^{6}a_k^2$ is the determinant of the quadratic form (1) and a is the least common multiple of its coefficients.

Denoting by $r(n;f)$ the number of representations of n by the form (1), we get

$$\prod_{k=1}^{6}\vartheta^2_{00}(\tau;0,2a_k)=1+\sum_{n=1}^{\infty}r(n;f)Q^n. \quad (1.9)$$

Further, put

$$\Theta(\tau;f)=1+\sum_{n=1}^{\infty}\rho(n;f)Q^n, \tag{1.10}$$

where $\rho(n;f)$ is the sum of singular series corresponding to the function $r(n;f)$.

Finally, let

$$\Gamma_0(4N)=\left\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,\middle|\;\alpha\delta-\beta\gamma=1,\;\;\gamma\equiv 0 \pmod{4N}\right\}$$

(nonhomogeneous congruence subgroup modulo $4N$).

1.2. For convenience we shall quote here some well-known results.

Lemma 1 (see, e.g., [4], Lemma 1, p. 114). *The entire modular form $F(\tau)$ of weight r for the congruence group $\Gamma_0(4N)$ is identically zero, if in the expansion into the series*

$$F(\tau)=\sum_{m=0}^{\infty}C_mQ^m$$

we have $C_m=0$ for all $m\le\frac{r}{12}4N\prod_{p|4N}(1+\frac{1}{p})$.

Lemma 2 ([5], Lemma 14, p. 18 and Remark to Lemma 18, p. 21). *The functions $\Theta(\tau;f)$ and $\prod_{k=1}^{6}\vartheta_{00}^2(\tau;0,2a_k)$ are entire modular forms of weight 6 and character $\chi(\delta)=(\frac{\Delta}{|\delta|})$ for $\Gamma_0(4a)$.*

Lemma 3 ([2], Theorem 2, pp. 70, 73, and [3], Theorem 2, pp. 196, 197). *For given N, the functions*

$$\begin{aligned}
&\Phi_2\big(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\big)=\\
&=\left\{\frac{1}{N_1^2}\,\vartheta^{(4)}_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)+\right.\\
&\quad+\frac{1}{N_2^2}\,\vartheta^{(4)}_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)-\\
&\left.-\frac{6}{N_1N_2}\,\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)\right\}\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k),
\end{aligned} \tag{1.11}$$

$$\begin{aligned}
&\Phi_3\big(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\big)=\\
&=\left\{\frac{1}{N_1}\,\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)-\right.
\end{aligned}$$

$$-\frac{1}{N_2}\,\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)\Big\}\prod_{k=3}^{4}\vartheta'_{g_kh_k}(\tau;0,2N_k), \quad (1.12)$$

$$\Phi_4\big(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\big)=$$
$$=\prod_{k=1}^{4}\vartheta'_{g_kh_k}(\tau;0,2N_k), \quad (1.13)$$

where

(a) $2\mid g_k$, $N_k\mid N$ $(k=1,2,3,4)$, $4\mid N\sum\limits_{k=1}^{4}\frac{h_k^2}{N_k}$, $4\mid\sum\limits_{k=1}^{4}\frac{g_k^2}{4N_k}$;

(b) *for all* α *and* δ *with* $\alpha\delta\equiv 1 \pmod{4N}$,

$$\left(\frac{\prod\limits_{k=1}^{4}N_k}{|\delta|}\right)\Phi_j\big(\tau;\alpha g_1,\dots,\alpha g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\big)=$$
$$=\left(\frac{\Delta}{|\delta|}\right)\Phi_j\big(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4\big)\quad(j=2,3,4),$$

are entire modular forms of weight 6 *and character* $\chi(\Delta)=(\frac{\Delta}{|\delta|})$ *for* $\Gamma_0(4N)$.

Remark. In what follows we shall write $\Phi_2(\tau;g_1,\dots,g_4;N_1,\dots,N_4)$ instead of $\Phi_2(\tau;g_1,\dots,g_4;0,\dots,0;N_1,\dots,N_4)$.

Lemma 4 ([1], Theorem, p. 383). *Let* $n=2^\alpha uv$, $u=\prod\limits_{p\mid n,\,p\nmid 2\Delta}p^\beta$, $v=\prod\limits_{\substack{p\mid n,\,p\mid\Delta\\ p>2}}p^\beta$ $(\alpha\ge 0,\ \beta\ge 0)$. *Then*

$$\rho(n;f)=\frac{2^{5\alpha+3}v^5}{\Delta^{1/2}}\,\chi_2\prod_{p\mid\Delta,\,p>2}\chi_p\prod_{p\mid\Delta,\,p>2}(1-p^{-6})^{-1}\sigma_5(u).$$

The values χ_2 *and* χ_p *are given in Lemmas* 5 *and* 6 *of* [1].

2. Formulas for $r(n;f)$

2.1. We consider the quadratic forms f_1, f_2, and f_3 from (2), whose determinants are respectively $\Delta=2^2$, $\Delta=2^6$, and $\Delta=2^{10}$.

Lemma 5. *The functions*

$$\psi(\tau;f_1)=\vartheta_{00}^{10}(\tau;0,2)\vartheta_{00}^{2}(\tau;0,4)-\Theta(\tau;f_1)+$$
$$+\,2\,\frac{1}{128\pi^4}\,X_1(\tau)-\frac{7}{2}\,\frac{1}{256\pi^4}\,\vartheta_{41}'^{\,4}(\tau;0,4), \quad (2.1)$$
$$\psi(\tau;f_2)=\vartheta_{00}^{6}(\tau;0,2)\vartheta_{00}^{6}(\tau;0,4)-\Theta(\tau;f_2)+$$

$$+\frac{3}{2}\,\frac{1}{128\pi^4}\,X_1(\tau)+\frac{1}{16}\,\frac{1}{16\pi^4}\,\vartheta_{21}'^{4}(\tau;0,2)-$$
$$-\frac{5}{2}\,\frac{1}{256\pi^4}\,\vartheta_{41}'^{4}(\tau;0,4), \tag{2.2}$$
$$\psi(\tau;f_3)=\vartheta_{00}^{2}(\tau;0,2)\vartheta_{00}^{10}(\tau;0,4)-\Theta(\tau;f_3)+$$
$$+\frac{1}{2}\,\frac{1}{128\pi^4}\,X_1(\tau)+\frac{1}{64}\,\frac{1}{16\pi^4}\,\vartheta_{21}'^{4}(\tau;0,2)-$$
$$-\frac{1}{256\pi^4}\,\vartheta_{41}'^{4}(\tau;0,4), \tag{2.3}$$

where, for brevity, we put

$$X_1(\tau)=\Phi_2\big(\tau;4,4,0,0;2,2,2,2\big), \tag{2.4}$$

are entire modular forms of weight 6 *and character* $\chi(\delta)=1$ *for* $\Gamma_0(8)$.

Proof. According to Lemma 2, the first two summands in (2.1)–(2.3) are entire modular forms of weight 6 and character $\chi(\delta)=1$ for $\Gamma_0(8)$.

In Lemma 3 put $N=2$. It is then obvious that the other summands in (2.1)–(2.3) satisfy condition (a) of this lemma.

If $\alpha\delta\equiv 1 \pmod 8$, then $\alpha\delta\equiv 1 \pmod 4$, i.e.,

$$\alpha\equiv\pm1 \pmod 4 \quad\text{and}\quad \delta\equiv\pm1 \pmod 4,\quad\text{respectively.} \tag{2.5}$$

It is easy to verify that the summands from (2.1)–(2.3) involve

$$\left(\frac{\prod\limits_{k=1}^{4}N_k}{|\delta|}\right)=1 \quad\text{and}\quad \left(\frac{\Delta}{|\delta|}\right)=1. \tag{2.6}$$

By (1.11), (1.4), (1.5), (2.5), and (1.8),

$$\Phi_2\big(\tau;4\alpha,4\alpha,0,0;2,2,2,2\big)=$$
$$=\left\{\frac{1}{2}\,\vartheta_{4\alpha,0}^{(4)}(\tau;0,4)\vartheta_{4\alpha,0}(\tau;0,4)-\frac{3}{2}\,\vartheta_{4\alpha,0}''^{2}(\tau;0,4)\right\}\vartheta_{00}^{2}(\tau;0,4)=$$
$$=\left\{\frac{1}{2}\,\vartheta_{\pm4,0}^{(4)}(\tau;2(\alpha\mp1),4)\vartheta_{\pm4,0}(\tau;2(\alpha\mp1),4)-\right.$$
$$\left.-\frac{3}{2}\,\vartheta_{\pm4,0}''^{2}(\tau;2(\alpha\mp1),4)\right\}\vartheta_{00}^{2}(\tau;0,4)=$$
$$=\left\{\frac{1}{2}\,\vartheta_{\pm4,0}^{(4)}(\tau;0,4)\vartheta_{\pm4,0}(\tau;0,4)-\frac{3}{2}\,\vartheta_{\pm4,0}''^{2}(\tau;0,4)\right\}\vartheta_{00}^{2}(\tau;0,4)=$$
$$=\left\{\frac{1}{2}\,\vartheta_{40}^{(4)}(\tau;0,4)\vartheta_{40}(\tau;0,4)-\frac{3}{2}\,\vartheta_{40}''^{2}(\tau;0,4)\right\}\vartheta_{00}^{2}(\tau;0,4)=$$
$$=\Phi_2\big(\tau;4,4,0,0;2,2,2,2\big).$$

Similarly, by (1.13), (1.4), (1.5), (2.5) and (1.8),

$$\vartheta'^4_{2\alpha,1}(\tau;0,2)=\vartheta'^4_{\pm 2,1}(\tau;\alpha\mp 1,2)=\vartheta'^4_{\pm 2,1}(\tau;0,2)=\vartheta'^4_{2,1}(\tau;0,2),$$
$$\vartheta'^4_{4\alpha,1}(\tau;0,4)=\vartheta'^4_{\pm 4,1}(\tau;2(\alpha\mp)1,4)=\vartheta'^4_{\pm 4,1}(\tau;0,4)=\vartheta'^4_{4,1}(\tau;0,4)$$

Hence by (2.6) the functions $X_1(\tau)$, $\vartheta'^4_{21}(\tau;0,2)$ and $\vartheta'^4_{41}(\tau;0,4)$ satisfy also condition (b) from Lemma 3. Thus they are entire modular forms of weight 6 and character $\chi(\delta)=1$ for $\Gamma_0(8)$. □

Theorem 1. $\vartheta^{10}_{00}(\tau;0,2)\vartheta^2_{00}(\tau;0,4)=$

$$=\Theta(\tau;f_1)-2\,\frac{1}{128\pi^4}\,X_1(\tau)+\frac{7}{2}\,\frac{1}{256\pi^4}\,\vartheta'^4_{41}(\tau;0,4), \tag{2.7}$$

$$\vartheta^6_{00}(\tau;0,2)\vartheta^6_{00}(\tau;0,4)=\Theta(\tau;f_2)-\frac{3}{2}\,\frac{1}{128\pi^4}\,X_1(\tau)-$$
$$-\frac{1}{16}\,\frac{1}{16\pi^4}\,\vartheta'^4_{21}(\tau;0,2)+\frac{5}{2}\,\frac{1}{256\pi^4}\,\vartheta'^4_{41}(\tau;0,4), \tag{2.8}$$

$$\vartheta^2_{00}(\tau;0,2)\vartheta^{10}_{00}(\tau;0,4)=\Theta(\tau;f_3)-\frac{1}{2}\,\frac{1}{128\pi^4}\,X_1(\tau)-$$
$$-\frac{1}{64}\,\frac{1}{16\pi^4}\,\vartheta'^4_{21}(\tau;0,2)+\frac{1}{256\pi^4}\,\vartheta'^4_{41}(\tau;0,4). \tag{2.9}$$

Proof. By Lemma 1, the functions $\psi(\tau;f_1)$, $\psi(\tau;f_2)$, and $\psi(\tau;f_3)$ will be identically zero, if all coefficients by Q^n $(n\le 6)$ in their expansion into the powers of Q are zero.

I. In Lemma 4, putting $n=2^\alpha u$, $v=1$, $\Delta=2^2$, 2^6, and 2^{10}, we respectively obtain

$$\rho(n;f_1)=\frac{2^{5\alpha+3}}{2}\,\chi_2\sigma_5(u),\quad \rho(n;f_2)=\frac{2^{5\alpha+3}}{2^3}\,\chi_2\sigma_5(u),$$
$$\rho(n;f_3)=\frac{2^{5\alpha+3}}{2^5}\,\chi_2\sigma_5(u).$$

In Lemma 5 from [1] we put $\gamma_6=1$, $\gamma_5=\cdots=\gamma_1=0$, $\gamma=1$, $b_1=\cdots=b_6=1$. Then

$$\chi_2=1 \quad\text{if}\quad \alpha=0,1,$$
$$\chi_2=2^{-5\alpha}\Big\{2^{5\alpha}-2\big(2^5(2^{5(\alpha-2)}-1)31^{-1}-1\big)\Big\}=$$
$$=\frac{2\cdot 9}{2^{5\alpha}31}\,\big(2^{5(\alpha-1)}55+7\big)\quad\text{if}\quad \alpha\ge 2.$$

Thus

$$\begin{aligned}\rho(n;f_1) &= 4\sigma_5(u) \quad \text{if} \quad \alpha=0,\\ &= 128\sigma_5(u) \quad \text{if} \quad \alpha=1,\\ &= 72\cdot 31^{-1}(2^{5(\alpha-1)}55+7)\sigma_5(u) \quad \text{if} \quad \alpha\ge 2.\end{aligned} \tag{2.10}$$

Calculating the values of $\rho(n;f_1)$ by (2.10) for all $n\le 6$, by virtue of (1.10) we obtain

$$\Theta(\tau;f_1)=$$
$$=1+4Q+128Q^2+976Q^3+4104Q^4+12504Q^5+31232Q^6+\cdots. \tag{2.11}$$

Now in Lemma 5 from [1] put $\gamma_6=\gamma_5=\gamma_4=1$, $\gamma_3=\gamma_2=\gamma_1=0$, $\gamma=3$, $b_1=\cdots=b_6=1$. Then

$$\begin{aligned}\chi_2 &= 1 \quad \text{if} \quad \alpha=0,1,\\ &= 2^{-5\alpha}\Big\{2^{5\alpha}-2^3\big(2^5(2^{5(\alpha-2)}-1)31^{-1}-1\big)\Big\}=\\ &= \frac{2^3 3}{2^{5\alpha}31}\big(2^{5(\alpha-1)}41+21\big) \quad \text{if} \quad \alpha\ge 2.\end{aligned}$$

Thus

$$\begin{aligned}\rho(n;f_2) &= \sigma_5(u) \quad \text{if} \quad \alpha=0,\\ &= 32\sigma_5(u) \quad \text{if} \quad \alpha=1,\\ &= 24\cdot 31^{-1}(2^{5(\alpha-1)}41+21)\sigma_5(u) \quad \text{if} \quad \alpha\ge 2.\end{aligned} \tag{2.12}$$

Calculating the values of $\rho(n;f_2)$ by (2.12) for all $n\le 6$, we get

$$\Theta(\tau;f_2)=$$
$$=1+Q+32Q^2+244Q^3+1032Q^4+3126Q^5+7808Q^6+\cdots. \tag{2.13}$$

Finally, in Lemma 5 from [1] put $\gamma_6=\cdots=\gamma_2=1$, $\gamma_1=0$, $\gamma=5$, $b_1=\cdots=b_6=1$. Then

$$\begin{aligned}\chi_2 &= 1 \quad \text{if} \quad \alpha=0,1,\\ &= 2^{-5\alpha}\Big\{2^{5\alpha}-2^5\big(2^5(2^{5(\alpha-2)}-1)31^{-1}-1\big)\Big\}=\\ &= \frac{2^5 3}{2^{5\alpha}31}\big(2^{5\alpha-4}5+21\big) \quad \text{if} \quad \alpha\ge 2.\end{aligned}$$

Thus

$$\begin{aligned}\rho(n;f_3) &= \frac{1}{4}\,\sigma_5(u) \quad \text{if} \quad \alpha=0,\\ &= 8\sigma_5(u) \quad \text{if} \quad \alpha=1,\\ &= 24\cdot 31^{-1}(2^{5\alpha-4}5+21)\sigma_5(u) \quad \text{if} \quad \alpha\ge 2.\end{aligned} \tag{2.14}$$

Calculating the values of $\rho(n;f_3)$ by (2.14) for all $n\leq 6$, we get

$$\Theta(\tau;f_3)=$$
$$=1+\frac{1}{4}\,Q+8Q^2+61Q^3+264Q^4+\frac{1563}{2}\,Q^5+1952Q^6+\cdots. \quad (2.15)$$

II. It follows from (1.6) that

$$\vartheta_{00}^{10}(\tau;0,2)\vartheta_{00}^{2}(\tau;0,4)=1+20Q+184Q^2+1040Q^3+$$
$$=4104Q^4+12344Q^5+30560Q^6+\cdots. \quad (2.16)$$

Equations (1.7) and (1.6) imply

$$\vartheta_{40}^{(4)}(\tau;0,4)\vartheta_{40}(\tau;0,4)=$$
$$=256\pi^4\sum_{m=-\infty}^{\infty}(2m+1)^4Q^{(2m+1)^2/2}\sum_{m=-\infty}^{\infty}Q^{(2m+1)^2/2}= \quad (2.17)$$
$$=256\pi^4(4Q)(1+82Q^4+81Q^8+62Q^{12}+\cdots),$$
$$\vartheta_{40}^{''2}(\tau;0,4)=\Big(16(\pi i)^2\sum_{m=-\infty}^{\infty}(2m+1)^2Q^{(2m+1)^2/2}\Big)^2= \quad (2.18)$$
$$=256\pi^4(4Q)(1+18Q^4+81Q^8+50Q^{12}+\cdots),$$
$$\vartheta_{00}^{2}(\tau;0,4)=\Big(\sum_{m=-\infty}^{\infty}Q^{2m^2}\Big)^2= \quad (2.19)$$
$$=1+4Q^2+4Q^4+4Q^8+8Q^{10}+4Q^{16}+\cdots.$$

Thus, by (2.4) and (1.11),

$$X_1(\tau)=\Big\{\frac{1}{2}\,\vartheta_{40}^{(4)}(\tau;0,4)\vartheta_{40}(\tau;0,4)-\frac{3}{2}\,\vartheta_{40}^{''2}(\tau;0,4)\Big\}\vartheta_{00}^{2}(\tau;0,4)= \quad (2.20)$$
$$=128\pi^4(4Q)\Big\{(1+82Q^4+81Q^8+626Q^{12}+\cdots)-$$
$$-3(1+18Q^4+81Q^8+50Q^{12}+\cdots)\Big\}\times \quad (2.20_1)$$
$$\times(1+4Q^2+4Q^4+4Q^8+8Q^{10}+\cdots)=$$
$$=128\pi^4(4Q)\big(-2-8Q^2+20Q^4+112Q^6-58Q^8-664Q^{10}-60Q^{12}+\cdots\big),$$

i.e,

$$\frac{1}{128\pi^4}\,X_1(\tau)=-8Q-32Q^3+80Q^5+448Q^7-$$
$$-232Q^9-2656Q^{11}-240Q^{13}+\cdots. \quad (2.21)$$

From (1.7) we have

$$\vartheta_{41}'^4(\tau;0,4) = \Big(4\pi i \sum_{m=-\infty}^{\infty} (-1)^m (2m+1) Q^{(2m+1)^2/2}\Big)^4 = \tag{2.22}$$
$$= 256\pi^4(16Q^2)(1 - 3Q^4 + 5Q^{12} + \cdots)^4,$$

i.e.,

$$\frac{1}{256\pi^4}\,\vartheta_{41}'^4(\tau;0,4) = 16Q^2 - 192Q^6 + 864Q^{10} - 1408Q^{14} + \cdots. \tag{2.23}$$

From (1.6) we find that

$$\vartheta_{00}^6(\tau;0,2)\vartheta_{00}^6(\tau;0,4) = 1 + 12Q + 72Q^2 + 304Q^3 +$$
$$+ 1032Q^4 + 2952Q^5 + 7328Q^6 + \cdots, \tag{2.24}$$

while (1.7) yields

$$\vartheta_{21}'^4(\tau;0,2) = \Big(2\pi i \sum_{m=-\infty}^{\infty} (-1)^m (2m+1) Q^{(2m+1)^2/4}\Big)^4 = \tag{2.25}$$
$$= 16\pi^4(16Q)(1 - 3Q^2 + 5Q^6 - 7Q^{12} + \cdots)^4,$$

i.e.,

$$\frac{1}{16\pi^4}\,\vartheta_{21}'^4(\tau;0,2) = 16Q - 192Q^3 + 864Q^5 - 1408Q^7 -$$
$$- 1584Q^9 + 8640Q^{11} - 6688Q^{13} + \cdots. \tag{2.26}$$

It follows from (1.6) that

$$\vartheta_{00}^2(\tau;0,2)\vartheta_{00}^{10}(\tau;0,4) = 1 + 4Q + 24Q^2 + 80Q^3 +$$
$$+ 264Q^4 + 728Q^5 + 1760Q^6 + \cdots \tag{2.27}$$

It is easy to verify that all coefficients of Q^n ($n \le 6$) in expansions of functions $\psi(\tau; f_i)$ ($i = 1, 2, 3$) into the powers of Q are zero. Thus identity (2.9) is proved. $\square$

Proof of formulas (I)–(III). Equating the coefficients by Q^n in both sides of identities (2.7)–(2.9), according to (1.9) and (1.10), we get

$$r(n; f_1) = \rho(n; f_1) - 2\nu_1(n) + \frac{7}{2}\,\widetilde{\nu}_1(n), \tag{2.28}$$

$$r(n; f_2) = \rho(n; f_2) - \frac{3}{2}\,\nu_1(n) - \frac{1}{16}\,\nu_{11}(n) + \frac{5}{2}\,\widetilde{\nu}_1(n), \tag{2.29}$$

$$r(n; f_3) = \rho(n; f_3) - \frac{1}{2}\,\nu_1(n) - \frac{1}{64}\,\nu_{11}(n) + \widetilde{\nu}_1(n), \tag{2.30}$$

where $\nu_1(n)$, $\widetilde{\nu}_1(n)$ and $\nu_{11}(n)$ denote the coefficients by Q^n in the expansion of the functions $\frac{1}{128\pi^4}\,X_1(\tau)$, $\frac{1}{256\pi^4}\,\vartheta'^4_{41}(\tau;0,4)$, and $\frac{1}{16\pi^4}\,\vartheta'^4_{21}(\tau;0,2)$ in powers of Q, respectively.

It follows from (2.20) and (2.17)–(2.19) that

$$\frac{1}{128\pi^4}\,X_1(\tau)=\Big(\sum_{m_1,m_2=-\infty}^{\infty}(2m_1+1)^4Q^{\{(2m_1+1)^2+(2m_2+1)^2\}/2}-$$
$$-3\sum_{m_1,m_2=-\infty}^{\infty}(2m_1+1)^2(2m_2+1)^2Q^{\{(2m_1+1)^2+(2m_2+1)^2\}/2}\Big)\times$$
$$\times\sum_{m_3,m_4=-\infty}^{\infty}Q^{2m_3^2+2m_4^2}=\sum_{n=1}^{\infty}\Big(\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2}}x_1^4-3x_1^2x_2^2\Big)Q^n,$$

i.e.,

$$\nu_1(n)=4\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2,x_1>0,x_2>0}}x_1^4-3x_1^2x_2^2. \tag{2.31}$$

Obviously, $\nu_1(n)\neq 0$ for $n\equiv 1 \pmod 2$ only.

From (2.22) we find that

$$\frac{1}{256\pi^4}\,\vartheta'^4_{41}(\tau;0,4)=\sum_{m_1,\dots,m_4=-\infty}^{\infty}(-1)^{m_1+\cdots+m_4}(2m_1+1)\cdots(2m_4+1)\times$$
$$\times Q^{\{(2m_1+1)^2+\cdots+(2m_4+1)^2\}/2}=$$
$$=\sum_{n=1}^{\infty}\Big(\sum_{\substack{x_1^2+\cdots+x_4^2=2n\\ 2\nmid x_1,\dots,2\nmid x_4}}(-1)^{(x_1-1)/2+\cdots+(x_4-1)/2}x_1x_2x_3x_4\Big)Q^n,$$

i.e.,

$$\widetilde{\nu}_1(n)=16\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=2n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\dots,x_4>0}}\Big(\frac{-1}{x_1x_2x_3x_4}\Big)x_1x_2x_3x_4. \tag{2.32}$$

Obviously, $\widetilde{\nu}_1(n)\neq 0$ for $n\equiv 2 \pmod 4$ only.

From (2.25) follows

$$\frac{1}{16\pi^4}\,\vartheta'^4_{21}(\tau;0,2)=\sum_{m_1,\dots,m_4=-\infty}^{\infty}(-1)^{m_1+\cdots+m_4}(2m_1+1)\cdots(2m_4+1)\times$$
$$\times Q^{\{(2m_1+1)^2+\cdots+(2m_4+1)^2\}/4}=$$

$$= \sum_{n=1}^{\infty} \Big(\sum_{\substack{x_1^2+\cdots+x_4^2=4n \\ 2\nmid x_1,\ldots,2\nmid x_4}} (-1)^{(x_1-1)/2+\cdots+(x_4-1)/2} x_1x_2x_3x_4 \Big) Q^n,$$

i.e,

$$\nu_{11}(n) = 16 \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=4n \\ 2\nmid x_1,\ldots,2\nmid x_4, x_1>0,\ldots,x_4>0}} \Big(\frac{-1}{x_1x_2x_3x_4}\Big) x_1x_2x_3x_4. \tag{2.33}$$

Obviously, $\nu_{11}(n) \neq 0$ for $n \equiv 1 \pmod 2$ only.

From (2.28), (2.10), (2.31), and (2.32) follows formula (I), from (2.29), (2.12), and (2.31)–(2.33) formula (II) and from (2.30), (2.14), and (2.31)–(2.33) formula (III).

2.2. Consider now the quadratic form

$$f_4 = x_1^2 + x_2^2 + \cdots + x_8^2 + 2(x_9^2 + x_{10}^2) + 4(x_{11}^2 + x_{12}^2)$$

whose determinant is $\Delta = 2^6$.

Lemma 6. *The function*

$$\begin{aligned}\psi(\tau; f_4) = {} & \vartheta_{00}^8(\tau;0,2)\vartheta_{00}^2(\tau;0,4)\vartheta_{00}^2(\tau;0,8) - \Theta(\tau; f_4) - \\ & - 2\,\frac{1}{128\pi^4}\, X_1(\tau) - \frac{15}{16}\,\frac{1}{16\pi^4}\,\vartheta_{21}'^4(\tau;0,2) + 16\,\frac{1}{512\pi^4}\, X_2(\tau) + \\ & + 2\,\frac{1}{128\pi^4}\, X_3(\tau) + \frac{7}{4}\,\frac{1}{256\pi^4}\,\vartheta_{41}'^4(\tau;0,4) + \\ & + 14\,\frac{1}{512\pi^4}\, X_4(\tau) - \frac{71}{2}\,\frac{1}{4096\pi^4}\,\vartheta_{81}'^4(\tau;0,8),\end{aligned}$$

where

$$X_1(\tau) = \Phi_2(\tau; 4,4,0,0; 2,2,2,2), \tag{2.34}$$

$$X_2(\tau) = \Phi_2(\tau; 8,8,4,4; 4,4,2,2), \tag{2.35}$$

$$X_3(\tau) = \Phi_2(\tau; 4,4,0,0; 2,2,4,4), \tag{2.36}$$

$$X_4(\tau) = \Phi_2(\tau; 8,8,0,0; 4,4,4,4), \tag{2.37}$$

is entire modular form of weight 6 *and character* $\chi(\delta) = 1$ *for* $\Gamma_0(16)$.

Proof. According to Lemma 2, the first two summands of the function $\psi(\tau; f_4)$ are entire modular forms of weight 6 and character $\chi(\delta) = 1$ for $\Gamma_0(16)$.

In Lemma 3 put $N = 4$. Then it is obvious that the other summands of $\psi(\tau; f_4)$ satisfy condition (a) of this lemma.

If $\alpha\delta \equiv 1 \pmod{16}$, then $\alpha\delta \equiv 1 \pmod 4$, i.e.,

$$\alpha \equiv \pm 1 \pmod 4 \quad \text{and} \quad \delta \equiv \pm 1 \pmod 4, \quad \text{respectively.} \tag{2.38}$$

It is easy to verify that all these summands involve

$$\left(\frac{\prod\limits_{k=1}^{4} N_k}{|\delta|}\right) = 1 \quad \text{and} \quad \left(\frac{\Delta}{|\delta|}\right) = 1. \tag{2.39}$$

Reasoning as in Lemma 5, by (1.11), (1.4), (1.5), (2.38) and (1.8), we get

$$\begin{gathered}
\Phi_2(\tau; 8\alpha, 8\alpha, 4\alpha, 4\alpha; 4, 4, 2, 2) = \Phi_2(\tau; 8, 8, 4, 4; 4, 4, 2, 2), \\
\Phi_2(\tau; 4\alpha, 4\alpha, 0, 0; 2, 2, 4, 4) = \Phi_2(\tau; 4, 4, 0, 0; 2, 2, 4, 4), \\
\Phi_2(\tau; 8\alpha, 8\alpha, 0, 0; 4, 4, 4, 4) = \Phi_2(\tau; 8, 8, 0, 0; 4, 4, 4, 4), \\
\vartheta'^4_{8\alpha,1}(\tau; 0, 8) = \vartheta'^4_{\pm 8,1}(\tau; 4(\alpha \mp 1), 8) = \vartheta'^4_{\pm 8,1}(\tau; 0, 8) = \vartheta'^4_{81}(\tau; 0, 8).
\end{gathered}$$

Similar relations for the other three summands of the function $\psi(\tau; f_4)$ have already been given in Lemma 5. Thus, according to (2.39), all these functions satisfy condition (b) from Lemma 3 as well. Consequently, they are entire modular forms of weight 6 and character $\chi(\delta) = 1$ for $\Gamma_0(16)$. □

Theorem 2. $\vartheta^8_{00}(\tau; 0, 2)\vartheta^2_{00}(\tau; 0, 4)\vartheta^2_{00}(\tau; 0, 8) =$

$$\begin{gathered}
= \Theta(\tau; f_4) + 2\,\frac{1}{128\pi^4}\,X_1(\tau) + \frac{15}{16}\,\frac{1}{16\pi^4}\,\vartheta'^4_{21}(\tau; 0, 2) - \\
-16\,\frac{1}{512\pi^4}\,X_2(\tau) - 2\,\frac{1}{128\pi^4}\,X_3(\tau) - \\
-\frac{7}{4}\,\frac{1}{256\pi^4}\,\vartheta'^4_{41}(\tau; 0, 4) - 14\,\frac{1}{512\pi^4}\,X_4(\tau) + \frac{71}{2}\,\frac{1}{4096\pi^4}\,\vartheta'^4_{81}(\tau; 0, 8).
\end{gathered} \tag{2.40}$$

Proof. By Lemma 1, the function $\psi(\tau; f_4)$ will be identically zero if all coefficients by Q^n $(n \le 12)$ in its expansion into the powers of Q are zero.

I. In Lemma 4 put $n = 2^\alpha u$, $v = 1$, $\Delta = 2^6$. Then $\rho(n; f_4) = 2^{5\alpha}\chi_2\sigma_5(u)$. In Lemma 5 from [1] we put $\gamma_6 = 2$, $\gamma_5 = 1$, $\gamma_4 = \cdots = \gamma_1 = 0$, $\gamma = 3$, $b_1 = \cdots = b_6 = 1$. Then

$$\begin{aligned}
\chi_2 &= 1 \quad \text{if} \quad \alpha = 0, 1, 2, \\
&= 2^{-5\alpha}\Big\{2^{5\alpha} - 2^3\big(2^5(2^{5(\alpha-3)} - 1)31^{-1} - 1\big)\Big\} = \\
&= \frac{8}{2^{5\alpha}31}\,(2^{5(\alpha-2)}3967 + 63) \quad \text{if} \quad \alpha \ge 3.
\end{aligned}$$

Thus

$$\begin{aligned}\rho(n;f_4) &= \sigma_5(u) \quad \text{if} \quad \alpha=0,\\ &= 32\sigma_5(u) \quad \text{if} \quad \alpha=1,\\ &= 1024\sigma_5(u) \quad \text{if} \quad \alpha=2,\\ &= \frac{8}{31}\,(2^{5(\alpha-2)}3967+63)\sigma_5(u) \quad \text{if} \quad \alpha\geq 3.\end{aligned} \tag{2.41}$$

Calculating the values of $\rho(n;f_4)$ by (2.41) for all $n\leq 12$, we get

$$\begin{aligned}\Theta(\tau;f_4) = 1 &+ Q + 32Q^2 + 244Q^3 + 1024Q^4 + 3126Q^5 +\\ &+ 7808Q^6 + 16808Q^7 + 32776Q^8 + 59293Q^9 +\\ &+ 100032Q^{10} + 161052Q^{11} + 249856Q^{12} + \cdots .\end{aligned} \tag{2.42}$$

II. From (1.6) follows

$$\begin{aligned}\vartheta_{00}^{8}(\tau;0,2)\vartheta_{00}^{2}(\tau;0,4)\vartheta_{00}^{2}(\tau;0,8) &= 1 + 16Q + 116Q^2 +\\ +512Q^3 + 1592Q^4 + 3936Q^5 &+ 8592Q^6 + 17408Q^7 + 32776Q^8 +\\ +57808Q^9 + 97400Q^{10} &+ 157184Q^{11} + 243040Q^{12} + \cdots .\end{aligned} \tag{2.43}$$

Equations (1.7) and (1.6) imply

$$\vartheta_{80}^{(4)}(\tau;0,8)\vartheta_{80}(\tau;0,8) =$$

$$= 4096\pi^4 \sum_{m=-\infty}^{\infty}(2m+1)^4Q^{(2m+1)^2}\sum_{m=-\infty}^{\infty}Q^{(2m+1)^2} = \tag{2.44}$$

$$= 4096\pi^4(4Q^2)(1+82Q^8+81Q^{16}+\cdots), \tag{2.45}$$

$$\vartheta_{80}^{\prime\prime 2}(\tau;0,8) = 4096\pi^4\Big(\sum_{m=-\infty}^{\infty}(2m+1)^2Q^{(2m+1)^2}\Big)^2 = \tag{2.46}$$

$$= 4096\pi^4(4Q^2)(1+18Q^8+81Q^{16}+\cdots), \tag{2.47}$$

$$\vartheta_{40}^{2}(\tau;0,4) = \Big(\sum_{m=-\infty}^{\infty}Q^{(2m+1)^2/2}\Big)^2 = \tag{2.48}$$

$$= 4Q(1+2Q^4+Q^8+2Q^{12}+\cdots). \tag{2.49}$$

Thus, by (2.35), (1.11), (2.45), (2.47), and (2.49),

$$X_2(\tau) = \Big\{\frac{1}{8}\,\vartheta_{80}^{(4)}(\tau;0,8)\vartheta_{80}(\tau;0,8) - \frac{3}{8}\,\vartheta_{80}^{\prime\prime 2}(\tau;0,8)\Big\}\vartheta_{40}^{2}(\tau;0,4) = \tag{2.50}$$

$$= 512\pi^4(4Q^2)\Big\{(1+82Q^8+81Q^{16}+\cdots) -$$

$$-3(1+18Q^8+81Q^{16}+\cdots)\Big\}\times \tag{2.50$_1$}$$
$$\times(4Q)(1+2Q^4+Q^8+2Q^{12}+\cdots)=$$
$$=512\pi^4(16Q^3)(-2-4Q^4+26Q^8+52Q^{12}+\cdots)$$

i.e.,

$$\frac{1}{512\pi^4}\,X_2(\tau)=-32Q^3-64Q^7+416Q^{11}+832Q^{15}+\cdots \tag{2.51}$$

From (1.6) we find that

$$\vartheta_{00}^2(\tau;0,8)=\Big(\sum_{m=-\infty}^{\infty}Q^{4m^2}\Big)^2= \tag{2.52}$$

$$=(1+2Q^4+2Q^{16}+\cdots)^2=1+4Q^4+4Q^8+4Q^{16}+\cdots \tag{2.53}$$

Thus, by (2.36), (1.11), (2.20$_1$), and (2.53),

$$X_3(\tau)=\Big\{\frac{1}{2}\,\vartheta_{40}^{(4)}(\tau;0,4)\vartheta_{40}(\tau;0,4)-\frac{3}{2}\,\vartheta_{40}''^2(\tau;0,4)\Big\}\vartheta_{00}^2(\tau;0,8)= \tag{2.54}$$
$$=128\pi^4(4Q)\Big\{(1+82Q^4+81Q^8+626Q^{12}+\cdots)-$$
$$-3(1+18Q^4+81Q^8+50Q^{12}+\cdots)\Big\}\times$$
$$\times(1+4Q^4+4Q^8+4Q^{16}+\cdots)=$$
$$=128\pi^4(4Q)(-2+20Q^4-58Q^8-60Q^{12}+\cdots),$$

i.e.,

$$\frac{1}{128\pi^4}\,X_3(\tau)=-8Q+80Q^5-232Q^9-240Q^{13}+\cdots. \tag{2.55}$$

From (2.37), (1.11), (2.50$_1$), and (2.53) we obtain

$$X_4(\tau)=\Big\{\frac{1}{8}\,\vartheta_{80}^{(4)}(\tau;0,8)\vartheta_{80}(\tau;0,8)-\frac{3}{8}\,\vartheta_{80}''^2(\tau;0,8)\Big\}\vartheta_{00}^2(\tau;0,8)= \tag{2.56}$$
$$=512\pi^4(4Q^2)\Big\{(1+82Q^8+81Q^{16}+\cdots)-$$
$$-3(1+18Q^8+81Q^{16}+\cdots)\Big\}\times$$
$$\times(1+4Q^4+4Q^8+4Q^{16}+\cdots)=$$
$$=512\pi^4(4Q^2)(-2-8Q^4+20Q^8+112Q^{12}+\cdots),$$

i.e.,

$$\frac{1}{512\pi^4}\,X_4(\tau)=-8Q^2-32Q^6+80Q^{10}+448Q^{14}+\cdots. \tag{2.57}$$

From (1.7) we have

$$\vartheta_{81}'^{4}(\tau;0,8) = \Big(8\pi i \sum_{m=-\infty}^{\infty} (-1)^m (2m+1)^{(2m+1)^2}\Big)^4 = \\ = 4096\pi^4 (16Q^4)(1 - 12Q^8 + 54Q^{16} + \cdots) \tag{2.58}$$

i.e.,

$$\frac{1}{4096\pi^4}\,\vartheta_{81}'^{4}(\tau;0,8) = 16Q^4 - 192Q^{12} + 864Q^{20} + \cdots . \tag{2.59}$$

According to (2.43), (2.42), (2.21), (2.25), (2.51), (2.55), (2.23), (2.57), and (2.59), it is easy to verify that all coefficients of Q^n ($n \le 12$) in the expansion of the function $\psi(\tau; f_4)$ into the powers of Q are zero.

Thus identity (2.40) is proved. □

Proof of formula (IV). Equating the coefficients by Q^n in both sides of identity (2.40), by (1.9) and (1.10) we obtain

$$r(n; f_4) = \rho(n; f_4) + 2\nu_1(n) + \frac{15}{16}\,\nu_{11}(n) - 16\nu_2(n) - \\ - 2\nu_3(n) - \frac{7}{4}\,\widetilde{\nu}_1(n) - 14\widetilde{\nu}_2(n) + \frac{71}{2}\,\widetilde{\nu}_3(n), \tag{2.60}$$

where $\nu_1(n)$, $\nu_{11}(n)$, $\nu_2(n)$, $\nu_3(n)$, $\widetilde{\nu}_1(n)$, $\widetilde{\nu}_2(n)$, and $\widetilde{\nu}_3(n)$ denote the coefficients of Q^n in the expansion of the functions $\frac{1}{128\pi^4}X_1(\tau)$, $\frac{1}{16\pi^4}\vartheta_{21}'^{4}(\tau;0,2)$, $\frac{1}{512\pi^4}X_2(\tau)$, $\frac{1}{128\pi^4}X_3(\tau)$, $\frac{1}{256\pi^4}\vartheta_{41}'^{4}(\tau;0,4)$, $\frac{1}{511\pi^4}X_4(\tau)$, and $\frac{1}{4096\pi^4}\vartheta_{81}'^{4}(\tau;0,8)$ into the powers of Q, respectively.

It follows from (2.50), (2.44), (2.46), and (2.49) that

$$\frac{1}{512\pi^4}\,X_2(\tau) = \Big(\sum_{m_1,m_2=-\infty}^{\infty} (2m_1+1)^4 Q^{(2m_1+1)^2+(2m_2+1)^2} - \\ - 3\sum_{m_1,m_2=-\infty}^{\infty} (2m_1+1)^2(2m_2+1)^2 Q^{(2m_1+1)^2+(2m_2+1)^2}\Big) \times \\ \times \sum_{m_3,m_4=-\infty}^{\infty} Q^{\{(2m_3+1)^2+(2m_4+1)^2\}/2} = \\ = \sum_{n=1}^{\infty}\Big(\sum_{\substack{2(x_1^2+x_2^2)+x_3^2+x_4^2=2n \\ 2\nmid x_1,\dots,2\nmid x_4}} x_1^4 - 3x_1^2x_2^2\Big) Q^n,$$

i.e.,

$$\nu_2(n) = 16 \sum_{\substack{2(x_1^2+x_2^2)+x_3^2+x_4^2=2n \\ 2\nmid x_1,\dots,2\nmid x_4, x_1>0,\dots,x_4>0}} x_1^4 - 3x_1^2x_2^2. \tag{2.61}$$

Obviously, $\nu_2(n) \neq 0$ for $n \equiv 3 \pmod 4$ only.

From (2.54), (2.17), (2.18) and (2.52) we find that

$$\frac{1}{128\pi^4} X_3(\tau) = \Big(\sum_{m_1,m_2=-\infty}^{\infty} (2m_1+1)^4 Q^{\{(2m_1+1)^2+(2m_2+1)^2\}/2} -$$

$$-3 \sum_{m_1,m_2=-\infty}^{\infty} (2m_1+1)^2(2m_2+1)^2 Q^{\{(2m_1+1)^2+(2m_2+1)^2\}/2} \Big) \times$$

$$\times \sum_{m_3,m_4=-\infty}^{\infty} Q^{4m_3^2+4m_4^2} = \sum_{n=1}^{\infty} \Big(\sum_{\substack{x_1^2+x_2^2+8(x_3^2+x_4^2)=2n \\ 2\nmid x_1, 2\nmid x_2}} x_1^4 - 3x_1^2x_2^2 \Big) Q^n,$$

i.e.,

$$\nu_3(n) = 4 \sum_{\substack{x_1^2+x_2^2+8(x_3^2+x_4^2)=2n \\ 2\nmid x_1, 2\nmid x_2, x_1>0, x_2>0}} x_1^4 - 3x_1^2x_2^2. \tag{2.62}$$

Obviously, $\nu_3(n) \neq 0$ for $n \equiv 1 \pmod 4$ only.

From (2.56), (2.44), (2.46), and (2.52) follows

$$\frac{1}{512\pi^4} X_4(\tau) = \Big(\sum_{m_1,m_2=-\infty}^{\infty} (2m_1+1)^4 Q^{(2m_1+1)^2+(2m_2+1)^2} -$$

$$-3 \sum_{m_1,m_2=-\infty}^{\infty} (2m_1+1)^2(2m_2+1)^2 Q^{(2m_1+1)^2+(2m_2+1)^2} \Big) \times$$

$$\times \sum_{m_3,m_4=-\infty}^{\infty} Q^{4m_3^2+4m_4^2} = \sum_{n=1}^{\infty} \Big(\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=n \\ 2\nmid x_1, 2\nmid x_2}} x_1^4 - 3x_1^2x_2^2 \Big) Q^n,$$

i.e.,

$$\widetilde{\nu}_2(n) = 4 \sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=n \\ 2\nmid x_1, 2\nmid x_2, x_1>0, x_2>0}} x_1^4 - 3x_1^2x_2^2. \tag{2.63}$$

Obviously, $\widetilde{\nu}_2(n) \neq 0$ for $n \equiv 2 \pmod 4$ only.

From (2.58) we find that

$$\frac{1}{4096\pi^4}\,\vartheta_{81}'^{4}(\tau;0,8)=\sum_{m_1,\dots,m_4=-\infty}^{\infty}(-1)^{\sum\limits_{k=1}^{4}m_k}\prod_{k=1}^{4}(2m_k+1)Q^{\sum\limits_{k=1}^{4}(2m_k+1)^2}=$$

$$=\sum_{n=1}^{\infty}\Bigl(\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=n\\ 2\nmid x_1,\dots,2\nmid x_4}}(-1)^{\sum\limits_{k=1}^{4}(x_k-1)/2}x_1x_2x_2x_4\Bigr)Q^n,$$

i.e.,

$$\widetilde{\nu}_3(n)=16\sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=n\\ 2\nmid x_1,\dots,2\nmid x_4,x_1>0,\dots,x_4>0}}\Bigl(\frac{-1}{x_1x_2x_3x_4}\Bigr)x_1x_2x_3x_4. \tag{2.64}$$

Obviously, $\widetilde{\nu}_3(n)\neq 0$ for $n\equiv 4 \pmod 8$ only.

The values $\nu_1(n)$, $\widetilde{\nu}_1(n)$, and $\nu_{11}(n)$ have already been defined by formulas (2.31), (2.32), and (2.33), respectively. Formula (IV) follows from (2.60), (2.41), (2.31)–(2.33), and (2.61)–(2.63). But in the case where $n\equiv 1 \pmod 4$, (2.60) implies that

$$r(n;f_4)=\rho(n;f_4)+2\nu_1(n)+\frac{15}{16}\,\nu_{11}(n)-2\nu_3(n),$$

since by (2.61), (2.32), (2.63), and (2.64)

$$\nu_2(n)=\widetilde{\nu}_1(n)=\widetilde{\nu}_2(n)=\widetilde{\nu}_3(n)=0.$$

However, one can verify that

$$\sum_{\substack{x_1^2+x_2^2+4(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2,x_1>0,x_2>0}}x_1^4-3x_1^2x_2^2=\sum_{\substack{x_1^2+x_2^2+8(x_3^2+x_4^2)=2n\\ 2\nmid x_1,2\nmid x_2,x_1>0,x_2>0}}x_1^4-3x_1^2x_2^2,$$

i.e., $2\nu_1(n)-2\nu_3(n)=0$.

2.3. Notice finally that one can derive in a similar way formulas for $r(n;f)$ for some other quadratic forms of type (1). In particular, one can derive the well-known Liouville formula obtained by him in 1864 for $r(n;f)$, when $f=x_1^2+x_2^2+\dots=x_{12}^2$, if we introduce the function

$$\begin{aligned}\psi(\tau;f)&=\vartheta_{00}^{12}(\tau;0,2)-\Theta(\tau;f)-\\&-\frac{1}{512\pi^4}\Bigl\{2\vartheta_{00}^{(4)}(\tau;0,2)\vartheta_{00}(\tau;0,2)-6\vartheta_{00}''^{2}(\tau;0,2)\Bigr\}\vartheta_{00}^{2}(\tau;0,2).\end{aligned}$$

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