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THERMOELASTIC EQUILIBRIUM OF BODIES IN GENERALIZED CYLINDRICAL COORDINATES

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ABSTRACT. Using the method of separation of variables, an exact solution is constructed for some boundary value and boundary-contact problems of thermoelastic equilibrium of one- and multilayer bodies bounded by the coordinate surfaces of generalized cylindrical coordinates ρ, α, z. ρ, α are the orthogonal coordinates on the plane and z is the linear coordinate. The body, occupying the domain $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < z_1\}$, is subjected to the action of a stationary thermal field and surface disturbances (such as stresses, displacements, or their combinations) for $z = 0$ and $z = z_1$. Special type homogeneous conditions are given on the remainder of the surface. The elastic body is assumed to be transversally isotropic with the plane of isotropy $z =$ const and nonhomogeneous along z. The same assumption is made for the layers of the multilayer body which contact along $z =$ const.

INTRODUCTION

Boundary value problems of elastic equilibrium of a homogeneous isotropic layer which are related to the problems considered in this paper were previously investigated by Lamé and Clapeyron. In the subsequent studies, the solutions obtained by these authors were simplified and generalized. A sufficiently complete bibliography on this topic is given in [1], [2].

In all the mentioned papers, solutions of the problem were constructed by means of double integral transformation formulas mostly for a homogeneous layer in the absence of thermal disturbance. In this paper, using the method of separation of variables and double series, we construct solutions of static boundary value and boundary-contact problems of thermoelasticity [3] for the linear coordinate parallelepiped $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < z_1\}$, where ρ, α, z are the generalized cylindrical coordinates (ρ, α are the orthogonal coordinates on the plane and z is a linear coordinate). In

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addition to thermal disturbance, either stresses or displacements or their combinations are given for $z = 0$ and $z = z_1$. Special type homogeneous boundary conditions are given on the lateral surfaces ($\rho = \rho_0$, $\rho = \rho_1$, $\alpha = \alpha_0$, $\alpha = \alpha_1$). If a multilayer body is considered, then its layers contact along the planes $z = \text{const}$. An elastic body or the layers of a multilayer body consist of a transtropic (transversally isotropic) material which is specially non-homogeneous along z ($z = \text{const}$ is the plane of isotropy).

It follows from the above discussion that in this paper the problem of elastic equilibrium of an infinite layer is generalized (despite special type homogeneous boundary conditions given on the lateral surfaces of the body) and solved by a simple method. The simplification is achieved by 1) transforming the thermal problem and constructing a general solution for the considered class of three-dimensional boundary value problems of thermoelasticity; 2) replacing the classical conditions on the boundary and contact surfaces by the equivalent ones; 3) using a double series instead of a double integral transformation. In conclusive Remarks 1 and 2, solutions of some nontrivial problems of thermoelasticity are given.

The effectiveness of the solutions can be characterized as follows.

Using the method of separation of variables, in the domain $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < z_1\}$ we can constuct an effective solution of the basic boundary value problems for the Laplace equation, with the zero conditions for $\rho = \rho_j$ and $\alpha = \alpha_j$, where $j = 0, 1$. Then, likewise effectively, in the same domain Ω and by the same method we can find a thermoelastic equilibrium of the considered bodies.

To conclude the introduction, note that the Lamé coefficients of the system ρ, α, z [2] are

$$h_\rho = h_\alpha = h = \sqrt{\Big(\frac{\partial z}{\partial \rho}\Big)^2 + \Big(\frac{\partial y}{\partial \rho}\Big)^2}, \quad h_z = 1,$$

and that

$$\frac{\partial x}{\partial \rho} - \frac{\partial y}{\partial \alpha} = 0, \quad \frac{\partial x}{\partial \alpha} + \frac{\partial y}{\partial \rho} = 0, \quad \frac{\partial}{\partial \rho}\Big(\frac{1}{h}\frac{\partial h}{\partial \rho}\Big) + \frac{\partial}{\partial \alpha}\Big(\frac{1}{h}\frac{\partial h}{\partial \alpha}\Big) = 0,$$

where x, y are the Cartesian coordinates.

§ 1. Equations of State, Boundary Conditions, a General Solution, the Uniqueness of a Solution

1.1. If the thermal field does not depend on time and no mass force is given, then the elastic equilibrium of a transtropic body which is nonhomogeneous along z can be described in terms of the generalized cylindrical coordinates by the known equations [3,4]. In particular, the equilibrium equations have the form

$$\left.\begin{aligned}
&\frac{\partial}{\partial\rho}(hR_\rho)+\frac{1}{h}\frac{\partial}{\partial\alpha}(h^2R_\alpha)+h^2\frac{\partial R_z}{\partial z}-\frac{\partial h}{\partial\rho}A_\alpha=0,\\
&\frac{\partial}{\partial\alpha}(hA_\alpha)+h^2\frac{\partial A_z}{\partial z}+\frac{1}{h}\frac{\partial}{\partial\rho}(h^2A_\rho)-\frac{\partial h}{\partial\alpha}R_\rho=0,\\
&h^2\frac{\partial Z_z}{\partial z}+\frac{\partial}{\partial\rho}(hZ_\rho)+\frac{\partial}{\partial\alpha}(hZ_\alpha)=0.
\end{aligned}\right\}\tag{1}$$

where R_ρ, A_α, Z_z are normal stresses; $R_\alpha=A_\rho$, $R_z=Z_\rho$, $A_z=Z_\alpha$ are tangential stresses. As for the physical law, it is written in the form

$$\left.\begin{aligned}
R_\rho={}&c_1e_{\rho\rho}+(c_1-2c_5)e_{\alpha\alpha}+c_3e_{zz}-k_{10}T=\frac{c_1}{h^2}\Big(\frac{\partial(hu)}{\partial\rho}+\frac{\partial(hv)}{\partial\alpha}\Big)-\\
&-2c_5\Big(\frac{1}{h}\frac{\partial v}{\partial\alpha}+\frac{1}{h^2}\frac{\partial h}{\partial\rho}u\Big)+c_3\frac{\partial w}{\partial z}-k_{10}T,\\
A_\alpha={}&c_1e_{\alpha\alpha}+(c_1-2c_5)e_{\rho\rho}+c_3e_{zz}-k_{10}T=\frac{c_1}{h^2}\Big(\frac{\partial(hu)}{\partial\rho}+\frac{\partial(hv)}{\partial\alpha}\Big)-\\
&-2c_5\Big(\frac{1}{h}\frac{\partial u}{\partial\rho}+\frac{1}{h^2}\frac{\partial h}{\partial\alpha}v\Big)+c_3\frac{\partial w}{\partial z}-k_{10}T,\\
Z_z={}&c_2e_{zz}+c_3(e_{\rho\rho}+e_{\alpha\alpha})-k_{20}T=\frac{c_3}{h^2}\Big(\frac{\partial(hu)}{\partial\rho}+\frac{\partial(hv)}{\partial\alpha}\Big)+\\
&+c_2\frac{\partial w}{\partial z}-k_{20}T,\ \ A_\rho=c_5e_{\alpha\rho}=c_5\Big[\frac{\partial}{\partial\rho}\Big(\frac{v}{h}\Big)+\frac{\partial}{\partial\alpha}\Big(\frac{u}{h}\Big)\Big],\\
Z_\rho={}&c_4e_{z\rho}=c_4\Big(\frac{\partial u}{\partial z}+\frac{1}{h}\frac{\partial w}{\partial\rho}\Big),\quad Z_\alpha=c_4e_{z\alpha}=c_4\Big(\frac{1}{h}\frac{\partial w}{\partial\alpha}+\frac{\partial v}{\partial z}\Big),
\end{aligned}\right\}\tag{2}$$

where u,v,w are the components of the displacement vector $\vec{U}$ along the tangents to the coordinate lines ρ,α,z; $e_{\rho,\rho},e_{\alpha\alpha},e_{zz},e_{z\rho}=e_{\rho z},e_{z\alpha}=e_{\alpha z},e_{\alpha\rho}=e_{\rho\alpha}$ are deformations; $c_j=c_j(z)$ $(j=1,2,\dots,5)$ are the elastic characteristics (for their expression in terms of the technical characteristics E_1,E_2,v_1,v_2,μ see [5]); $k_{10}=[2(c_1-c_5)k_1+c_3k_2]$, $k_{20}=(2c_3k_1+c_2k_2)$; $k_1=k_1(z)$ and $k_2=k_2(z)$ are the coefficients of linear thermal expansion in the plane of isotropy and along z; T is the elastic body temperature defined by the equation

$$\Delta_2T+\frac{1}{\lambda_1}\frac{\partial}{\partial z}\Big(\lambda_2\frac{\partial T}{\partial z}\Big)=0\tag{3}$$

with the corresponding boundary conditions. Here $\lambda_1=\lambda_1(z)$ and $\lambda_2=\lambda_1(z)$ are the heat conduction coefficients in the plane of isotropy and along z [3]; $\Delta_2=\frac{1}{h^2}\big(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial\alpha^2}\big)$; in the case of circular cylindrical coordinates r,α,z we have $h=r$ and the operation $\frac{\partial}{\partial\rho}$ is replaced by the operation $\tau\frac{\partial}{\partial r}$.

Using (1) and (2), we can obtain the following system with respect to $D, Z_\rho, Z_\alpha, K, u, v, w$:

$$\left.\begin{aligned}
&\text{a) } \frac{\partial}{\partial z}\Big(\frac{c_3}{c_1}D+\frac{c_1c_2-c_3^2}{c_1}\frac{\partial w}{\partial z}\Big)+\frac{1}{h^2}\Big[\frac{\partial}{\partial\rho}(hZ_\rho)+\frac{\partial}{\partial\alpha}(hZ_\alpha)\Big]=\\
&\qquad\qquad=\frac{\partial}{\partial z}\Big(\frac{c_1k_{20}-c_3k_{10}}{c_1}T\Big),\\
&\text{b) } \frac{\partial D}{\partial\rho}-\frac{\partial K}{\partial\alpha}+\frac{\partial(hZ_\rho)}{\partial z}=0,\quad \text{c) } \frac{\partial D}{\partial\alpha}+\frac{\partial(hZ_\alpha)}{\partial z}+\frac{\partial K}{\partial\rho}=0,\\
&\text{d) } \frac{1}{h^2}\Big[\frac{\partial(Z_\alpha)}{\partial\rho}-\frac{\partial(hZ_p)}{\partial\alpha}\Big]-c_4\frac{\partial}{\partial z}\Big(\frac{K}{c_5}\Big)=0,
\end{aligned}\right\}\tag{4}$$

$$\left.\begin{aligned}
&\text{a) } \frac{1}{h^2}\Big[\frac{\partial(hu)}{\partial\rho}+\frac{\partial(hv)}{\partial\alpha}\Big]+\frac{c_3}{c_1}\frac{\partial w}{\partial z}-\frac{k_{10}}{c_1}T=\frac{D}{c_1},\\
&\text{b) } \frac{\partial(hu)}{\partial z}+\frac{\partial w}{\partial\rho}=\frac{hZ_\rho}{c_4},\\
&\text{c) } \frac{\partial w}{\partial\alpha}+\frac{\partial(hu)}{\partial z}=\frac{hZ_\alpha}{c_4},\quad \text{d) } \frac{\partial(hv)}{\partial\rho}-\frac{\partial(hu)}{\partial\alpha}=\frac{h^2K}{c_5}.
\end{aligned}\right\}\tag{5}$$

By virtue of (5) it is easy to verify that equality (4d) is the identity.

Next, we shall consider thermoelastic equilibrium of the curvilinear coordinate parallelepiped (CCP) occupying the domain $\Omega=\{\rho_0<\rho<\rho_1,\ \alpha_0<\alpha<\alpha_1,\ 0<z<z_1\}$. We shall use boundary conditions of the form

$$\left.\begin{aligned}
&\text{for } \rho=\rho_j:\ a)\ \frac{\partial T}{\partial\rho}=0,\ u=0,\ K=0,\ Z_\rho=0,\ \text{or}\\
&\qquad\qquad b)\ T=0,\ D=0,\ v=o,\ w=0.
\end{aligned}\right\}\tag{6}$$

$$\left.\begin{aligned}
&\text{for } \alpha=\alpha_j:\ a)\ \frac{\partial T}{\partial\alpha}=0,\ v=0,\ Z_\alpha=0,\ K=0,\ \text{or}\\
&\qquad\qquad b)\ T=0,\ D=0,\ w=o,\ u=0.
\end{aligned}\right\}\tag{7}$$

$$\left.\begin{aligned}
&\text{for } z=z_j:\ a)\ T=\tau_j(\rho,\alpha)\ \text{or}\ b)\ \frac{\partial T}{\partial z}=\tilde\tau_j(\rho,\alpha),\ \text{or}\\
&\qquad\qquad c)\ \frac{\partial T}{\partial z}+\Theta_jT=\tilde\tau_j(\rho,\alpha).
\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned}
&\text{for } z=z_j:\ a)\ Z_z=F_{j1}(\rho,\alpha),\ hZ_\rho=F_{j2}(\rho,\alpha),\ hZ_\alpha=F_{j3}(\rho,\alpha)\ \text{or}\\
&\qquad\qquad b)\ w=f_{j1}(\rho,\alpha),\ hu=f_{j2}(\rho,\alpha),\ hv=f_{j3}(\rho,\alpha),\ \text{or}\\
&\qquad\qquad c)\ w=f_{j1}(\rho\alpha),\ hZ_\rho=F_{j2}(\rho,\alpha),\ hZ_\alpha=F_{j3}(\rho\alpha),\ \text{or}\\
&\qquad\qquad d)\ Z_z=F_{j1}(\rho,\alpha),\ hu=f_{j2}(\rho,\alpha),\ hv=f_{j3}(\rho,\alpha),
\end{aligned}\right\}\tag{9}$$

where $j=0,1$, and $z_0=0$; Θ_j are the given constants. The conditions imposed on the functions $\tau_j(\rho,\alpha)$, $\tilde\tau_j(\rho\alpha)$, $f_{jl}(\rho,\alpha)$ and $F_{jl}(\rho,\alpha)$ $(l=1,2,3)$

will be discussed below; we only note that that these functions are chosen so that the compatibility conditions hold on the CCP edges.

We shall give a brief technical interpretation of the boundary conditions (6a), (7a), (9c) for $f_{j1} = 0$ and $F_{jl} = 0$ $(l = 2, 3)$ – conditions a_1 and (6b), (7b), (9d) for $F_{j1} = 0$ and $f_{jl} = 0$ $(l = 2, 3)$ – conditions a_2.

In the case of the conditions a_1 it can be assumed that the cylindrical or plane boundary S of the CCP is connected, respectively, with an absolutely smooth cylindrical or plane boundary surface S of an absolutely rigid body which is a thermal insulator.

Since the body is absolutely rigid, the normal to the S component of the displacement vector vanishes and, since S is absolutely smooth, we have $K = 0,\ Z_\rho = 0$ or $K = 0,\ Z_\alpha = 0$, or $Z_\rho = 0,\ Z_\alpha = 0$

In the case of the conditions a_2 it is assumed that an absolutely flexible but absolutely nontensile and noncompressible thin plate is glued onto the cylindrical or plane boundary surface S of the CCP (naturally, the plate takes the shape of S).

Since the plate is absolutely nontensile and noncompressible, we have $v = 0,\ w = 0$ or $u = 0,\ w = 0$, or $u = 0,\ v = 0$, and since it is absolutely flexible, we have $D = 0$ (the condition $T = 0$ for $\rho = \rho_j$ and $\alpha = \alpha_j$ is satisfied by other technical means).

Note that the less the curvature of the cylindrical boundary surface $\rho = \rho_j$, the less conditions (6a), (6b) differ, respectively, from the conditions

$$\left.\begin{aligned} &a)\ \frac{\partial T}{\partial \rho} = 0,\ u = 0,\ A_\rho = 0,\ Z_\rho = 0 \text{ and} \\ &b)\ T = 0,\ R_r = 0,\ v = 0,\ w = 0 \end{aligned}\right\} \tag{10}$$

for $\rho = \rho_j$. Conditions (6a) and (6b) are equivalent to conditions (10a) and (10b) when $\rho = \rho_j$ is a plane. A similar reasoning holds for the surface $\alpha = \alpha_j$ and conditions (7).

1.2. When $\lambda_1 = \text{const}$ and $\lambda_2 = \text{const}$, in a thermally homogeneous medium the heat conduction equation (3) takes the form

$$\Delta_2 T + \lambda_0 \frac{\partial^2 T}{\partial z^2} = 0, \tag{11}$$

where $\lambda_0 = \lambda_2/\lambda_1$. Now, using the method of separation of variables, the function T in the domain $\Omega = \{\rho < \rho_0 < \rho_1,\ \alpha_0 < \alpha < \alpha_1,\ 0 < z < z_1\}$ is written as

$$T = t_0 + t_1 z + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(A_{Tmn} e^{-p_r z} + B_{Tmn} e^{p_r (z - z_1)}\right) \psi_{mn}(\rho, \alpha), \tag{12}$$

where $t_0, t_1, p_r = \lambda_0^{-0.5} p(m,n) \geq 0,\ A_{Tmn},\ B_{Tmn}$ are constants; $\psi_{mn}(\rho,\alpha)$ is a nontrivial solution of the Sturm–Liouville problem [7]

$$\Delta_2 \psi_{mn} + p^2 \psi_{mn} = 0; \tag{13}$$

$$\text{for } \rho = \rho_j: \ a)\ \psi_{mn} = 0 \text{ or } b)\ \frac{\partial}{\partial \rho} \psi_{mn} = 0; \tag{14}$$

$$\text{for } \alpha = \alpha_j: \ a)\ \psi_{mn} = 0 \text{ or } b)\ \frac{\partial}{\partial \alpha} \psi_{mn} = 0. \tag{15}$$

Conditions (14) and (15) follow from conditions (6) and (7).

For the Cartesian coordinate system, when $\rho = x,\ \alpha = y,\ h = 1,\ \psi_{mn}$ is the product of trigonometric functions; in the case of cylindrical coordinates, when $\rho = r,\ \alpha = \alpha,\ x = r\cos\alpha,\ y = r\sin\alpha,\ h = r,\ \psi_{mn}$ is the product of trigonometric and Bessel functions; for a cylindrical-elliptic coordinate system, when $x = c\cosh\rho\cos\alpha,\ y = c\sinh\rho\sin\alpha,\ h = c\sqrt{0,5(\cosh 2\rho - \cos 2\alpha)}$, and c is a scale factor, ψ_{mn} is the product of Mattieu functions; in a cylindrical-parabolic coordinate system, when $x = c\frac{\rho^2-\alpha^2}{2},\ y = \rho\alpha,\ h = c\sqrt{\rho^2+\alpha^2},\ \psi_{mn}$ is the product of Weber functions [8,9].

If the medium is thermally nonhomogeneous ($\lambda_1 = \lambda_1(z)$ and $\lambda_2 = \lambda_2(z)$), then

$$T = \eta_{T0}(z) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \eta_{Tmn}(z) \psi_{mn}(\rho,\alpha),$$

where $\eta_{T0}(z)$ and $\eta_{Tmn}(z)$ are solutions of the equations

$$\frac{d}{dz}\Big(\lambda_2 \frac{d\eta_{T0}}{dz}\Big) = 0 \text{ and } \frac{1}{\lambda_1}\frac{d}{dz}\Big(\lambda_2 \frac{d\eta_{Tmn}}{dz}\Big) - \rho^2 \eta_{Tmn} = 0.$$

In the generalized cylindrical coordinates, the solution of equation (11) can be written in the form

$$T = \sum_{s=0}^{\infty} (p_{1s} T_{1s} + p_{2s} T_{2s}),$$

where

$$T_{1s} = z^s \sum_{q=0}^{[\frac{s}{2}]} \lambda_0^{q-\frac{s}{2}} a_{1q} \Big(\frac{r}{z}\Big)^{2q},\ T_{2s} = z^s \sum_{q=0}^{\infty} \lambda_0^{q-\frac{s}{2}} a_{1q} (\ln r - a_{2q}) \Big(\frac{r}{z}\Big)^{2q};$$

$$a_{1q} = \frac{(-1)^q \cdot s!}{4^q q! q! (s-2q)!}, \quad a_{2q} = \frac{1}{2}\Big[\frac{1}{q+1} - \sum_{\tilde q=1}^{q} \frac{2\tilde q + 1}{\tilde q(\tilde q+1)}\Big],$$

$\left[\frac{s}{2}\right]$ is the integer part of the number $\frac{s}{2}$, $0! = 1$, $a_{10} = 1$, $a_{20} = 0,5$, p_{1s}, p_{2s} are arbitrary constants. The following relations hold:

$$T_{1(s-1)} = \frac{1}{\lambda_0}\frac{\partial}{\partial z}T_{1s},\ T_{2(s-1)} = \frac{1}{\lambda_0}\frac{\partial}{\partial z}T_{2s}.$$

In the Cartesian coordinate system $r = \sqrt{x^2+y^2}$; in the cylindrical coordinate system r is one of the coordinates and thus we obtain an axially symmetric solution; in the cylindrical-elliptic coordinate system $r = c\sqrt{0,5(\cosh 2\rho + \cos 2\alpha)}$, where c is a scale factor; in the cylindrical-parabolic coordinate system $r = 0,5c \cdot (\rho^2 + \alpha^2)$; in the cylindrical-bipolar coordinate system $r = c\sqrt{\frac{\cosh\rho - \cos\alpha}{\cosh\rho + \cos\alpha}}$.

Next, for a thermally homogeneous medium we assume that

$$T = \frac{\partial^2}{\partial z^2}\tilde{T}, \tag{16}$$

where

$$\tilde{T} = \tilde{T}_0 + \tilde{T}_1 = \frac{t_0}{2}\left(z^2 - \frac{\lambda_0}{2}r^2\right) + \frac{t_1}{6}\left(z^3 - \frac{3\lambda_0}{2}zr^2\right) + \\ + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{Tmn}e^{-p_T z} + B_{Tmn}e^{p_T(z-z_1)}\right)\frac{\psi_{mn}(\rho,\alpha)}{p_T^2}. \tag{17}$$

It can be easily verified that $\tilde{T}$ satisfies the same equation as T. In the expression for $\tilde{T}, \tilde{T}_0$ is the polynomial part of T (i.e., the terms with the coefficients t_0 and t_1) and $\tilde{T}_1$ is the remaining part of $\tilde{T}$.

1.3. To apply the method of separation of variables for solving the considered boundary value problems, we represent the boundary conditions (9) as for $z = z_j$:

$$\begin{aligned} &a) Z_z = F_{j1}(\rho,\alpha),\ \Gamma_1(hZ_\rho, hZ_\alpha) = \tilde{F}_{j2}(\rho,\alpha),\ \Gamma_2(hZ_\alpha, hZ_\rho) = \tilde{F}_{j3}(\rho,\alpha) \text{ or} \\ &b) w = f_{j1}(\rho,\alpha),\ \Gamma_1(hu, hv) = \tilde{f}_{j2}(\rho,\alpha),\ \Gamma_2(hv, hu) = \tilde{f}_{j3}(\rho,\alpha) \text{ or} \\ &c) w = f_{j1}(\rho,\alpha),\ \Gamma_1(hZ_\rho, hZ_\alpha) = \tilde{F}_{j2}(\rho,\alpha),\ \Gamma_2(hZ_\alpha, hZ_\rho) = \tilde{F}_{j3}(\rho,\alpha) \text{ or} \\ &d) Z_z = F_{j1}(\rho,\alpha),\ \Gamma_1(hu, hv) = \tilde{f}_{j2}(\rho,\alpha),\ \Gamma_2(hv, hu) = \tilde{f}_{j3}(\rho,\alpha), \end{aligned} \tag{18}$$

where $\Gamma_1(g_1, g_2) = \frac{1}{h^2}\left(\frac{\partial g_1}{\partial\rho} + \frac{\partial g_2}{\partial\alpha}\right), \Gamma_2(g_2, g_1) = \frac{1}{h^2}\left(\frac{\partial g_2}{\partial\rho} - \frac{\partial g_1}{\partial\alpha}\right)$, $g_1 = hZ_\rho$ or $g_1 = hu$, $g_2 = hZ_\alpha$ or $g_2 = hv$. It is assumed that the functions $\tilde{\tau}_j(\rho,\alpha)$, $\tilde{F}_{j2}(\rho,\alpha)$, and $\tilde{F}_{j3}(\rho,\alpha)$, the functions $\tau_j(\rho,\alpha)$, $F_{j1}(\rho,\alpha), \tilde{f}_{j2}(\rho,\alpha)$, and $\tilde{f}_{j3}(\rho,\alpha)$ with their first derivatives, the function $f_{j1}(\rho,\alpha)$ with its first and second derivatives are expanded into absolutely and uniformly converging Fourier series with respect to the eigenfunctions of problem (13), (14), (15). The expansion with respect to functions ψ_{mn} can also be assumed valid, at least formally, when in equation (13) the variables are not separated (into cylindrical-bipolar coordinates).

The equivalence of the boundary conditions (9) and (18) will be discussed below.

The aim of this study is to construct a regular solution of the boundary value problems (3), (4), (5), (6), (7), (8), (9) or (3), (4), (5), (6), (7), (8), (18). For this we must define the notion of regularity.

A solution of system (4),(5) defined by the functions u, v, w will be called regular if the functions u, v, w are three times continuously differentiable in the domain Ω, where $\tilde{\Omega}$ is the domain Ω with the boundaries $\rho = \rho_j$ and $\alpha = \alpha_j$ on the surface $z = z_j$ can be represented together with their first and second derivatives by absolutely and uniformly converging Fourier series with respect to the eigenfunctions of problem (13), (14), (15). Moreover, it is assumed that the equilibrium equations hold for $\rho = \rho_j$ and $\alpha = \alpha_j$.

1.4. By virtue of the compatibility conditions on the CCP edges we conclude that the boundary conditions (9) and (18) will be equivalent if in the domain $\Omega = \{\rho_0 < \rho < \rho_1,\ \alpha_0 < \alpha < \alpha_1\}$ the boundary value problem

$$\frac{\partial g_1}{\partial \rho} + \frac{\partial g_2}{\partial \alpha} = 0,\ \frac{\partial g_2}{\partial \rho} - \frac{\partial g_1}{\partial \alpha} = 0; \tag{19}$$

$$\text{for } \rho = \rho_j :\ a)\ g_2 = 0,\ \frac{\partial g_1}{\partial \rho} = 0 \text{ or } b)\ g_1 = 0,\ \frac{\partial g_2}{\partial \rho} = 0; \tag{20}$$

$$\text{for } \alpha = \alpha_j :\ a)\ g_1 = 0,\ \frac{\partial g_2}{\partial \alpha} = 0 \text{ or } b)\ g_2 = 0,\ \frac{\partial g_1}{\partial \alpha} = 0 \tag{21}$$

has only the trivial solution.

According to Keldysh–Sedov's theorem [10], the boundary value problem (19), (20), (21), except for problems (19), (20a), (21b) and (19), (20b), (21a), has a solution

$$g_1 = 0,\ g_2 = 0.$$

The boundary value problem (19), (20a), (21b) has a solution

$$g_1 = g_{10} = \text{const}, \quad g_2 = 0, \tag{22}$$

while the boundary value problem (19), (20b), (21a) has a solution

$$g_1 = 0, \quad g_2 = g_{20} = \text{const}. \tag{23}$$

A difficulty created by the nonzero solution of problem (19), (20a), (21b) and (19), (20b), (21a) can be overcome as follows: To the solution of the boundary value problem (3), (4), (5), (8), (6a), (7b), (18) we add the solution

$$hu = 0, \quad w = 0, \quad hv = b_1 + b_2 \cdot l_z, \tag{24}$$

while to the solution of the boundary value problem (3), (4), (5), (8), (6b), (7a), (18) we add the solution

$$hv = 0, \quad w = 0, \quad hu = b_3 + b_4 \cdot l_z. \tag{25}$$

In (24) and (25), $l_z = \int c_4^{-1} dz$ (for $c_4 = \text{const}$ $l_z = c_4^{-1} \cdot z$); b_1, b_2, b_3, b_4 are constants.

1.5. It follows from (4b,c,d) that

$$\Delta_2 K + \frac{\partial}{\partial z}\Big[c_4 \frac{\partial}{\partial z}\Big(\frac{1}{c_5} K\Big)\Big] = 0. \tag{26}$$

Then the boundary conditions (6), (7) imply that on the lateral surfaces $\rho = \rho_j$ and $\alpha = \alpha_j$ of the CCP the function K or its normal derivative (see equations (4b,c,d)) is equal to zero. As for the surfaces $z = z_j$, by (18) we have

$$\Gamma_2(hZ_\alpha, hZ_\rho) = c_4 \frac{\partial}{\partial z}\Big(\frac{1}{c_5} K\Big), \quad \Gamma_2(hv, hu) = \frac{1}{c_5} K.$$

Hence for the function K we obtain a classical problem of mathematical physics which consists in defining K by equation (26) when either the function K or its normal derivative is defined on the surface of the domain of definition of K, or K is given on one part of the surface and its normal derivative (for $c_5 = c_5(z)$) on the other; the expression $\frac{1}{c_5}\frac{\partial K}{\partial z} + \big(\frac{1}{c_5}\big)' \cdot K$ can be given on $z = z_j$.

Thus a general elastic field corresponding to the considered boundary value problems can be represented as a sum (superposition) of a solenoidal field for $\operatorname{div} \vec{U} = 0$, $w = 0$, $T = 0$ and a thermoelastic field with a plane rotor of the displacement vector for $\operatorname{rot}_z \vec{U} = 0$.

Using the method of separation of variables the function K can be written in the form

$$K = b_{10} + b_{11} \cdot l_z + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K_{mn}(z) \cdot \bar{\psi}_{mn}(\rho, \alpha), \tag{27}$$

where b_{10}, b_{11} are constants and $K_{mn}(z)$ is a solution of the equation

$$\frac{d}{dz}\Big[c_4 \frac{d}{dz}\Big(\frac{1}{c_5} K_{mn}\Big)\Big] - p_1^2 \cdot K_{mn} = 0,$$

where $p_1 = p_1(m, n)$. $\bar{\psi}_{mn}(\rho, \alpha)$ is a solution of problem (13), (14), (15). The function $\bar{\psi}_{mn}(\rho, \alpha)$ is conjugate to function $\psi_{mn}(\rho, \alpha)$ in the sense that if $\psi_{mn}|_{\rho=\rho_j} = 0$, then $\big(\frac{\partial}{\partial \rho}\bar{\psi}_{mn}\big)_{\rho=\rho_j} = 0$, but if $\big(\frac{\partial}{\partial \rho}\psi_{mn}\big)_{\rho=\rho_j} = 0$, then $\bar{\psi}_{mn}|_{\rho=\rho_j} = 0$, and vice versa; we have a similar situation for $\alpha = \alpha_j$.

It might seem that the constants b_{10} and b_{11} in (27) are nonzero when conditions (6b) are given for $\rho = \rho_j$ and conditions (7b) for $\alpha = \alpha_j$ (in all other cases b_{10} and b_{11} are equal to zero), but in that case we also have $b_{10} = 0$ and $b_{11} = 0$, since, as can be easily verified,

$$\int_{\rho_0}^{\rho_1}\int_{\alpha_0}^{\alpha_1} K \cdot h^2 d\rho d\alpha = 0.$$

Finally, for the considered class of boundary value problems of thermoelasticity we obtain

$$K = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K_{mn}(z) \cdot \bar{\psi}_{mn}(\rho,\alpha). \tag{28}$$

Without loss of generality the function K can be represented as

$$K = \frac{\partial}{\partial z}\Big[c_4 \frac{\partial}{\partial z}\Big(\frac{1}{c_5}\varphi_1\Big)\Big], \tag{29}$$

where

$$\Delta_2\varphi_1 + \frac{\partial}{\partial z}\Big[c_4 \frac{\partial}{\partial z}\Big(\frac{1}{c_5}\varphi_1\Big)\Big] = 0,$$

and, with (28) taken into account,

$$\varphi_1 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \varphi_{1mn}(z) \cdot \bar{\psi}_{mn}(\rho,\alpha), \tag{30}$$

where $\varphi_{1mn}(z)$ is a solution of the equation

$$\frac{d}{dz}\Big[c_4 \frac{d}{dz}\Big(\frac{1}{c_5}\varphi_{1mn}\Big)\Big] - p_1^2 \cdot \varphi_{1mn} = 0.$$

The convenience of representation (29) will be seen in our further discussion.

1.6.

Theorem 1. *For the considered class of boundary value problems of thermoelasticity, a general solution in the class of regular functions can be represented as*

$$\left.\begin{aligned} hu &= \frac{\partial}{\partial\rho}\Big(\varphi_3 + \frac{1}{2c_4}\varphi_2\Big) + \frac{1}{c_5}\frac{\partial\varphi_1}{\partial\alpha}, \quad hv = \frac{\partial}{\partial\alpha}\Big(\varphi_3 + \frac{1}{2c_4}\varphi_2\Big) - \frac{1}{c_5}\frac{\partial\varphi_1}{\partial\rho}, \\ w &= -\frac{\partial}{\partial z}\Big(\varphi_3 + \frac{1}{2c_4}\varphi_2\Big) + \frac{1}{c_4}\frac{\partial\varphi_2}{\partial z}. \end{aligned}\right\} \tag{31}$$

Here

$$\left.\begin{aligned}
&a)\ \frac{\partial}{\partial z}\Big[c_4\frac{\partial}{\partial z}\Big(\frac{1}{c_5}\varphi_1\Big)\Big]+\Delta_2\varphi_1=0,\\
&b)\ \frac{\partial^2\varphi_3}{\partial z^2}+\gamma_1\Delta_2\varphi_3+\gamma_2\Delta_2\varphi_2+\frac{1}{2}\Big(\frac{1}{c_4}\Big)''\cdot\varphi_2=\gamma_4 T,\\
&c)\ \frac{\partial^2\varphi_2}{\partial z^2}+\gamma_1\Delta_2\varphi_2+\gamma_3\Delta_2\varphi_3=\gamma_5 T,
\end{aligned}\right\}\tag{32}$$

where $\gamma_1=\frac{c_1c_2-c_3(c_3+2c_4)}{2c_2c_4}$, $\gamma_2=\frac{c_1c_2-(c_3+2c_4)^2}{4c_2c_4^2}$, $\gamma_3=\frac{c_1c_2-c_3^2}{c_2}$, $\gamma_4=\frac{c_2k_{10}-(c_3+2c_4)k_{20}}{2c_2c_4}$, $\gamma_5=\frac{c_2k_{10}-c_3k_{20}}{c_2}$.

Proof. By virtue of representation (29) we can rewrite (4b, c, d) in the form

$$\left.\begin{aligned}
&\frac{\partial D}{\partial\rho}+\frac{\partial}{\partial z}\Big[hZ_\rho-c_4\frac{\partial^2}{\partial\alpha\partial z}\Big(\frac{\varphi_1}{c_5}\Big)\Big]=0,\\
&\frac{\partial D}{\partial\alpha}+\frac{\partial}{\partial z}\Big[hZ_\alpha+c_4\frac{\partial^2}{\partial\rho\partial z}\Big(\frac{\varphi_1}{c_5}\Big)\Big]=0,\\
&\frac{\partial}{\partial\rho}\Big[hZ_\alpha+c_4\frac{\partial^2}{\partial\rho\partial z}\Big(\frac{\varphi_1}{c_5}\Big)\Big]-\frac{\partial}{\partial\alpha}\Big[hZ_\rho-c_4\frac{\partial^2}{\partial\alpha\partial z}\Big(\frac{\varphi_1}{c_5}\Big)\Big]=0,
\end{aligned}\right\}\tag{33}$$

from which it follows that there exists a function $\frac{\partial}{\partial z}\varphi_2$ such that

$$\left.\begin{aligned}
&D=-\frac{\partial^2\varphi_2}{\partial z^2},\quad hZ_\rho=\frac{\partial^2\varphi_2}{\partial\rho\partial z}+c_4\frac{\partial^2}{\partial\rho\partial z}\Big(\frac{\varphi_1}{c_5}\Big),\\
&hZ_\alpha=\frac{\partial^2\varphi_2}{\partial\alpha\partial z}-c_4\frac{\partial^2}{\partial\rho\partial z}\Big(\frac{\varphi_1}{c_5}\Big).
\end{aligned}\right\}\tag{34}$$

By substituting (34) and (29) into (5b,c,d) we obtain

$$\left.\begin{aligned}
&\frac{\partial}{\partial z}\Big[hu-\frac{\partial}{\partial\alpha}\Big(\frac{\varphi_1}{c_5}\Big)\Big]+\frac{\partial}{\partial\rho}\Big[w-\frac{1}{c_4}\frac{\partial\varphi_2}{\partial z}\Big]=0,\\
&\frac{\partial}{\partial\alpha}\Big[w-\frac{1}{c_4}\frac{\partial\varphi_2}{\partial z}\Big]+\frac{\partial}{\partial z}\Big[hv+\frac{\partial}{\partial\rho}\Big(\frac{\varphi_1}{c_5}\Big)\Big]=0,\\
&\frac{\partial}{\partial\rho}\Big[hv+\frac{\partial}{\partial\rho}\Big(\frac{\varphi_1}{c_5}\Big)\Big]-\frac{\partial}{\partial\alpha}\Big[hu-\frac{\partial}{\partial\alpha}\Big(\frac{\varphi_1}{c_5}\Big)\Big]=0,
\end{aligned}\right\}\tag{35}$$

from which it follows that there exists a function φ_3 such that

$$\left.\begin{aligned}
hu&=\frac{\partial}{\partial\rho}\Big(\varphi_3+\frac{\varphi_2}{2c_4}\Big)+\frac{1}{c_5}\frac{\partial\varphi_1}{\partial\alpha},\\
hv&=\frac{\partial}{\partial\alpha}\Big(\varphi_3+\frac{\varphi_2}{2c_4}\Big)-\frac{1}{c_5}\frac{\partial\varphi_1}{\partial\rho},\\
w&=-\frac{\partial}{\partial z}\Big(\varphi_3+\frac{\varphi_2}{2c_4}\Big)+\frac{1}{c_4}\frac{\partial\varphi_2}{\partial z}.
\end{aligned}\right\}\tag{36}$$

The substitution of (34) and (36) into (4a) and (5a) gives

$$\left.\begin{aligned}
&a)\ \frac{\partial}{\partial z}\Big[\Delta_2\varphi_2-\frac{c_3}{c_1}\frac{\partial^2\varphi_2}{\partial z^2}-\frac{c_1c_2-c_3^2}{c_1}\frac{\partial}{\partial z}\Big(\frac{\partial\varphi_3}{\partial z}-\frac{1}{2c_4}\frac{\partial\varphi_2}{\partial z}+\\
&\qquad+\Big(\frac{1}{2c_4}\Big)'\varphi_2\Big)-\frac{c_1k_{20}-c_3k_{10}}{c_1}T\Big]=0,\\
&b)\ \Delta_2\varphi_3+\frac{1}{2c_4}\Delta_2\varphi_2+\frac{1}{c_1}\frac{\partial^2\varphi_2}{\partial z^2}-\frac{c_3}{c_1}\frac{\partial}{\partial z}\Big(\frac{\partial\varphi_3}{\partial z}-\frac{1}{2c_4}\frac{\partial\varphi_2}{\partial z}+\\
&\qquad+\Big(\frac{1}{2c_4}\Big)'\varphi_2\Big)-\frac{k_{10}}{c_1}T=0,
\end{aligned}\right\}\tag{37}$$

where the prime is the derivative with respect to z.

After integrating equation (37) with respect to z (37a) we obtain

$$\left.\begin{aligned}
&\frac{\partial^2\varphi_3}{\partial z^2}+\gamma_1\Delta_2\varphi_3+\gamma_2\Delta_2\varphi_2+\Big(\frac{1}{2c_4}\Big)''\varphi_2-\gamma_4T=-\frac{c_3+2c_4}{2c_2c_4}f(\rho,\alpha),\\
&\frac{\partial^2\varphi_2}{\partial z^2}+\gamma_1\Delta_2\varphi_2+\gamma_3\Delta_2\varphi_3-\gamma_5T=-\frac{c_3}{c_2}f(\rho,\alpha),
\end{aligned}\right\}\tag{38}$$

where $f(\rho,\alpha)$ is the function appearing as a result of the integration of (37a) with respect to z.

A particular solution of system (38) can be written in the form

$$\varphi_2^*=\chi^*,\ \varphi_3^*=\frac{1}{2c_4}\chi^*,\ T^*=0,\tag{39}$$

where χ^* is in turn a particular solution of the equation $\Delta_2\chi=f(\rho,\alpha)$. The substituion of (39) into (36) shows that $w^*=0,\ v^*=0,\ u^*=0$, so that without loss of generality it can be assumed that $f(\rho,\alpha)=0$.

Therefore $\varphi_1,\ \varphi_2$, and φ_3 satisfy equations (32). Since (36) and (31) coincide, this proves Theorem 1. □

It is interesting to note that if $T=0$ and in (18) $F_{j1}=0$ or $f_{j1}=0$, $\tilde F_{j2}=0$ or $\tilde f_{j2}=0$, then $\varphi_2=0,\ \varphi_3=0$ and thus the solution of the problem of elastic equilibrium of the CCP is reduced to finding φ_1, i.e., to integrating equation (32a) with the corresponding boundary conditions.

1.7. For a homogeneous transtropic medium (which is assumed to be thermally homogeneous) we assume that

$$\varphi_2=\gamma_3\cdot\Phi_2,\ \varphi_3=\Phi_3+\gamma_3\cdot a\cdot\Phi_2,\tag{40}$$

where $a=\sqrt{-\gamma_2/\gamma_3}\cdot i$, if $\gamma_2<0$, $i=\sqrt{-1}$; $a=\sqrt{\gamma_2/\gamma_3}$, if $\gamma_2\geq 0$. On substituting (40) and (16) into (32) and performing some transformations, we obtain

$$\left.\begin{aligned} &a)\ \frac{c_4}{c_5}\frac{\partial^2\varphi_1}{\partial z^2}+\Delta_2\varphi_1=0,\ b)\ \frac{1}{\gamma_1-a\gamma_3}\frac{\partial^2\Phi_3}{\partial z^2}+\Delta_2\Phi_3=\frac{\gamma_4-a\gamma_5}{\gamma_4-a\gamma_3}\frac{\partial^2\tilde{T}}{\partial z^2},\\ &c)\ \frac{1}{\gamma_1+a\gamma_3}\frac{\partial^2\Phi_2}{\partial z^2}+\Delta_2\Phi_2-\frac{1}{\gamma_1^2-\gamma_2\gamma_3}\frac{\partial^2\Phi_3}{\partial z^2}=\frac{\gamma_1\gamma_5-\gamma_3\gamma_4}{\gamma_3(\gamma_1^2-\gamma_2\gamma_3)}\frac{\partial^2\tilde{T}}{\partial z^2}. \end{aligned}\right\}\quad(41)$$

In the considered case (30) implies

$$\varphi_1=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{1mn}e^{-\tilde{p}_1 z}+B_{1mn}e^{\tilde{p}_1(z-z_1)}\right)\bar{\psi}_{mn}(\rho,\alpha), \qquad (42)$$

where $\tilde{p}_1=\sqrt{c_5/c_4}\cdot p_1(m,n)$, A_{1mn} and B_{1mn} are constants.

Let us now construct a general solution of system (41b,c) for different values of $\gamma_1,\gamma_2,\dots,\gamma_5$ (for every value the function φ_1 is given by formula (42)) and denote by $\tilde{\Phi}_2$ and $\tilde{\Phi}_3$ the functions satisfying the equations

$$\frac{1}{\gamma_1+a\gamma_3}\frac{\partial^2\tilde{\Phi}_2}{\partial z^2}+\Delta_2\tilde{\Phi}_2=0,\ \frac{1}{\gamma_1-a\gamma_3}\frac{\partial^2\tilde{\Phi}_3}{\partial z^2}+\Delta_2\tilde{\Phi}_3=0.$$

(a) If $a\neq 0$, $\lambda_0(\gamma_1\pm a\gamma_3)\neq 1$, and $\gamma_5\neq\lambda_0(\gamma_1\gamma_5-\gamma_3\gamma_4)$, then the solution of system (41b,c) has the form

$$\Phi_3=\tilde{\Phi}_3+G_1\tilde{T},\ \Phi_2=\tilde{\Phi}_2+G_2\tilde{\Phi}_3+G_3\tilde{T}, \qquad (43)$$

where

$$G_1=\frac{\gamma_4-a\gamma_5}{1-\lambda_0(\gamma_1-a\gamma_3)},\ G_2=-\frac{1}{2a\gamma_3},\ G_3=\frac{\gamma_3G_1+\gamma_1\gamma_5-\gamma_3\gamma_4}{\gamma_3(\gamma_1-a\gamma_3)[1-\lambda_0(\gamma_1+a\gamma_3)]}.$$

(b) If $a\neq 0$, $\lambda_0(\gamma_1-a\gamma_3)=1$, $\gamma_5\neq\lambda_0(\gamma_1\gamma_5-\gamma_3\gamma_4)$, then the solution of system (41b,c) has the form

$$\Phi_3=\tilde{\Phi}_3+G_1z\frac{\partial\tilde{T}}{\partial z},\ \Phi_2=\tilde{\Phi}_2+G_2\tilde{\Phi}_3+G_3z\frac{\partial\tilde{T}}{\partial z}+G_4\tilde{T}, \qquad (44)$$

where

$$G_1=\frac{\gamma_4-a\gamma_5}{2},\ G_2=-\frac{1}{2a\gamma_3},\ G_3=-\frac{G_1}{2a\gamma_3},\ G_4=\frac{\lambda_0\gamma_5(\gamma_2-a\gamma_1)-\gamma_4}{2\lambda_0\gamma_2\gamma_3}.$$

(c) If $a\neq 0$, $\lambda_0(\gamma_1+a\gamma_3)=1$, $\gamma_5\neq\lambda_0(\gamma_1\gamma_5-\gamma_3\gamma_4)$, then the solution of system (41b,c) has the form

$$\Phi_3=\tilde{\Phi}_3+G_1\tilde{T},\ \Phi_2=\tilde{\Phi}_2+G_2\tilde{\Phi}_3+G_3z\frac{\partial\tilde{T}}{\partial z}, \qquad (45)$$

where

$$G_1=\frac{\gamma_4-a\gamma_5}{2a\gamma_3},\ G_2=-\frac{1}{2a\gamma_3},\ G_3=\frac{\gamma_3G_1+\gamma_1\gamma_5-\gamma_3\gamma_4}{2\gamma_3(\gamma_1-a\gamma_3)}.$$

(d) If $a \neq 0,\ \gamma_5 = \lambda_0(\gamma_1\gamma_5 - \ \gamma_3\gamma_4)$, then no matter whether any of the conditions is fulfilled or not the solution of system (41b,c) has the form

$$\Phi_3 = \tilde{\Phi}_3 + G_1\tilde{T},\ \Phi_2 = \tilde{\Phi}_2 + G_2\tilde{\Phi}_3, \tag{46}$$

where

$$G_1 = \frac{\gamma_1\gamma_5 - \gamma_3\gamma_4}{\gamma_3},\ G_2 = -\frac{1}{2a\gamma_3}.$$

(e) If $a = 0$, (or $\gamma_2 = 0), \lambda_0\gamma_1 \neq 1,\ \gamma_5 \neq \lambda_0(\gamma_1\gamma_5 - \gamma_3\gamma_4)$, then the solution of system (41b, c) has the form

$$\Phi_3 = \tilde{\Phi}_3 + G_1\tilde{T},\ \Phi_2 = \tilde{\Phi}_2 + G_2 z\frac{\partial\tilde{\Phi}_3}{\partial z} + G_3\tilde{T}, \tag{47}$$

where

$$G_1 = \frac{\gamma_4}{1-\lambda_0\gamma_1},\ G_2 = \frac{1}{2\gamma_1},\ G_3 = \frac{\gamma_5(1-\lambda_0\gamma_1) + \lambda_0\gamma_3\gamma_4}{\gamma_3(1-\lambda_0\gamma_1)^2}.$$

(f) If $a = 0$, $(\gamma_2 = 0), \lambda_0\gamma_1 = 1,\ \gamma_5 \neq \lambda_0(\gamma_1\gamma_5 - \gamma_3\gamma_4)$, then the solution of system (1b,c) has the form

$$\Phi_3 = \tilde{\Phi}_3 + G_1 z\frac{\partial\tilde{T}}{\partial z},\ \Phi_2 = \tilde{\Phi}_2 + G_2 z\frac{\partial\tilde{\Phi}_3}{\partial z} + G_3 z^2\frac{\partial^2\tilde{T}}{\partial z^2} + G_4 z\frac{\partial\tilde{T}}{\partial z}, \tag{48}$$

where

$$G_1 = \frac{\gamma_4}{2},\ G_2 = \frac{1}{2\gamma_1},\ G_3 = \frac{\gamma_4}{8\gamma_1},\ G_4 = \frac{4\gamma_5 - \lambda_0\gamma_3\gamma_4}{8\gamma_3}.$$

(g) If $a = 0$ $(\gamma_2 = 0), \gamma_5 = \lambda_0(\gamma_1\gamma_5 - \gamma_3\gamma_4)$, then no matter whether the condition $\lambda_0\gamma_1 = 1$ is fulfilled or not the solution of system (41b,c) has the form

$$\Phi_3 = \tilde{\Phi}_3 + G_1\tilde{T},\ \Phi_2 = \tilde{\Phi}_2 + G_2 z\frac{\partial\tilde{\Phi}_3}{\partial z}, \tag{49}$$

where

$$G_1 = -\frac{\gamma_1\gamma_5 - \gamma_3\gamma_4}{\gamma_3},\ G_2 = \frac{1}{2\gamma_1}.$$

(h) In the isotropic case with $\lambda_0 = 1,\ k_1 = k_2 = k,\ \gamma_1 = 1,\ \gamma_2 = 0,\ \gamma_3 = \frac{E}{1-\nu^2},\ \gamma_4 = 0,\ \gamma_5 = \frac{E}{1-\nu}k$, where E is the elasticity modulus and ν is the Poisson coefficient, the solution of system (41b.c) takes the form

$$\Phi_3 = \tilde{\Phi}_3 - (1+\nu)k\tilde{T}, \quad \Phi_2 = \tilde{\Phi}_2 + \frac{z}{2}\frac{\partial\tilde{\Phi}_3}{\partial z}. \tag{50}$$

1.8. By imposing certain restrictions on the nonhomogeneity and anisotropy, we can simplify the integration of system (32) in some cases. However we shall not discuss this here but shall consider an isotropic nonhomogeneous medium with a constant shear modulus [11], which is interesting from the standpoint of thermal effects. The elastic equilibrium of the medium will be expressed in terms of harmonic functions. We assume that

$$\lambda_1=\lambda_2=\lambda=\text{const},\quad k_1=k_2=k=\frac{k_0(a_0z+1)}{2a_0z+2-\nu_0},$$
$$E=\frac{E_0(2a_0z+2-\nu_0)}{a_0z+1},\ \ \nu=\frac{a_0z+1-\nu_0}{a_0z+1},$$

where k_0, E_0, ν_0, and a_0, are physically admissible constants, and rewrite equation (32) in the form

$$\Delta\varphi_1=0,\ \Delta\varphi_3=0,\quad \Delta\varphi_2=\frac{E_0(a_0z+1)}{\nu_0}\frac{\partial^2}{\partial z^2}(\varphi_3+k_0\tilde{T}),\tag{51}$$

where $\Delta=\frac{\partial^2}{\partial z^2}+\Delta_2$.

The solution of system (51) can, in turn, be represented as

$$\varphi_1=\tilde{\varphi}_1,\varphi_3=\tilde{\varphi}_3-k_0\tilde{T},\ \varphi_2=\tilde{\varphi}_2+\frac{E_0}{4\nu_0}\Big(a_0z^2\frac{\partial\tilde{\varphi}_3}{\partial z}+2z\frac{\partial\tilde{\varphi}_3}{\partial z}-a_0z\tilde{\varphi}_3\Big),\tag{52}$$

where $\tilde{\varphi}_1$, $\tilde{\varphi}_2$, $\tilde{\varphi}_3$ and $\tilde{T}$ are harmonic functions.

1.9. The expressions for the functions T, $\tilde{T}$, and φ_1 are given by the respective formulas. We shall now write the expressions for the functions $\tilde{\Phi}_2$ and $\tilde{\Phi}_3$ appearing in Subsection 1.7, and for the functions $\tilde{\varphi}_2$ and $\tilde{\varphi}_3$ appearing in Subsection 1.8:

$$\tilde{\Phi}_j=\tilde{\Phi}_{j0}+\tilde{\Phi}_{j1}=b_{j1}+b_{j2}z+b_{j3}\Big[z^2-\frac{r^2}{2(\gamma_1+(-1)^ja\gamma_3)}\Big]+b_{j4}\Big[z^3-$$
$$-\frac{3zr^2}{2(\gamma_1+(-1)^ja\gamma_3)}\Big]+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\big(A_{jmn}e^{-\tilde{p}_jz}+B_{jmn}e^{\tilde{p}_j(z-z_1)}\big)\psi_{mn},\tag{53}$$

where $j=2,3$; b_{j1}, b_{j2}, b_{j3}, b_{j4}, A_{jmn}, B_{jmn}, and $\tilde{p}_j=(\gamma_1+(-1)^ja\gamma_3)^{0,5}\times p(m,n)$ are constants. Putting $a=0$, $\gamma_1=1$, $\tilde{p}_j=p$ in (53), we obtain the representations for the functions $\tilde{\varphi}_2$ and $\tilde{\varphi}_3$ figuring in (52).

1.10.

Theorem 2. *The homogeneous boundary value problems* (3), (4), (5), (6), (7), (8), (9) *have only the trivial solution in a class of regular functions.*

Proof. Let us prove the theorem for the boundary value problem (3), (4), (5), (6a), (7a) when $\tau_j = 0$, $\tilde{\tau}_j = 0$ and $f_{jl} = 0$, $F_{jl} = 0$. The energy equality for such a boundary value problem can be written in the form

$$\int_{\rho_0}^{\rho_1}\int_{\alpha_0}^{\alpha_1}\int_{0}^{z_1} Wh^2 d\rho d\alpha dz = -\int_{\alpha_0}^{\alpha_1}\int_{0}^{z_1}\Big[\Big(\frac{c_5}{h}\frac{\partial h}{\partial\rho}v^2\Big)_{\rho=\rho_0} + \Big(\frac{c_5}{h}\frac{\partial h}{\partial\rho}v^2\Big)_{\rho=\rho_1}\Big] d\alpha dz - \\ -\int_{\rho_0}^{\rho_1}\int_{0}^{z_1}\Big[\Big(\frac{c_5}{h}\frac{\partial h}{\partial\alpha}u^2\Big)_{\alpha=\alpha_0} + \Big(\frac{c_5}{h}\frac{\partial h}{\partial\alpha}u^2\Big)_{\alpha=\alpha_1}\Big] d\rho dz, \tag{54}$$

where W is the potential energy accumulated by unit volume of the CCP, $\frac{c_5}{h}\frac{\partial h}{\partial\rho} \geq 0$, $\frac{c_5}{h}\frac{\partial h}{\partial\alpha} \geq 0$. $\square$

Equation (54) implies that the considered homogeneous boundary value problem has only the trivial solution. Therefore the boundary value problem (3), (4), (5), (6a), (7a), (8), (9) admits one regular solution at most. In a similar manner one can prove that other homogeneous problems also have only the trivial solution, which completes the proof of Theorem 2. Therefore the boundary value problems (3), (4), (5), (6), (7), (8), (9) admit one regular solution at most.

§ 2. An Analytic Solution of Some Boundary Value Problems of Thermoelasticity

2.1. The representation of thermoelastic problems by the functions $\varphi_1, \tilde{\Phi}_2, \tilde{\Phi}_3, \tilde{\varphi}_2, \tilde{\varphi}_3$, and $\tilde{T}$ (see §1) enables us to write analytic solutions of quite a number of boundary value problems. For simplicity, this will be illustrated on homogeneous isotropic bodies.

Using formulas (50) and (40), we write the following expressions for displacements

$$\left.\begin{aligned} 2(1-\nu)hu &= z\frac{\partial^2\tilde{\Phi}_3}{\partial z\partial\rho} + 2(1-\nu)\frac{\partial\tilde{\Phi}_3}{\partial\rho} + 2\frac{\partial\tilde{\Phi}_2}{\partial\rho} + 4\gamma\frac{\partial\varphi_1}{\partial\alpha} - 2(1-\nu^2)k\frac{\partial\tilde{T}}{\partial\rho},\\ 2(1-\nu)hv &= z\frac{\partial^2\tilde{\Phi}_3}{\partial z\partial\alpha} + 2(1-\nu)\frac{\partial\tilde{\Phi}_3}{\partial\alpha} + 2\frac{\partial\tilde{\Phi}_2}{\partial\alpha} - 4\gamma\frac{\partial\varphi_1}{\partial\rho} - 2(1-\nu^2)k\frac{\partial\tilde{T}}{\partial\alpha},\\ 2(1-\nu)w &= z\frac{\partial^2\tilde{\Phi}_3}{\partial z^2} - (1-2\nu)\frac{\partial\tilde{\Phi}_3}{\partial z} + 2\frac{\partial\tilde{\Phi}_2}{\partial z} + 2(1-\nu^2)k\frac{\partial\tilde{T}}{\partial z}, \end{aligned}\right\} \tag{55}$$

where $\gamma = \frac{1-\nu^2}{E}$;

$$a)\ \Delta\varphi_1 = 0,\ b)\ \Delta\tilde{\Phi}_2 = 0,\ c)\ \Delta\tilde{\Phi}_3 = 0,\ d)\ \Delta\tilde{T} = 0. \tag{56}$$

By virtue of (17), (53), and (42) the functions $\tilde{T}, \tilde{\Phi}_2, \tilde{\Phi}_3$, and φ_1 take the form

$$\left.\begin{aligned}
&a)\ \tilde{T} = \tilde{T}_0 + \tilde{T}_1 = \frac{t_0}{2}\Big(z^2 - \frac{r^2}{2}\Big) + \frac{t_1}{6}\Big(z^3 - \frac{3zr^2}{2}\Big) + \\
&\quad + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\big(A_{Tmn}e^{-pz} + B_{Tmn}e^{p(z-z_1)}\big)\frac{1}{p_T^2}\psi_{mn}, \\
&b)\ \tilde{\Phi}_j = \tilde{\Phi}_{j0} + \tilde{\Phi}_{j1} = b_{j1} + b_{j2}z + b_{j3}\Big(z^2 - \frac{r^2}{2}\Big) + b_{j4}\Big(z^3 - \frac{3zr^2}{2}\Big) + \\
&\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\big(A_{jmn}e^{-pz} + B_{jmn}e^{p(z-z_1)}\big)\psi_{mn},\ j = 2, 3, \\
&c)\ \varphi_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\big(A_{1mn}e^{-p_1 z} + B_{1mn}e^{p_1(z-z_1)}\big)\bar{\psi}_{mn}.
\end{aligned}\right\} \quad (57)$$

The necessity of replacing conditions (9) by conditions (18) in the case of a homogeneous isotropic body is confirmed by the formulas

$$\begin{aligned}
&w = \frac{z}{2(1-\nu)}\frac{\partial^2\tilde{\Phi}_3}{\partial z^2} - \frac{1}{2(1-\nu)}\frac{\partial}{\partial z}\big[(1-2\nu)\tilde{\Phi}_3 - 2\tilde{\Phi}_2 - 2(1-\nu^2)k\cdot\tilde{T}\big], \\
&\Gamma_1(hu, hv) = -\frac{z}{2(1-\nu)}\frac{\partial^3\tilde{\Phi}_3}{\partial z^3} - \frac{1}{1-\nu}\frac{\partial^2}{\partial z^2}\big[(1-\nu)\tilde{\Phi}_3 + \tilde{\Phi}_2 - (1-\nu^2)k\tilde{T}\big], \\
&\Gamma_2(hv, hu) = \frac{2(1+\nu)}{E}\frac{\partial^2\varphi_1}{\partial z^2},\ \ Z_z = \frac{E}{1-\nu}\Big(\frac{z}{2}\frac{\partial^3\tilde{\Phi}_3}{\partial z^3} + \frac{\partial^2\tilde{\Phi}_2}{\partial z^2}\Big), \\
&\Gamma_1(hZ_\rho, hZ_\alpha) = -\frac{E}{1-\nu^2}\frac{\partial}{\partial z}\Big(\frac{z}{2}\frac{\partial^3\tilde{\Phi}_3}{\partial z^3} + \frac{\partial^2\tilde{\Phi}_2}{\partial z^2}\Big),\ \Gamma_2(hZ_\alpha, hZ_\rho) = \frac{\partial^3\varphi_1}{\partial z^3}.
\end{aligned}$$

Using similar formulas, one can show that it is also necessary to replace conditions (9) by conditions (18) in the case of both transtropic and non-homogeneous bodies.

Naturally, when considering boundary value problems admitting a rigid displacement of the elastic body, it is required of the boundary condition that the principal vector and principal moment be equal to zero. This will always be assumed in what follows.

2.2. Let a temperature field T act on the CCP occupying the domain $\Omega = \{\rho_0 < \rho < \rho_1, \alpha_0 < \alpha < \alpha_1, 0 < z < z_1\}$, and, on the CCP surface, let the following conditions be given: (8a), (6b) with $j = 0$, (6a) with $j = 1$, (7b) with $j = 0$, (7a) with $j = 1$, (9a). Then to find an elastic equilibrium of the CCP we write the functions $\tilde{T}, \tilde{\Phi}_2, \tilde{\Phi}_3$, and φ_1 in the form

$$\begin{gathered}\tilde{T} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_T(z)\frac{1}{p^2}\psi_{mn}(\rho,\alpha),\ \ \tilde{\Phi}_j = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_j(z)\psi_{mn}(\rho,\alpha),\\ \varphi_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_1(z)\bar{\psi}_{mn}(\rho,\alpha),\end{gathered} \tag{58}$$

where $j = 2, 3$; $H_l(z) = A_{lmn}e^{-\bar{p}z} + B_{lmn}e^{\bar{p}(z-z_1)}$ with $\bar{p} = p$ for $l = T, 2, 3$, and $\bar{p} = p_1$ for $l = 1$ $\bar{p} = p_1$; $\psi_{mn}(\rho, \alpha)$ are the eigenfunctions of problem (13), (14a) with $j = 0$, of (14b) with $j = 1$, and (15a) with $j = 0$, of (15b) with $j = 1$. In the considered case, in (57) $\tilde{T}_0 = 0$, $\tilde{\Phi}_{j0} = 0$.

By virtue of (8a), (18a), (55) and (58) the constants A_{lmn}, B_{lmn} are defined by the following systems of linear algebraic equations:

$$H_T(0) = \tau_{0mn}, \quad H_T(z_1) = \tau_{1mn}, \tag{59}$$

$$\Big[\frac{d^3}{dz^3}H_1(z)\Big]_{z=0} = \tilde{F}_{03mn}, \quad \Big[\frac{d^3}{dz^3}H_1(z)\Big]_{z=z_1} = \tilde{F}_{13mn}, \tag{60}$$

$$\left.\begin{aligned}&\Big[\frac{z}{2}\frac{d^3}{dz^3}H_3(z) + \frac{d^2}{dz^2}H_2(z)\Big]_{z=0} = \gamma F_{01mn},\\ &\Big[\frac{z}{2}\frac{d^3}{dz^3}H_3(z) + \frac{d^2}{dz^2}H_2(z)\Big]_{z=z_1} = \gamma F_{11mn},\\ &\Big[\frac{d}{dz}\Big(\frac{z}{2}\frac{d^3}{dz^3}H_3(z) + \frac{d^2}{dz^2}H_2(z)\Big)\Big]_{z=0} = -\gamma\tilde{F}_{02mn},\\ &\Big[\frac{d}{dz}\Big(\frac{z}{2}\frac{d^3}{dz^3}H_3(z) + \frac{d^2}{dz^2}H_2(z)\Big)\Big]_{z=z_1} = -\gamma\tilde{F}_{12mn},\end{aligned}\right\} \tag{61}$$

where $\tau_{0mn}, \tau_{1mn}, F_{01mn}, F_{11mn}, \tilde{F}_{02mn}, \tilde{F}_{12mn}$ are the Fourier coefficients of the functions $\tau_0(\rho,\alpha), \tau_1(\rho,\alpha), F_{01}(\rho,\alpha), F_{11}(\rho,\alpha), \tilde{F}_{02}(\rho,\alpha), \tilde{F}_{12}(\rho,\alpha)$, respectively, expanded in a Fourier series with respect to the functions ψ_{mn}, $\tilde{F}_{03mn}$ and $\tilde{F}_{13mn}$ are the Fourier coefficients of the functions $F_{03}(\rho,\alpha)$ and $F_{13}(\rho,\alpha)$, respectively, expanded in a Fourier series with respect to the function $\bar{\psi}_{mn}$ [7].

It readily follows that (59) and (60) are systems with a second-order matrix and (61) is a system with a fourth-order matrix.

It is not difficult to prove that the corresponding functional series converge in a closed domain $\bar{\Omega}$ if we construct a uniformly converging numerical series majorizing these functional series in $\bar{\Omega}$. Indeed, for definiteness, let us consider the above problem in the Cartesian coordinate system in the domain $\Omega = \{0 < x < x_1, 0 < y < y_1, -z_1 < z < z_1\}$ for $T = 0$. It is assumed that $Z_z = F_{11}(x,y)$, $\Gamma_1(Z_x, Z_y) = -\tilde{F}_{12}(x,y)$, $\Gamma_2(Z_y, Z_x) = -\tilde{F}_{13}(x,y)$ for $z = -z_1$, and $Z_z = F_{11}(x,y)$, $\Gamma_1(Z_x, Z_y) = -\tilde{F}_{12}(x,y)$, $\Gamma_2(Z_y, Z_x) =$

$-\tilde{F}_{13}(x,y)$ for $z = z_1$ (the load is symmetrical with respect to the plane $z = 0$). In this case, too, the functions $\tilde{\Phi}_2, \tilde{\Phi}_3$ and φ_1 will take the form

$$\tilde{\Phi}_j = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{jmn}e^{-pz_1}\cosh(pz)\sin\Big[\frac{\pi(2m-1)}{2x_1}x\Big]\cdot\sin\Big[\frac{\pi(2n-1)}{2y_1}y\Big],$$
$$\varphi_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{1mn}e^{-pz_1}\cosh(pz)\cos\Big[\frac{\pi(2m-1)}{2x_1}x\Big]\cdot\cos\Big[\frac{\pi(2n-1)}{2y_1}y\Big],$$

where $j = 2,3$, $p = \sqrt{\left[\frac{\pi(2m-1)}{2x_1}\right]^2 + \left[\frac{\pi(2n-1)}{2y_1}\right]^2}$, and equations (60), (61) will be rewritten as

$$p^3\cdot A_{1mn}e^{-pz_1}\sinh(pz_1) = \tilde{F}_{13mn};$$

$$\begin{cases} pz_1\sinh(pz_1)A_{3mn} + 2\cosh(pz_1)\cdot A_{2mn} = p^{-2}\cdot 2\gamma\cdot e^{pz_1}\cdot F_{11mn}, \\ \big[pz_1\cosh(pz_1)+\sinh(pz_1)\big]A_{3mn} + 2\sinh(pz_1)\cdot A_{2mn} = -p^{-3}2\gamma e^{pz_1}F_{12mn}. \end{cases}$$

Using these equalities we obtain

$$A_{1mn} = \frac{e^{pz_1}}{\sinh(pz_1)}\cdot p^{-3}\cdot\tilde{F}_{13mn},$$
$$A_{3mn} = -\frac{4\gamma}{p^3}\frac{e^{pz_1}\cosh(pz_1)\cdot\tilde{F}_{12mn} + e^{pz_1}\cdot\sinh(pz_1)\cdot p\cdot F_{11mn}}{\sinh(2pz_1)+2pz_1}$$
$$A_{2mn} = \frac{\gamma e^{pz_1}}{\cosh(pz_1)}p^{-2}\cdot F_{11mn} - 0,5\cdot th(pz_1)\cdot pz_1\cdot A_{3mn}.$$

The latter equalities imply that there exists a positive constant A_0 such that the uniformly converging numerical series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_0\big(|\tilde{F}_{13mn}| + |\tilde{F}_{12mn}| + p|F_{11mn}|\big)$$

will majorize, in $\bar{\Omega}$, the functional series representing displacements and stresses by formulas (55) (in $\bar{\Omega}$ the displacements u, v, and w are the analytic functions of each coordinate).

We have thus obtained a regular solution of the considered boundary value problem (we think it reasonable that such solutions are sometimes called exact solutions [2]). Applying a similar technique, one can solve any of the boundary value problems (11), (4), (5), (6), (7), (8), (9) for a homogeneous isotropic CCP.

If the domain Ω is infinite, then in some cases the solution of the problem can be constructed by an integral transformation [1, 2].

To conclude this subsection, note that although the coordinate surfaces of this system of coordinates or another make it possible to consider a thermoelastic equilibrium of bodies of various shape, the mathematical tool of the solution remains the same. The geometric shape of an elastic body is defined only by the form of the parameters h, r and of the functions $\psi_{mn}, \bar{\psi}_{mn}$.

2.3. In the previous subsection, the solution of the boundary value problems was reduced to defining the constants A_{lmn} and B_{lmn} by means of two systems of linear equations with a second-order matrix and a system of linear equations with a fourth-order matrix. It appears that in most cases the solution of boundary value problems can be reduced to defining A_{lmn} and B_{lmn} by equations and systems of linear equations with a second-order matrix. We can illustrate this for a homogeneous isotropic medium by solving the boundary value problem (11), (4), (5), (6b), (7b), (8a), (9a) in the domain $\Omega = \{\rho_0 < \rho < \rho_1, \alpha_0 < \alpha < \alpha_1, -z_1 < z < z_1\}$. It is assumed that for $z = -z_1$ we have the conditions that are fulfilled in (9a) for $z = 0$. Let us denote this problem by l and write it in the form

$$I = I_1 + I_2.$$

Problem I_1 can be obtained from problem I if in the latter we set

$$T = 0,5(\tau_1 + \tau_0) = \tau^{(1)}, \ Z_z = 0,5(F_{11} + F_{01}) = F_1^{(1)},$$
$$hZ_\rho = 0,5(F_{12} - F_{02}) = F_2^{(1)}, \ hZ_\alpha = 0,5(F_{13} - F_{03}) = F_3^{(1)}$$

for $z = z_1$ and

$$T = 0,5(\tau_1 + \tau_0) = \tau^{(1)}, \ Z_z = 0,5(F_{11} + F_{01}) = F_1^{(1)},$$
$$hZ_\rho = -0,5(F_{12} - F_{02}) = -F_2^{(1)}, \ hZ_\alpha = -0,5(F_{13} - F_{03}) = -F_3^{(1)}$$

for $z = -z_1$. Problem I_2 can be obtained from problem I if in the latter we set

$$T = 0,5(\tau_1 - \tau_0) = \tau^{(2)}, \ Z_z = 0,5(F_{11} - F_{01}) = F_1^{(2)},$$
$$hZ_\rho = 0,5(F_{12} + F_{02}) = F_2^{(2)}, \ hZ_\alpha = 0,5(F_{13} + F_{03}) = F_3^{(2)}$$

for $z = z_1$ and

$$T = -0,5(\tau_1 - \tau_0) = -\tau^{(2)}, \ Z_z = -0,5(F_{11} - F_{01}) = -F_1^{(2)},$$
$$hZ_\rho = 0,5(F_{12} + F_{02}) = F_2^{(2)}, \ hZ_\alpha = 0,5(F_{13} + F_{03}) = F_3^{(2)}$$

for $z = -z_1$.

For problem I_1

$$\varphi_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{1mn} \cdot e^{-p_1 z_1} \cosh(p_1 z) \cdot \bar{\psi}_{mn}(\rho, \alpha),$$
$$\tilde{\Phi}_j = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{jmn} \cdot e^{p z_1} \cdot \cosh(pz) \cdot \psi_{mn}(\rho, \alpha),$$

where $j = T, 2, 3$ and the constants $A_{Tmn}, A_{1mn}, A_{2mn}$, and A_{3mn} are defined by the equations

$$A_{Tmn} \cdot e^{-pz_1} \cdot \cosh(pz_1) = \tau_{mn}^{(1)}, \quad p_1^3 \cdot A_{1mn} \cdot e^{-p_1 z_1} \cdot \sinh(p_1 z_1) = \tilde{F}_{3mn}^{(1)},$$

$$\begin{cases} pz_1 \cdot \sinh(pz_1) \cdot A_{3mn} + 2\cosh(pz_1) \cdot A_{2mn} = 2\gamma \cdot p^{-2} \cdot e^{pz_1} \cdot \tilde{F}_{1mn}^{(1)}, \\ \left[pz_1 \cdot \cosh(pz_1) + \sinh(pz_1)\right] \cdot A_{3mn} + 2\sinh(pz_1) \cdot A_{2mn} = \\ \qquad = -2\gamma \cdot p^{-3} e^{pz_1} \tilde{F}_{2mn}^{(1)}. \end{cases}$$

In the case of Problem I_2

$$\varphi_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{1mn} e^{-p_1 z_1} \sinh(p_1 z) \bar{\psi}_{mn}(\rho, \alpha),$$

$$\tilde{\Phi}_j = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{jmn} e^{pz_1} \sinh(pz) \psi_{mn}(\rho, \alpha),$$

where $j = T, 2, 3$, and the constants $B_{Tmn}, B_{1mn}, B_{2mn}$, and B_{3mn} are defined by the equations

$$B_{Tmn} e^{-pz_1} \sinh(pz_1) = \tau_{mn}^{(2)}, \quad p_1^3 \cdot B_{1mn} \cdot e^{-p_1 z_1} \cosh(p_1 z_1) = \tilde{F}_{3mn}^{(2)},$$

$$pz_1 \cosh(pz_1) \cdot B_{3mn} + 2\sinh(pz_1) \cdot B_{2mn} = 2\gamma p^{-2} e^{pz_1} \tilde{F}_{1mn}^{(2)},$$

$$\left[pz_1 \sinh(pz_1) + \cosh(pz_1)\right] B_{3mn} + 2\cosh(pz_1) A_{2mn} = -2\gamma \cdot p^{-3} e^{pz_1} \tilde{F}_{2mn}^{(2)}.$$

The homogeneous conditions (9c) and (9d) provide a continuous extension of the solution [12] and therefore problem I_1 is equivalent to the boundary value problem (11), (4), (5), (6b), (7b), (8a) for $z = 0$ and $\tilde{\tau}_0 = 0$; to (8a) for $z = z_1$ and $f_{01} = 0$, $F_{02} = 0$, $F_{03} = 0$, to (9a) for $z = z_1$. Problem I_2 is equivalent to (8a) for $z = 0$, $\tau_0 = 0$; for $z = z_1$, to (9d) for $z = 0$ and $F_{01} = 0$, $f_{02} = 0$. $f_{03} = 0$, to (9a) for $z = z_1$.

The method considered simplifies the investigation and solution of boundary value problems. It can be used (1) if the boundary conditions for $z = 0$ are of the same type as for $z = z_1$; (2) if the homogeneous conditions (9c) or (9d) are fulfilled for $z = 0$ or $z = z_1$; (3) if one of conditions (9c), (9d) is fulfilled for $z = 0$, and the other for $z = z_1$.

The above arguments also hold for a nonhomogeneous transtropic CCP occupying the domain $\Omega = \{\rho_0 < \rho < \rho_1, \alpha_0 < \alpha < \alpha_1, -z_1 < z < z_1\}$ provided that $\lambda_1(z), \lambda_2(z)$, and $c_j(z)$ $(j = 1, 2, 3, 4, 5)$ are even functions of the coordinate z.

2.4. Consider a CCP having layers along z and occupying the domain Ω_z. Here Ω_z is the union of the domains $\Omega_{z_1} = \{\rho_0 < \rho < \rho_1, \alpha_0 < \alpha < \alpha_1, 0 < z < z_1\}$, $\Omega_{z_2} = \{\rho_0 < \rho < \rho_1, \alpha_0 < \alpha < \alpha_1, z_1 < z < z_2\}, \dots$, $\Omega_{z_\beta} = \{\rho_0 < \rho < \rho_1, \alpha_0 < \alpha < \alpha_1, z_{\beta-1} < z < z_\beta\}$ contacting one another along the planes $z = z_j$, where $j = 1, 2, \dots, \beta - 1$, and β is the number of layers. Each layer has its own elastic and thermal characteristics. For $\rho = \rho_j$ $(j = 0, 1)$ some of conditions (6) are fulfilled for all layers simultaneously, and for $\alpha = \alpha_j$ some of conditions (7).

If the body occupies the domain Ω_z, then conditions (8), (18), (6), (7) are fulfilled on its boundaries after replacing z_1 by z_β in (8) and (18). On the contact surfaces $z = z_j$ $(j = 1, 2, \dots, \beta - 1;$ $z = z_j$ is the contact surface of the jth layer contacting the $(j+1)$th layer) we give the conditions

$$T_j - T_{j+1} = \tau_{j1}(\rho, \alpha), \ \frac{\partial T_j}{\partial z} - \frac{\partial T_{j+1}}{\partial z} = \tau_{j2}(\rho, \alpha); \tag{62}$$

$$\left.\begin{aligned}
&w_j - w_{j+1} = q_{j1}(\rho, \alpha), \ Z_{z_j} - Z_{z_{j+1}} = Q_{j1}(\rho, \alpha), \\
&\Gamma_1(hu_j, hv_j) - \Gamma_1(hu_{j+1}, hv_{j+1}) = \tilde{q}_{j2}(\rho, \alpha), \\
&\Gamma_1(hZ_{\rho_j}, hZ_{\alpha_j}) - \Gamma_1(hZ_{\rho_{(j+1)}}, hZ_{\alpha_{(j+1)}}) = \tilde{Q}_{j2}(\rho, \alpha), \\
&\Gamma_2(hv_j, hu_j) - \Gamma_2(hv_{j+1}, hu_{j+1}) = \tilde{q}_{j3}(\rho, \alpha), \\
&\Gamma_2(hZ_{\alpha_j}, hZ_{\rho_j}) - \Gamma_2(hZ_{\alpha_{(j+1)}}, hZ_{\rho_{(j+1)}}) = \tilde{Q}_{j3}(\rho, \alpha)
\end{aligned}\right\} \tag{63}$$

or

$$\left.\begin{aligned}
&w_j - w_{j+1} = q_{j1}(\rho, \alpha), \ Z_{z_j} - Z_{z_{j+1}} = Q_{j1}(\rho, \alpha), \\
&\Gamma_1(hZ_{\rho_j}, hZ_{\alpha_j}) = \tilde{Q}_{j2}(\rho, \alpha), \ \Gamma_2(hZ_{\alpha_j}, hZ_{\rho_j}) = \tilde{Q}_{j3}(\rho, \alpha), \\
&\Gamma_1(hZ_{\rho_{(j+1)}}, hZ_{\alpha_{(j+1)}}) = \tilde{Q}_{(j+1)2}(\rho, \alpha), \\
&\Gamma_2(hZ_{\alpha_{(j+1)}}, hZ_{\rho_{(j+1)}}) = \tilde{Q}_{(j+1)3}(\rho, \alpha),
\end{aligned}\right\} \tag{64}$$

where $\tau_{j1}(\rho, \alpha), \tau_{j2}(\rho, \alpha), \dots, \tilde{Q}_{(j+1)3}(\rho, \alpha)$ are the known functions.

To find a thermoelastic balance of the multilayer CCP, for the jth layer we must write, using conditions (6), (7), the expressions of the functions $\tilde{T}^{(j)}, \tilde{\Phi}_2^{(j)}, \tilde{\Phi}_3^{(j)}, \varphi_1^{(j)}$. Applying the arguments of Subsection 2.2, similarly to systems (59), (60), and (61) we obtain two systems of 2β equations with 2β unknowns and one system of 4β equations with 4β unknowns. Then we prove the solvability of the systems, convergence of the corresponding series, and uniqueness of the obtained regular solutions of the corresponding boundary-contact problems of thermoelasticity (solutions for the multilayer CCP are called regular if each of the solutions u_j, v_j and w_j is such).

In addition to the above contact conditions, we can consider some other contact conditions enabling us to solve boundary-contact problems of thermoelasticity with the same effectiveness.

2.5. All the arguments we used in this section for the homogeneous isotropic CCP or the multilayer CCP with isotropic homogeneous layers can be applied to transtropic bodies (see formulas (43)–(49) from Subsection 1.7) and to bodies with a special nonhomogeneity (see formulas (52)). As for composite bodies, we can find a thermoelastic equilibrium of the multilayer CCP every layer of which has its own thermal and transtropic characteristics (a piecewise-transtropic body) or its own elastic and thermal nonhomogeneity (a piecewise-nonhomogeneous body).

Note that if the multilayer CCP with transtropic layers is considered in the Cartesian coordinate system, then the class of solvable boundary-contact problems of thermoelasticity will become much broader. This is, in fact, the only coordinate system in which the problems can be solved using the boundary conditions (6), (7), (8), (9) and natural (but not of type (64)) contact conditions, i.e., without performing any transformations of the boundary and contact conditions. In that case, we shall have a greater number of possible contact and boundary conditions for $z = z_j$.

2.6. If the CCP is subjected only to the action of a thermal field and the condition $\gamma_5 = \lambda_0(\gamma_1\gamma_5 - \gamma_3\gamma_4)$ is satisfied, then by virtue of formulas (31), (2), (40), (46), (49), and (52) one can easily verify that the following two remarks are true.

Remark 1. When $\gamma_5 = \lambda_0(\gamma_1\gamma_5 - \gamma_3\gamma_4)$ and the boundary conditions (8), (6), (7), and (9a) are fulfilled for $F_{jl} = 0$, the thermoelastic equilibrium of a transtropic homogeneous CCP has the form

$$w = \frac{\gamma_5}{\lambda_0\gamma_3}\frac{\partial \tilde{T}}{\partial z}, \; hv = -\frac{\gamma_5}{\lambda_0\gamma_3}\frac{\partial \tilde{T}}{\partial \alpha}, \; hu = -\frac{\gamma_5}{\lambda_0\gamma_3}\frac{\partial \tilde{T}}{\partial \rho}.$$

For an isotropic body $\frac{\gamma_5}{\lambda_0\gamma_3} = k(1+\nu)$.

Remark 2. When the boundary conditions (8), (6), (7) and (9a) are fulfilled for $F_{jt} = 0$, the thermoelastic equilibrium of the nonhomogeneous CCP described in Subsection 1.8 has the form

$$w = k_0\frac{\partial \widetilde{T}}{\partial z}, \quad hv = -k_0\frac{\partial \widetilde{T}}{\partial \alpha}, \quad hu = -k_0\frac{\partial \widetilde{T}}{\partial \rho}.$$

The function $\widetilde{T}$ figuring in both remarks is defined by formula (17) when $\widetilde{T}_0 = 0$.

If the thermoelastic equilibrium of the CCP is considered when the boundary conditions (8), (6a), (7a), (9a) are fulfilled for $F_{ji} = 0$, then the theorems hold only provided that

$$\int_{\rho_0}^{\rho_1}\int_{\alpha_0}^{\alpha_1} r_j(\rho,\alpha)h^2 d\rho\, d\alpha = 0 \quad \text{or} \quad \int_{\rho_0}^{\rho_1}\int_{\alpha_0}^{\alpha_1} \tilde{r}_j(\rho,\alpha)h^2 d\rho\, d\alpha = 0, \qquad (65)$$

where $j = 0, 1$.

Remark 1 implies that if (65) and the equality $\gamma_5 = \lambda_0(\gamma_1\gamma_5 - \gamma_3\gamma_4)$ are satisfied, the boundary value problem (11), (41), (8), (6), (7), (9a) can be represented by combining the boundary value problem (11), (41), (8), (6), (7), (9a) for $F_{jt} = 0$ and the ordinary ($T = 0$) boundary value problem (41), (6), (7), (9a). Remark 2 leads to a similar conclusion.

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