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ASYMPTOTIC EXPANSION OF SOLUTIONS OF PARABOLIC EQUATIONS WITH A SMALL PARAMETER

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ABSTRACT. The heat equation with a small parameter,

$$(1+\varepsilon^{-m}\chi(\tfrac{x}{\varepsilon}))u_t = u_{xx},$$

is considered, where $\varepsilon \in (0,1)$, $m < 1$ and χ is a finite function. A complete asymptotic expansion of the solution in powers ε is constructed.

In [1], [2] E. Sanchez-Palencia and H. Tchatat noted, for the first time, problems in which a small parameter is contained not only in the equation but also in the characteristics of the domain itself. In subsequent years such problems were studied by O. A. Oleynik, S. A. Nazarov, Yu. D. Golovatii, and G. S. Sobolev [3]–[5].

In this paper we consider a problem on heat conduction in a medium whose density has a perturbation concentrated in a small neighborhood of the origin.

In the domain $\Omega = (1,1) \times (0,T)$ let us consider the initial boundary value problem for a heat equation of the form

$$\Big(1+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with the boundary conditions

$$u(-1,t) = u(1,t) = 0 \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{3}$$

where $\varepsilon \in (0,1)$, $m < 1$ is some real number, and the function χ satisfies the following conditions: $\chi(\xi) = 0$ for $|\xi| > 1$, $\chi(\xi) > 0$ for $|\xi| < 1$, and

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$\int_{-1}^{1} \chi(\xi)d\xi = M = \text{const} > 0$. Assume that the initial function u_0 is continuous on $[-1,1]$, satisfies the condition $u_0(1) = u_0(1) = 0$, and is holomorphic in the neigborhood of $x = 0$.

In that case it readily follows that

$$\lim_{\varepsilon \to 0} \varepsilon^{-m} \int_{-\varepsilon}^{\varepsilon} \chi\Big(\frac{x}{\varepsilon}\Big)\, dx = 0$$

and therefore such a perturbation ($m < 1$) will be called "weak".

By a solution of problem (1)–(3) we shall understand a function u which satisfies equation (1) in Ω for $x \neq \pm\varepsilon$, conditions (2) and (3), and at the discontinuity points $x = \pm\varepsilon$ of the function χ (there are no other discontinuity points) satisfies the conditions of "continuous sewing"

$$\begin{aligned} u(\varepsilon+0,t) &= u(\varepsilon-0,t), \quad \frac{\partial u}{\partial x}(\varepsilon+0,t) = \frac{\partial u}{\partial u}\partial x(\varepsilon-0,t), \\ u(-\varepsilon+0,t) &= u(-\varepsilon-0,t), \quad \frac{\partial u}{\partial x}(-\varepsilon+0,t) = \frac{\partial u}{\partial u}\partial x(-\varepsilon-0,t). \end{aligned} \tag{4}$$

According to O. A. Oleynik's paper [6] problem (1)–(3) is uniquely solvable in the domain Ω.

Let m be a rational number and $m = \frac{l}{p}$, where p is a natural number and $\ell < p$ is some integer number. We introduce the notation $\xi = \frac{x}{\varepsilon}$, $\Omega_+^\varepsilon = (\varepsilon,1) \times (0,T)$, $\Omega_-^\varepsilon = (-1,-\varepsilon) \times (0,T)$, $\Omega_+ = (0,1) \times (0,T)$, $\Omega_- = (-1,0) \times (0,T)$.

Now we shall construct a complete asymptotic expansion of the solution u_ε of problem (1)–(3) in powers of value $\delta = \varepsilon^{\frac{1}{p}}$ when $\varepsilon \to 0$.

We shall seek for a solution of the form

$$u_\varepsilon(x,t) \sim \begin{cases} \sum\limits_{i=0}^{\infty} \delta^i v_i^\pm(x,t), & (x,t) \in \Omega_\pm^\varepsilon, \\ \sum\limits_{i=0}^{\infty} \delta^i w_i(\frac{x}{\varepsilon},t), & (x,t) \in (-\varepsilon,\varepsilon) \times (0,T). \end{cases} \tag{5}$$

First we shall find out which conditions the functions $v_i^\pm$ and w_i must satisfy when $t = 0$.

By virtue of expansion (5) and condition (3) we have

$$u_0(x) \sim \sum_{i=0}^{\infty} \delta^i v_i^\pm(x,0), \quad |x| > \varepsilon.$$

Hence it follows that

$$v_0^\pm(x,0) = u_0(x), \quad v_i^\pm(x,0) = 0, \quad i \geq 1. \tag{6}$$

Using expansion (5) and condition (3), for the function w_i we obtain

$$u_0(x) \sim \sum_{i=0}^{\infty} \delta^i w_i(\xi, 0), \quad |\xi| < 1,$$

which, after expanding u_0 into a Taylor series, gives

$$\sum_{i=0}^{\infty} \delta^i w_i(\xi, 0) \sim \sum_{i=0}^{\infty} \delta^{pi} \frac{\xi^i}{i!} \frac{d^i}{dx^i} u_0(0).$$

Having equated the coefficients at the same powers of δ, we obtain

$$\begin{gathered} w_{ip}(\xi, 0) = \frac{\xi^i}{i!} \frac{d^i}{dx^i} u_0(0), \quad i = 0, 1, 2, \dots, \\ w_j(\xi, 0) = 0, \quad j \neq pk, \quad k = 0, 1, 2, \dots. \end{gathered} \tag{7}$$

It is easy to verify that condition (2) implies

$$v_i^{\pm}(\pm 1, t) = 0, \quad i \geq 0. \tag{8}$$

By substituting the formal expansion (5) into equation (1) we obtain

$$\begin{aligned} &\sum_{i=0}^{\infty} \delta^i \Big(\frac{\partial}{\partial t} v_i^{\pm}(x, t) - \frac{\partial^2}{\partial x^2} v_i^{\pm}(x, t) \Big) \sim 0, \\ &\sum_{i=0}^{\infty} \delta^{i-2p} \Big(\frac{\partial}{\partial t} w_{i-2p}(\xi, t) + \chi(\xi) \frac{\partial}{\partial t} w_{i-2p+\ell}(\xi, t) - \frac{\partial^2}{\partial \xi^2} w_i(\xi, t) \Big) \sim 0, \end{aligned}$$

which yields

$$\frac{\partial}{\partial t} v_i^{\pm}(x, t) - \frac{\partial^2}{\partial x^2} v_i^{\pm}(x, t) = 0, \quad |x| > \varepsilon, \tag{9}$$

$$\frac{\partial^2}{\partial \xi^2} w_i(\xi, t) = \frac{\partial}{\partial t} w_{i-2p}(\xi, t) + \chi(\xi) \frac{\partial}{\partial t} w_{i-2p+\ell}(\xi, t), \quad |\xi| < 1, \tag{10}$$

where there are no terms with negative indices.

In what follows we shall write v_i instead of $v_i^{\pm}$, assuming that for $x > \varepsilon$ we mean v_i^+, and for $x < -\varepsilon$ we mean v_i^-. Let, in the neighborhood of the point $(0, 1)$, the functions v_i be holomorphic with respect to x.

By the formal expansion (5) we obtain

$$\begin{aligned} u_\varepsilon(x, t) &\sim \sum_{i=0}^{\infty} \delta^i \sum_{s=0}^{\infty} \frac{x^s}{s!} \frac{\partial^s}{\partial x^s} v_i(\pm 0, t), \quad |x| > \varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x}(x, t) &\sim \sum_{i=0}^{\infty} \delta^i \sum_{s=1}^{\infty} \frac{x^{s-1}}{(s-1)!} \frac{\partial^s}{\partial x^s} v_i(\pm 0, t), \quad |x| > \varepsilon, \end{aligned}$$

$$u_\varepsilon(x,t) \sim \sum_{i=0}^{\infty} \delta^i w_i(\xi,t), \quad |\xi| < 1,$$
$$\frac{\partial u_\varepsilon}{\partial x}(x,t) \sim \sum_{i=0}^{\infty} \delta^{i-p} \frac{\partial w_i}{\partial \xi}(\xi,t), \quad |\xi| < 1.$$

But the solution u_ε must satisfy the condition of "continuous sewing" (4). Therefore for $x = \varepsilon$ and $\xi = 1$ we have

$$\begin{aligned}
&\sum_{i=0}^{\infty} \delta^i \sum_{s=0}^{[\frac{i}{p}]} \frac{1}{s!} \frac{\partial^s}{\partial x^s} v_{i-ps}(+0,t) \sim \sum_{i=0}^{\infty} \delta^i w_i(1,t), \\
&\sum_{i=0}^{\infty} \delta^i \sum_{s=0}^{[\frac{i}{p}]} \frac{1}{s!} \frac{\partial^{s+1}}{\partial x^{s+1}} v_{i-ps}(+0,t) \sim \sum_{i=0}^{\infty} \delta^i \frac{\partial w_{i+p}}{\partial \xi}(1,t).
\end{aligned} \tag{11}$$

Hence

$$w_i(1,t) = \sum_{s=0}^{[\frac{i}{p}]} \frac{1}{s!} \frac{\partial^s}{\partial x^s} v_{i-ps}(+0,t), \quad i \ge 0.$$

Therefore

$$w_i(1,t) - v_i(+0,t) = \sum_{s=1}^{[\frac{i}{p}]} \frac{1}{s!} \frac{\partial^s}{\partial x^s} v_{i-ps}(+0,t).$$

Thus we obtain

$$w_i(1,t) - v_i(+0,t) = F_i^+, \quad i \ge 0, \tag{12}$$

where the value F_i^+ is defined by the values of $v_j(+0,t)$ for $j \le i-p$.

In the same manner we obtain

$$w_i(-1,t) - v_i(-0,t) = F_i^-, \quad i \ge 0, \tag{13}$$

where F_i^- is defined by the values of $v_j(-0,t)$ for $j \le i-p$.

For the functions w_i we obtain

$$\frac{\partial w_i}{\partial \xi}(1,t) = 0, \quad i = 0,1,\dots,(p-1),$$
$$\frac{\partial w_i}{\partial \xi}(1,t) = \sum_{s=0}^{[\frac{i}{p}]-1} \frac{1}{s!} \frac{\partial^{s+1}}{\partial x^{s+1}} v_{i-p-ps}(+0,t), \quad i \ge p.$$

As a result, we have

$$\frac{\partial w_i}{\partial \xi}(1,t) - \frac{\partial}{\partial x} v_{i-p}(+0,t) = \Phi_i^+, \tag{14}$$

where Φ_i^+ depend on v_j for $j \le i-2p$.

In the same manner we obtain

$$\frac{\partial w_i}{\partial \xi}(-1,t) - \frac{\partial}{\partial x}\, v_{i-p}(-0,t) = \Phi_i^-, \tag{15}$$

where Φ_i^- depends on v_j for $j \le i - 2p$.

It will be shown now how one can construct successively all the functions v_i and w_i.

I. *Step* 1. By equation (10) we have $\frac{\partial^2 w_i}{\partial \xi^2}(\xi,t) = 0$, $i = 0,1,\dots,(p-1)$. Then $\frac{\partial w_i}{\partial \xi}(\xi,t) = a_i(t)$, $i = 0,1,\dots,(p-1)$. But condition (14) implies $\frac{\partial w_i}{\partial \xi}(\pm 1,t) = 0$, $i=0,1,\dots,(p-1)$. Then $\frac{\partial w_i}{\partial \xi}(\xi,t) = 0$ for $i = 0,1,\dots,(p-1)$. Therefore $w_i(\xi,t) = C_i(t)$, $\quad i = 0,1,\dots,(p-1)$. By equation (10) we obtain $\frac{\partial^2 w_p}{\partial \xi^2}(\xi,t) = 0$ and $\frac{\partial w_p}{\partial \xi}(\xi,t) = a_p(t)$. But (14) implies that $\frac{\partial w_p}{\partial \xi}(\pm 1,t) = \frac{\partial v_0}{\partial x}(\pm 0,t)$. Then for the function v_0 we obtain the condition $\frac{\partial v_0}{\partial x}(+0,t) = \frac{\partial v_0}{\partial x}(-0,t)$. Condition (12) obviously implies $v_0(+0,t) = v_0(-0,t)$. Thus to define the function v_0 we obtain the problem

$$\begin{aligned}
&\frac{\partial v_0}{\partial t}(x,t) = \frac{\partial^2 v_0}{\partial x^2}(x,t), \quad x \ne 0,\\
&v_0(x,0) = u_0(x),\\
&v_0(-1,t) = v_0(1,t) = 0,\\
&v_0(+0,t) = v_0(-0,t),\\
&\frac{\partial v_0}{\partial x}(+0,t) = \frac{\partial v_0}{\partial x}(-0,t),
\end{aligned}$$

which, as follows from [6], is uniquely solvable. Moreover, the solution coincides with the solution of the problem

$$\begin{aligned}
&\frac{\partial v_0}{\partial t}(x,t) = \frac{\partial^2 v_0}{\partial x^2}(x,t), \quad x \ne 0,\\
&v_0(-1,t) = v_0(1,t) = 0,\\
&v_0(x,0) = u_0(x).
\end{aligned}$$

Thus the function v_0 is defined uniquely. But in that case the condition $w_0(\pm 1,t) = v_0(\pm 0,t)$ implies $w_0(\xi,t) = C_0(t) = v_0(0,t)$.

Therefore, by performing step 1, we uniquely define the functions v_0 and w_0, while the functions $w_1, w_2, \dots, w_p$ are defined to within the functions C_i depending only on t.

II. *Step* 2. By equation (10) we obtain

$$\frac{\partial^2 w_{p+1}}{\partial \xi^2}(\xi,t) = \chi(\xi)\, \frac{\partial w_{1+\ell-p}}{\partial t}(\xi,t),$$

where $\ell < p$ and the right-hand part is absent if $1 + \ell - p < 0$.

Thus for w_{p+1} we obtain the equation

$$\frac{\partial^2 w_{p+1}}{\partial \xi^2}(\xi,t) = f_0(\xi,t),$$

where f_0 is the known function. Hence it follows that

$$\frac{\partial w_{p+1}}{\partial \xi}(\xi,t) = \int_{\xi_0}^{\xi} f_0(s,t)\,ds + a_{p+1}(t)$$

and $\frac{\partial w_{p+1}}{\partial \xi}$ is defined to within a term of the form $a_{p+1}(t)$. In step 1 we have defined w_1 to within the term $C_1(t)$. By condition (12) we have

$$w_1(\pm 1,t) - v_1(\pm 0,t) = 0 \quad \text{if} \quad p > 1;$$
$$w_1(\pm 1,t) - v_1(\pm 0,t) = \frac{\partial}{\partial x} v_0(\pm 0,t) \quad \text{if} \quad p = 1.$$

In both cases it is easy to verify that $v_1(+0,t) - v_1(-0,t) = h_1(t)$, where h_1 is uniquely defined.

By condition (14) we have

$$\frac{\partial w_{p+1}}{\partial \xi}(\pm 1,t) = \frac{\partial v_1}{\partial x}(\pm 0,t) \quad \text{if} \quad p > 1;$$
$$\frac{\partial w_{p+1}}{\partial \xi}(\pm 1,t) = \frac{\partial v_1}{\partial x}(\pm 0,t) + \frac{\partial^2 v_0}{\partial x^2}(\pm 0,t) \quad \text{if} \quad p = 1.$$

In both cases this readily yields

$$\frac{\partial v_1}{\partial x}(+0,t) - \frac{\partial v_1}{\partial x}(-0,t) = \frac{\partial w_{p+1}}{\partial \xi}(+1,t) - \frac{\partial w_{p+1}}{\partial \xi}(-1,t) + \widetilde{h}_0(t),$$

where $\widetilde{h}_0$ depends on v_0. Therefore

$$\frac{\partial v_1}{\partial x}(+0,t) - \frac{\partial v_1}{\partial x}(-0,t) = H_1(t),$$

where H_1 is uniquely defined.

Thus to define the function v_1 we obtain the problem

$$\begin{aligned}
&\frac{\partial v_1}{\partial t}(x,t) = \frac{\partial^2 v_1}{\partial x^2}(x,t), \quad x \neq 0,\\
&v_1(x,0) = 0,\\
&v_1(-1,t) = v_1(1,t) = 0,\\
&v_1(+0,t) - v_1(-0,t) = h_1(t),\\
&\frac{\partial v_1}{\partial x}(+0,t) - \frac{\partial v_1}{\partial x}(-0,t) = H_1(t),
\end{aligned}$$

where h_1 and H_1 are the known functions. This problem is uniquely solvable according to [6].

Thus the function v_1 is defined uniquely. Since the function $w_1(\xi,t) = C_1(t)$, by the condition $w_1(1,t) - v_1(+0,t) = 0$ for $p > 1$ and the condition $w_1(1,t) - v_1(-0,t) = \frac{\partial}{\partial x} v_0(+0,t)$ for $p = 1$ the function w_1 is also defined uniquely.

The function w_{p+1} can be represented by

$$\frac{\partial w_{p+1}}{\partial \xi}(\xi,t) = \int\limits_{\xi_0}^{\xi} f_0(s,t)\,ds + a_{p+1}(t).$$

Then, by the conditions

$$\frac{\partial w_{p+1}}{\partial \xi}(1,t) = \frac{\partial v_1}{\partial x}(+0,t) \quad \text{for} \quad p > 1,$$
$$\frac{\partial w_{p+1}}{\partial \xi}(1,t) = \frac{\partial v_1}{\partial x}(+0,t) + \frac{\partial^2 v_0}{\partial x^2}(+0,t) \quad \text{for} \quad p = 1,$$

the function a_{p+1} is uniquely defined. Therefore the function $\frac{\partial w_{p+1}}{\partial \xi}$ is uniquely defined and $\frac{\partial w_{p+1}}{\partial \xi}(\xi,t) = f_1(\xi,t)$. Hence we obtain

$$w_{p+1}(\xi,t) = \int\limits_{\xi_0}^{\xi} f_1(s,t)\,ds + C_{p+1}(t)$$

and the function w_{p+1} is defined to within the term C_{p+1} depending on t.

Thus in step 2 we have defined the functions w_1 and v_1 uniquely, while the function w_{p+1} was defined to within the function C_{p+1} depending on t.

III. *Step* $n+1$. Let the functions v_i and w_i be uniquely defined for all $i \le n$, and the functions $w_{n+1},\dots,w_{n+p}$ be defined to within the terms $C_{n+1},\dots,C_{n+p}$ depending on t.

Consider the equation for the function w_{n+p+1}

$$\frac{\partial^2 w_{n+p+1}}{\partial \xi^2}(\xi,t) = \frac{\partial w_{n+1-p}}{\partial t}(\xi,t) + \chi(\xi)\,\frac{\partial w_{n+1+\ell-p}}{\partial t}(\xi,t),$$

where the right-hand part has no terms with negative indices. In any case the right-hand part of the equation if defined uniquely since $\ell < p$ and $p \ge 1$, and therefore $n+1-p \le n$ and $n+1+\ell-p \le n$. As a result we obtain the equation

$$\frac{\partial^2 w_{n+p+1}}{\partial \xi^2}(\xi,t) = f_n(\xi,t)$$

which readily implies that

$$\frac{\partial w_{n+p+1}}{\partial \xi}(\xi,t)=\int\limits_{\xi_0}^{\xi} f_n(s,t)\,ds+a_{n+p+1}(t)$$

and the function $\frac{\partial w_{n+p+1}}{\partial \xi}$ is defined to within the term a_{n+p+1} depending only on t.

By condition (12) we have $w_{n+1}(\pm 1,t)-v_{n+1}(\pm 0,t)=F^{\pm}_{n+1}$, where $F^{\pm}_{n+1}$ is defined by means of the functions v_j for $j\leq n+1-p$. But $n+1-p\leq n$ and therefore the difference $v_{n+1}(+0,t)-v_{n+1}(-0,t)$ is defined by the difference $w_{n+1}(1,t)-w_{n+1}(-1,t)$, which is uniquely defined. Thus $v_{n+1}(+0,t)-v_{n+1}(-0,t)=h_{n+1}(t)$, where h_{n+1} is the known function.

By condition (14) we have

$$\frac{\partial w_{n+1+p}}{\partial \xi}(\pm 1,t)-\frac{\partial v_{n+1}}{\partial x}(\pm 0,t)=\Phi^{\pm}_{n+p+1}(t),$$

where $\Phi^{\pm}_{n+p+1}$ is defined by means of the functions v_j for $j\leq n+1-p$. But $n+1-p\leq n$ and therefore the difference $\frac{\partial}{\partial x}v_{n+1}(+0,t)-\frac{\partial}{\partial x}v_{n+1}(-0,t)$ is defined by the difference

$$\frac{\partial w_{n+p+1}}{\partial \xi}(1,t)-\frac{\partial w_{n+p+1}}{\partial \xi}(-1,t)=\int\limits_{-1}^{1} f_n(s,t)\,ds$$

and hence is defined uniquely. Thus

$$\frac{\partial v_{n+1}}{\partial x}(+0,t)-\frac{\partial v_{n+1}}{\partial x}(-0,t)=H_{n+1}(t),$$

where H_{n+1} is the known value.

Finally, to define the function v_{n+1} we obtain the problem

$$\begin{aligned}
&\frac{\partial v_{n+1}}{\partial t}(x,t)=\frac{\partial^2 v_{n+1}}{\partial x^2}(x,t),\quad x\neq 0,\\
&v_{n+1}(x,0)=0,\\
&v_{n+1}(-1,t)=v_{n+1}(1,t)=0,\\
&v_{n+1}(+0,t)-v_{n+1}(-0,t)=h_{n+1}(t),\\
&\frac{\partial v_{n+1}}{\partial x}(+0,t)-\frac{\partial v_{n+1}}{\partial x}(-0,t)=H_{n+1}(t),
\end{aligned}$$

where h_{n+1} and H_{n+1} are the known functions. This problem is uniquely solvable according to [6].

Thus the function v_{n+1} has been defined uniquely. Since the function w_{n+1} has been defined to within the term C_{n+1} depending only on

t, the function w_{n+1} is defined uniquely by the condition $w_{n+1}(1,t) - v_{n+1}(+0,t) = F^{+}_{n+1}(t)$.

The function w_{n+p+1} can be represented as

$$\frac{\partial w_{n+p+1}}{\partial \xi}(\xi,t) = \int\limits_{\xi_0}^{\xi} f_n(s,t)\,ds + a_{n+p+1}(t).$$

Then the function a_{n+p+1} is defined uniquely by the condition

$$\frac{\partial w_{n+p+1}}{\partial \xi}(1,t) - \frac{\partial v_{n+1}}{\partial x} = \Phi^{+}_{n+p+1}(t).$$

Therefore

$$\frac{\partial w_{n+p+1}}{\partial \xi}(\xi,t) = f_{n+1}(\xi,t)$$

and w_{n+p+1} is defined by the formula

$$w_{n+p+1}(\xi,t) = \int\limits_{\xi_0}^{\xi} f_{n+1}(s,t)\,ds + C_{n+p+1}(t)$$

to within the term C_{n+p+1}.

Therefore, if it is assumed that the functions v_i and w_i have the known exact values for all $i \le n$, and the fucntions $w_{n+1}, \dots, w_{n+p}$ are known to within the terms $C_{n+1}, \dots, C_{n+p}$ depending only on the variable t, then we shall define the functions v_{n+1} and w_{n+1} uniquely, while the function w_{n+1+p} will be defined to within the term C_{n+p+1} depending only on t.

Thus, using the arguments of I, II, and III, we conclude by induction that the functions v_i and w_i can be defined uniquely for arbitrary i. We have therefore formally constructed the asymptotic series (5).

Consider a partial sum of series (5)

$$u_N(x,t) = \begin{cases} \sum\limits_{i=0}^{N} \delta^i v_i^{\pm}(x,t), & |x| > \varepsilon, \\ \sum\limits_{i=0}^{N} \delta^i w_i(\frac{x}{\varepsilon},t), & |x| < \varepsilon, \end{cases} \tag{16}$$

and evaluate the difference $u_N(\varepsilon+0,t) - u_N(\varepsilon-0,t)$.

We readily obtain

$$u_N(\varepsilon-0,t) = \sum_{i=0}^{N} \delta^i w_i(1,t);$$

$$u_N(\varepsilon+0,t) = \sum_{i=0}^{N} \delta^i \sum_{s=0}^{[\frac{i}{p}]} \frac{1}{s!}\,\frac{\partial^s}{\partial x^s}\,v_{i-sp}(+0,t) + O(\delta^{N+1}).$$

But then conditions (11) imply that $u_N(\varepsilon+0,t)-u_N(\varepsilon-0,t)=O(\delta^{N+1})$. In the same manner we find that

$$\begin{gathered}
u_N(-\varepsilon+0,t)-u_N(-\varepsilon-0,t)=O(\delta^{N+1});\\
\frac{\partial u_N}{\partial x}(\varepsilon+0,t)-\frac{\partial u_N}{\partial x}(\varepsilon-0,t)=O(\delta^{N});\\
\frac{\partial u_N}{\partial x}(-\varepsilon+0,t)-\frac{\partial u_N}{\partial x}(-\varepsilon-0,t)=O(\delta^{N}).
\end{gathered}$$

Thus the function u_N has discontinuities at the points $x=\pm\varepsilon$. Let us correct this function at the points of discontinuity. We introduce the notation

$$\begin{aligned}
C_1&=u_N(\varepsilon+0,t)-u_N(\varepsilon-0,t),\\
C_2&=u_N(-\varepsilon+0,t)-u_N(-\varepsilon-0,t),\\
B_1&=\frac{\partial u_N}{\partial x}(\varepsilon+0,t)-\frac{\partial u_N}{\partial x}(\varepsilon-0,t),\\
B_2&=\frac{\partial u_N}{\partial x}(-\varepsilon+0,t)-\frac{\partial u_N}{\partial x}(-\varepsilon-0,t).
\end{aligned}$$

Let φ be a smooth function, $\varphi(x)\equiv 1$ for $|x|\le\frac{1}{2}$, and $\varphi(-1)=\varphi(1)=0$. Assume that $\varphi_N(x,t)$ for $|x|<\varepsilon$, $\varphi_N(x,t)=(B_1(x-\varepsilon)+C_1)\varphi(x)$ for $x\ge\varepsilon$, and $\varphi_N(x,t)=(B_2(x+\varepsilon)+C_2)\varphi(x)$ for $x<-\varepsilon$. Consider the function V_N defined by the formula $V_N(x,t)=u_N(x,t)-\varphi_N(x,t)$. It is easy to obtain

$$\begin{gathered}
\Big(\frac{\partial V_N}{\partial t}+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\,\frac{\partial V_N}{\partial x^2}\Big)=O(\delta^{N-1}),\\
V_N(x,0)=u_0(x)+O(\delta^{N+1}).
\end{gathered}$$

Now for the function $\overline{v}=u_\varepsilon-V_N$ we obtain a problem of the form

$$\begin{aligned}
&\frac{\partial\overline{v}}{\partial t}+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\,\frac{\partial\overline{v}}{\partial t}-\frac{\partial^2\overline{v}}{\partial x^2}=F_\delta,\\
&\overline{v}(-1,t)=\overline{v}(1,t)=0,\\
&\overline{v}(x,0)=\varphi_\delta,
\end{aligned}\tag{17}$$

where $F_\delta(x,t)=O(\delta^{N-1})$ and $\varphi_\delta(x)=O(\delta^{N+1})$.

By multiplying the equation by $\overline{v}$ and integrating the resulting equality over the domain $[-1,1]\times[0,\tau_0]$, where $\tau_0\in(0,T]$, we obtain

$$\int\limits_{-1}^{1}\int\limits_{0}^{\tau_0}\bigg(\Big(1+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)\frac{\partial\overline{v}}{\partial t}\,\overline{v}^2-\frac{\partial^2\overline{v}}{\partial x^2}\,\overline{v}\bigg)dx\,dt=\int\limits_{-1}^{1}\int\limits_{0}^{\tau_0}F_\delta(x,t)\overline{v}\,dx\,dt.$$

Hence, after integration by parts, we have

$$\frac{1}{2}\int\limits_{-1}^{1}\Big(1+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)\overline{v}^2\,dx+\int\limits_{-1}^{1}\int\limits_{0}^{\tau_0}\Big(\frac{\partial\overline{v}}{\partial x}\Big)^2\,dx\,dt=$$

$$=\frac{1}{2}\int\limits_{-1}^{1}\Big(1+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)\varphi_\delta^2(x)\,dx+\int\limits_{-1}^{1}\int\limits_{0}^{\tau_0}F_\delta(x,t)\overline{v}(x,t)\,dx\,dt,$$

which implies

$$\int\limits_{-1}^{1}\overline{v}^2(x,\tau_0)\,dx\le\int\limits_{-1}^{1}\varphi_\delta^2(x)\,dx+\varepsilon^{-m}\int\limits_{-\varepsilon}^{\varepsilon}\chi\Big(\frac{x}{\varepsilon}\Big)\varphi_\delta^2(x)\,dx+$$

$$+2\bigg|\int\limits_{-1}^{1}\int\limits_{0}^{\tau_0}F_\delta(x,t)\overline{v}(x,t)\,dx\,dt\bigg|.$$

Taking into account the estimates of the functions φ_δ and F_φ and using the known inequality $2ab\le\varepsilon a^2+\frac{1}{\varepsilon}b^2$, we obtain

$$\int\limits_{-1}^{1}\int\limits_{0}^{T}\overline{v}^2(x,t)\,dx\,dt\le\widetilde{C}\delta^{2(N-1)},$$

where $\widetilde{C}$ does not depend on δ and N.

Thus we have established that $\|u_\varepsilon-V_{N_1}\|_{L_2(\Omega)}\le\widetilde{C}\delta^{N_1-1}$ for any N_1.

Let $N_1=N+2$. Then $\|u_\varepsilon+V_{N+2}\|_{L_2(\Omega)}\le\widetilde{C}\delta^{N+1}$. On the other hand, $\|V_{N+2}-u_{N+2}\|_{L_2(\Omega)}\le\overline{C}\delta^{N+2}$. Hence it follows that $\|u_\varepsilon-u_{N+2}\|_{L_2(\Omega)}\le C_1\delta^{N+1}$. This immediately implies $\|u_\varepsilon-u_N\|_{L_2(\Omega)}\le\widetilde{M}\delta^{N+1}$. Thus we have proved

Theorem. *Let u_ε be the solution of problem* (1)–(3) *and u_N be a partial sum of the formal asymptotic series* (5) *defined by formula* (16). *Then the inequality $\|u_\varepsilon-u_N\|_{L_2(\Omega)}\le\widetilde{M}\delta^{N+1}$ holds, where the constant $\widetilde{M}$ does not depend on δ and N.*

The construction of the functions v_j and w_j enables us to make several conclusions. In particular, let $m_1=\frac{\ell_1}{p}$ and $m_2=\frac{\ell_2}{p}$, where $\ell_1<\ell_2$. It is easy to see that in both cases the functions v_0 and w_0 are defined in the same manner. Moreover, the functions v_j and w_j are also defined in the same manner if $j<p-1-\ell_2$. Thus for such m_1 and m_2 the asymptotic expansions coincide in the first several terms.

The theorem and the above remarks give rise to

Corollary. *Let $m < 1$ be a real number. Then the limit function $\overline{u} = \lim\limits_{\varepsilon\to 0} u_\varepsilon$, where u_ε is the solution of problem* (1)–(3)*, is the solution of the problem*

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t}(x,t) = \frac{\partial^2 \overline{u}}{\partial x^2}(x,t), \quad x \in (-1,1),\\
&\overline{u}(-1,t) = \overline{u}(1,t) = 0,\\
&\overline{u}(x,0) = u_0(x).
\end{aligned}$$

Remark. The corollary can also be proved without using asymptotic expansions. We intend to do this in future papers.

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