GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 5, 1998, 483-500

VERTICAL COHOMOLOGIES AND THEIR APPLICATION TO COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS

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Dedicated to the memory of Roin Nadiradze

Abstract. Some functorial and topological properties of vertical cohomologies and their application to completely integrable Hamiltonian systems are studied.

§ 1. Introduction

If on a smooth Riemannian manifold M^n we have a distribution V of dimension k, which is actually a smooth section of the Grassman fiber bundle $G_k(TM^n) \to M^n$ adjoint to the tangent fibration TM^n to the manifold M^n, then by means of the Riemannian metric we obtain $TM^n = V \oplus N$, where N is a normal fiber bundle to V. Let $P : TM^n \to V \subset TM^n$ be a natural projection. The operator P defines the mapping $P^* : \Lambda^*(M^n) \to \Lambda^*(M^n)$ ($(\Lambda^*(M^n), d^*)$ is the de Rham differential complex) by the formula $(P^*\alpha)(X_1, \dots, X_q) = \alpha(PX_1, \dots, PX_q)$, where $\alpha \in \Lambda^q(M^n)$ and $X_1, \dots, X_q \in S(M^n)$ are the smooth vector fields on M^n.

Denote by $\Lambda^*_V(M^n)$ all fixed points of the operator P^*. In what follows we shall consider the case where V is integrable, i.e., where M^n is partitioned into leaves and the tangent space to the leaf that passes through the point $x \in M^n$ is V_x. Then the pair $(\Lambda^*_V(M^n), d^*_V)$ forms a differential complex with the differential $d^*_V = P^* \circ d^*$.

The cohomologies of the complex $(\Lambda^*_V(M^n), d^*_V)$ are called vertical and denoted by $H^*_V(M^n)$ ([1]). It has turned out that these cohomologies coincide with those of the classical BRST operator ([2], [3]).

In §2 the vertical cohomologies are defined without fixing the metric on M^n, and some of their functorial properties are studied. The FOL category

1991 *Mathematics Subject Classification.* 53C12, 70H05.

Key words and phrases. Foliation, vertical cohomology, Hamiltonian systems, isoenergetic surfaces, Liouville torus.

1072-947X/98/0900-0483\$15.00/0

of smooth foliations and leaf-to-leaf transforming mappings is introduced, and a natural transformation of the de Rham functor Λ^* to the functor $\Lambda^*_{\mathcal{F}}$ is constructed (Proposition 2.2). The notion of leaf-to-leaf transforming homotopic mappings is introduced, and the homotopy axiom for vertical cohomologies is proved (Theorem 2.5). The notion of a relative group of vertical cohomologies is introduced by analogy with de Rham's theory, and the long exact cohomologic sequences (2.6) and (2.7) are derived. Moreover, for a leaf-to-leaf transforming mapping $f : (M^n, \mathcal{F}_1) \to (N^m, \mathcal{F}_2)$, the cohomology groups $H^*(f)$ are constructed and proved (Theorem 2.8) to be isomorphic for leaf-to-leaf transforming homotopic mappings. Finally, a double complex (K^{**}, D^*) is constructed for the countable covering $U = \{u_\alpha\}_{\alpha \in A}$ of the manifold M^n. It is shown that the cohomologies of (K^{**}, D^*) are isomorphic to the vertical cohomologies. A combinatorial definition of vertical comologies in the Čech sense (Theorems 2.10 and 2.12) is also given.

In §3 some of the main facts from the topological theory of integrable Hamiltonian systems ([5], [6]) are presented. Using the notion of vertical cohomologies, the groups corresponding to nonresonance Hamiltonian systems are constructed (Theorems 3.3 and 3.4).

In §4 the case of a spherical pendulum is considered as an example.

§ 2. Vertical Cohomologies

2.1. Definition of Vertical Cohomologies. Let M^n be a smooth n-dimensional manifold, and V be a k-dimensional involutive distribution on M^n whose foliation is denoted by $\mathcal{F}$. The bundle of exterior p-forms on V is denoted by $A^p(V)$, and the set of smooth sections of the bundle $A^p(V)$ by $\Lambda^p_{\mathcal{F}}(M^n)$. Then $\Lambda^p_{\mathcal{F}}(M^n)$ is a module over the algebra of infinitely differentiable functions $C^\infty(M^n)$ on M^n.

Let $\alpha \in \Lambda^p_{\mathcal{F}}(M^n)$, and let $X_1, \dots, X_p$ be the smooth vector fields on M^n which are tangent to the leaves of the foliation $\mathcal{F}$, i.e., they are the smooth sections of the bundle $V \overset{p_{\mathcal{F}}}{\to} M^n$. Then the mapping defined by the formula

$$\alpha(X_1, \dots, X_p) : x \overset{j}{\longmapsto} \alpha(X_1(x), \dots, X_p(x)), \tag{2.1}$$

which on the module of sections $S(V)$ of V assigns an exterior p-form to an element $\alpha \in \Lambda^p_{\mathcal{F}}(M^n)$, is an isomorphism.

Let us now define the operator

$$d^p_{\mathcal{F}} : \Lambda^p_{\mathcal{F}}(M^n) \to \Lambda^{p+1}_{\mathcal{F}}(M^n)$$

by the relation

$$(d^p_{\mathcal{F}}\alpha)(X_1,\dots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i-1}X_i\alpha(X_1,\dots,\widehat{X}_i,X_{i+1},\dots,X_{p+1}) + \\ + \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{p+1}),$$

where $\alpha \in \Lambda^p_{\mathcal{F}}(M^n)$, $X_i \in S(V)$, $i = \overline{1,p+1}$.

It is easy to verify that the embedding $i : S(V) \hookrightarrow S(TM^n)$ induces the projection $i^*_q : A^q(S(TM)) \to A^q(S(V))$. Indeed, let $\alpha \in A^q(S(V))$. Define $\overline{\alpha} \in A^q(S(TM))$ so that $i^*_q\overline{\alpha} = \alpha$. Any Riemannian metric defines a smooth section $P : M^n \to \mathrm{End}(TM^n)$ of the bundle of homomorphisms $\mathrm{End}(TM^n)$ of the tangent bundle TM^n, where $P(x) : T_xM^n \to V_x \subset T_xM^n$ is the orthogonal projection onto a leaf of the foliation $\mathcal{F}$ which passes through the point $x \in M^n$. The element $\overline{\alpha}$ can be defined by the formula

$$\overline{\alpha}(X_1,\dots,X_q) = \alpha(PX_1,\dots,PX_q), \quad X_i \in S(TM^n), \quad i = \overline{1,q}.$$

The following diagram is commutative:

$$\begin{array}{ccc} A^q(S(TM^n)) & \xrightarrow{i^*_q} & A^q_{\mathcal{F}}(M^n) \\ \downarrow{\scriptstyle d^q} & & \downarrow{\scriptstyle d^q_{\mathcal{F}}} \\ A^{q+1}(S(TM^n)) & \xrightarrow{i^*_{q+1}} & A^{q+1}_{\mathcal{F}}(M^n) \end{array},$$

where by $A^*_{\mathcal{F}}(M^n)$ is denoted $A^*(S(V))$. Indeed,

$$(d^q_{\mathcal{F}}i^*_q\alpha)(X_1,\dots,X_{q+1}) = \sum_{j=1}^{q+1}(-1)^{j-1}(X_j)(i^*_q\alpha)(X_1,\dots,\widehat{X}_j,\dots,X_{q+1}) + \\ + \sum_{j<t}(-1)^{j+t}(i^*_q\alpha)([X_j,X_t],X_1,\dots,\widehat{X}_j,\dots,\widehat{X}_t,\dots,X_{q+1}) = \\ = (d^q\alpha)(X_1,\dots,X_{q+1}) = (i^*_{q+1}d^q\alpha)(X_1,\dots,X_{q+1}).$$

The commutativity $d^q_{\mathcal{F}} \circ i^*_q = i^*_{q+1} \circ d^q$ implies $d^q_{\mathcal{F}} \circ d^{q-1}_{\mathcal{F}} = 0$. Thus i^* is a cochain mapping between the differential complexes $(A^*(S(TM^n),d^*)$ and $(A^*_{\mathcal{F}}(M^n),d^*_{\mathcal{F}})$.

Definition 2.1. Cohomology groups of the complex $(A^*_{\mathcal{F}}(M),d^*_{\mathcal{F}})$ are called vertical cohomologies of the foliation $(M^n,\mathcal{F})$, and we denote them by $H^*_{\mathcal{F}}(M^n)$.

It is easy to verify that $d^*_{\mathcal{F}}$ is an antidifferentiation of order 1, i.e., if $\alpha \in A^q_{\mathcal{F}}(M^n)$, $\beta \in A^l_{\mathcal{F}}(M^n)$, we have

$$d^{q+l}_{\mathcal{F}}(\alpha \wedge \beta) = (d^q_{\mathcal{F}}\alpha) \wedge \beta + (-1)^q \alpha \wedge (d^l_{\mathcal{F}}\beta),$$

where $\wedge$ is an exterior product. This implies that the homomorphism i^* induces a homomorphism between the de Rham cohomology algebra and the cohomology algebra $H^*_{\mathcal{F}}(M^n)$. We denote this homomorphism by the same symbol i^*. Since we already know that the cochain mapping i^* is an epimorphism, we have a short exact sequence of cochain differential complexes

$$0 \to Z_{\mathcal{F}} \xrightarrow{j^*} A^*(S(TM^n)) \xrightarrow{i^*} A^*_{\mathcal{F}}(M^n) \to 0, \tag{2.2}$$

where by $(Z^*_{\mathcal{F}}, d^*)$ is denoted the kernel of the mapping i^*. Sequence (2.2) induces a long exact sequence of cohomology groups

$$0 \to \widetilde{H}^0_{\mathcal{F}}(M^n) \xrightarrow{j^*} H^0(M^n) \xrightarrow{i^*} H^0_{\mathcal{F}}(M^n) \xrightarrow{\delta} \widetilde{H}^1_{\mathcal{F}}(M^n) \xrightarrow{j^*} \cdots . \tag{2.3}$$

Here by $\widetilde{H}^*_{\mathcal{F}}(M^n)$ are denoted the cohomology groups of the complex $(Z^*_{\mathcal{F}}, d)$. Since $H^{k+m}_{\mathcal{F}}(M^n) = 0$, $m > 0$, from (2.3) we get $\widetilde{H}^{m+k}_{\mathcal{F}}(M^n) \approx H^{m+k}(M^n)$, $m > 1$.

Denote by FOL a category whose objects are smooth foliations $(M^n, \mathcal{F})$, and morphisms from $(M^n_1, \mathcal{F}_1)$ to $(M^m_2, \mathcal{F}_2)$ are leaf-to-leaf transforming mappings, i.e., smooth mappings h from M^n_1 to M^n_2 which preserve the foliation structure by performing a leaf-to-leaf transformation.

If h is the leaf-to-leaf transforming mapping between the foliations $(M^n_1, \mathcal{F}_1)$ and $(M^n_2, \mathcal{F}_2)$, then h defines the morphism h^* between the $C^\infty(M^m_2)$-module $\Lambda^*_{\mathcal{F}_2}(M^m_2)$ and the $C^\infty(M^n_1)$-module $\Lambda^*_{\mathcal{F}_1}(M^n_1)$ by the formula

$$(h^*\alpha)(X_1, \dots, X_p)(x) = \alpha(h^\top X_1(x), \dots, h^\top X_p(x)), \tag{2.4}$$

where $\alpha \in \Lambda^p_{\mathcal{F}_2}(M_2)$ and $X_1, \dots, X_p \in S(V_1)$, V_1 is the distribution associated with $\mathcal{F}_1$, and $h^\top$ is the tangent mapping to h. We have the commutative diagram

$$\begin{array}{ccc} \Lambda^*(M^m_2) & \xrightarrow{h^*} & \Lambda^*(M^n_1) \\ \big\downarrow{\scriptstyle i^*_2} & & \big\downarrow{\scriptstyle i^*_1} \\ \Lambda^*_{\mathcal{F}_2}(M^m_2) & \xrightarrow{h^*} & \Lambda^*_{\mathcal{F}_1}(M^n_1) \end{array} .$$

A direct calculation shows that h^* is a cochain mapping, i.e., $d^*_{\mathcal{F}_1} \circ h^* = h^{*+1} \circ d^*_{\mathcal{F}_2}$. Hence we have

Proposition 2.2. *The mapping i^* is a natural transformation of the de Rham functor Λ^* to the functor $\Lambda^*_{\mathcal{F}}$, where Λ^* and $\Lambda^*_{\mathcal{F}}$ are the contravariant functors from the* FOL *category to the category of differential graded algebras and their homomorphisms.*

2.2. A Homotopy Axiom for Vertical Cohomologies. Let $(M^n, \mathcal{F})$ be a foliation of dimension k. On the manifold $M^n \times \mathbb{R}$ we define naturally a foliation $\widehat{\mathcal{F}}$ of dimension $k+1$ whose leaves are manifolds $L_\alpha \times \mathbb{R}$, $\alpha \in A$, where L_α, $\alpha \in A$, are the leaves of the foliation $\mathcal{F}$.

Lemma 2.3. *The projection $\pi : M^n \times \mathbb{R} \to M^n$ defines an isomorphism in vertical cohomologies.*

Proof. Consider the zero section s of the trivial bundle $M^n \times \mathbb{R} \xrightarrow{\pi} M^n$, i.e., $s(x) = (x, 0)$, $x \in M^n$. Then the mappings π and s are the leaf-to-leaf transforming mappings which define the cochain mappings $\pi^* : (\Lambda^*_{\mathcal{F}}(M^n), d^*_{\mathcal{F}}) \to (\Lambda^*_{\widehat{\mathcal{F}}}(M^n \times \mathbb{R}), d^*_{\widehat{\mathcal{F}}})$ and $s^* : (\Lambda^*_{\widehat{\mathcal{F}}}(M^n \times \mathbb{R}), d^*_{\widehat{\mathcal{F}}}) \to (\Lambda^*_{\mathcal{F}}(M^n), d^*_{\mathcal{F}})$. Since $s^* \circ \pi^* = 1$, π^* is an monomorphism. We shall show that π^* induces an isomorphism at the cohomology level. To this end, we shall construct a cochain equivalence of the mappings 1 and $\pi^* \circ s^*$.

Note that each form from $\Lambda^*_{\mathcal{F}}(M^n \times \mathbb{R})$ can be uniquely represented by linear combinations of the following two types of forms:

$$\begin{aligned} &\text{(I)}\ \ (\pi^*\varphi) \cdot f, \quad \varphi \in \Lambda^*_{\mathcal{F}}(M^n), \quad f \in C^\infty(M^n \times \mathbb{R}); \\ &\text{(II)}\ \ (\pi^*\varphi) \wedge dt \cdot f, \quad \varphi \in \Lambda^*_{\mathcal{F}}(M^n), \quad f \in C^\infty(M^n \times \mathbb{R}), \end{aligned}$$

where t is the coordinate on the straight line $\mathbb{R}$. Define the operator $K^* : \Lambda^*_{\widehat{\mathcal{F}}}(M^n \times \mathbb{R}) \to \Lambda^{*-1}_{\widehat{\mathcal{F}}}(M^n \times \mathbb{R})$ as follows:

$$K^*((\pi^*\varphi) \cdot f) = 0,$$

$$K^*((\pi^*\varphi) \wedge dt \cdot f) = \pi^*\varphi \cdot \int_0^t f\, dt.$$

A direct calculation shows that the relation

$$1 - \pi^* \circ s^* = (-1)^{q-1}(d^{q-1}_{\widehat{\mathcal{F}}} K^q - K^{q+1} d^q_{\widehat{\mathcal{F}}})$$

is fulfilled on the forms of types (I) and (II). $\square$

Definition 2.4. Two leaf-to-leaf transforming mappings $f, g : (M_1^n, \mathcal{F}_1) \to (M_2^m, \mathcal{F}_2)$ between the foliations $(M_1^n, \mathcal{F}_1)$ and $(M_2^m, \mathcal{F}_2)$ are called leaf-to-leaf transforming homotopic if there exists a leaf-to-leaf transforming mapping

$$F : (M_1^n \times \mathbb{R}, \widehat{\mathcal{F}}_1) \to (M_2^m, \mathcal{F}_2)$$

such that
$$\begin{cases} F(x,t)=f(x), & t\geq 1,\\ F(x,t)=g(x), & t\leq 0,\end{cases} \quad x\in M,\quad t\in\mathbb{R}.$$

Theorem 2.5 (A Homotopy Axiom). *Leaf-to-leaf transforming homotopic mappings induce identical mappings in vertical cohomologies.*

Proof. Let $f,g:(M_1^n,\mathcal{F}_1)\to(M_2^m,\mathcal{F}_2)$ be the leaf-to-leaf transforming homotopic mappings and $F:(M_1^n\times\mathbb{R},\widehat{\mathcal{F}}_1)\to(M_2^m,\mathcal{F}_2)$ be the homotopy between f and g. Denote by s_0 and s_1 the sections $s_0,s_1:(M_1^n,\mathcal{F}_1)\to(M_1^n\times\mathbb{R},\widehat{\mathcal{F}}_1)$, $s_0(x)=(x,0)$, $s_1(x)=(x,1)$, $x\in M_1^n$. Then $f=F\circ s_1$ and $g=F\circ s_0$. Hence we have $f^*=s_1^*\circ F^*$ and $g^*=s_0^*\circ F^*$. From the proof of Lemma 2.3 it follows that $s_1^*=s_0^*=(\pi_1^*)^{-1}$, where $\pi_1:M_1^n\times\mathbb{R}\to M_1^n$ is the projection. Therefore $f^*=g^*$. $\square$

The foliations $(M_1^n,\mathcal{F}_1)$ and $(M_2^m,\mathcal{F}_2)$ will be said to be of the same homotopy type if there are leaf-to-leaf transforming smooth mappings $f:M_1^n\to M_2^m$ and $g:M_2^m\to M_1^n$ such that $g\circ f$ and $f\circ g$ are leaf-to-leaf transforming homotopic mappings to the identical mappings of the foliations $(M_1^n,\mathcal{F}_1)$ and $(M_2^m,\mathcal{F}_2)$, respectively.

Corollary 2.6. *If two foliations $(M_1^n,\mathcal{F}_1)$ and $(M_2^m,\mathcal{F}_2)$ are of the same homotopy type, then their vertical cohomologies are isomorphic.*

2.3. Relative Vertical Cohomologies. Let $(M^n,\mathcal{F}_1)$ and $(N^m,\mathcal{F}_2)$ be two foliations, and let f be a leaf-to-leaf transforming smooth mapping $f:M^n\to N^m$. Define the differential complex
$$(\Lambda^*(f),\overline{d}^*),\quad \Lambda^*(f)=\underset{q\geq 0}{\oplus}\Lambda^q(f),$$
where
$$\Lambda^q(f)=\Lambda^q_{\mathcal{F}_2}(N^m)\oplus\Lambda^{q-1}_{\mathcal{F}_1}(M^n),\quad \overline{d}^*(\omega,\theta)=(-d^*_{\mathcal{F}_2}\omega,f^*\omega+d^*_{\mathcal{F}_1}\theta).$$

We easily verify that $\overline{d}^2=0$ and denote the cohomology groups of this complex by $H^*(f)$. Note that the complex $(\Lambda^*(f),\overline{d}^*)$ is the cone of the cochain mapping $f^*:\Lambda^*_{\mathcal{F}_2}(N^m)\to\Lambda^*_{\mathcal{F}_1}(M^n)$. If we regraduate the complex $\Lambda^*_{\mathcal{F}_1}(M^n)$ as $\widetilde{\Lambda}^p_{\mathcal{F}_1}(M_1^n)\equiv\Lambda^{p-1}_{\mathcal{F}_1}(M^n)$, then we obtain an exact sequence of differential complexes
$$0\to\widetilde{\Lambda}^*_{\mathcal{F}_1}(M^n)\xrightarrow{\alpha}\Lambda^*(f)\xrightarrow{\beta}\Lambda^*_{\mathcal{F}_2}(N^m)\to 0 \tag{2.5}$$
with the obvious mappings α and β: $\alpha(\theta)=(0,\theta)$, $\beta(\omega,\theta)=\omega$. From (2.5) we have an exact sequence in cohomologies
$$\cdots\to H^{q-1}_{\mathcal{F}_1}(M^n)\xrightarrow{\alpha^*}H^q(f)\xrightarrow{\beta^*}H^q_{\mathcal{F}_2}(N^m)\xrightarrow{\delta^*}H^q_{\mathcal{F}_1}(M^n)\to\cdots.$$

It is easily seen that $\delta^* = f^*$. Let $\omega \in \Lambda^q_{\mathcal{F}_2}(N^m)$ be the closed form, and $(\omega, \theta) \in \Lambda^q(f)$. Then $\overline{d}(\omega, \theta) = (0, f^*\omega + d^{q-1}_{\mathcal{F}_1}\theta)$, and by the definition of the operator δ^* we have $\delta^*[\omega] = [f^*\omega + d^{q-1}_{\mathcal{F}_1}\theta] = f^*[\omega]$. Hence we finally get a long exact sequence

$$\cdots \to H^{q-1}_{\mathcal{F}_1}(M^n) \xrightarrow{\alpha^*} H^q(f) \xrightarrow{\beta^*} H^q_{\mathcal{F}_2}(N^m) \xrightarrow{f^*} H^q_{\mathcal{F}_1}(M^n) \xrightarrow{\alpha^*} \cdots . \tag{2.6}$$

Corollary 2.7. *If the foliations $(M^n, \mathcal{F}_1)$ and $(N^m, \mathcal{F}_2)$ are of the p-th and q-th dimension, respectively, then*

(i) $\beta^* : H^{p+1}(f) \to H^{p+1}_{\mathcal{F}_2}(N^m)$ *is an epimorphism,*
$\alpha^* : H^q_{\mathcal{F}_1}(M^n) \to H^{q+1}(f)$ *is an epimorphism,*
$\beta^* : H^i(f) \to H^i_{\mathcal{F}_2}(N^m)$ *is an isomorphism for* $i > p+1$,
$\alpha^* : H^i_{\mathcal{F}_1}(M^n) \to H^{i+1}(f)$ *is an isomorphism for* $i > q$;

(ii) $H^i(f) = 0$ *for* $i > \max\{p+1, q\}$.

Theorem 2.8. *If $f, g : (M^n, \mathcal{F}_1) \to (N^m, \mathcal{F}_2)$ are leaf-to-leaf transforming homotopic mappings, then $H^*(f) = H^*(g)$.*

Proof. Let $F : (M^n \times \mathbb{R}, \widehat{\mathcal{F}}_1) \to (N^m, \mathcal{F}_2)$ be the homotopy mapping between f and g. Let s_0 and s_1 be the zero and the unit section, respectively, of the trivial bundle $M^n \times \mathbb{R} \xrightarrow{\pi} M^n$. Then $F \circ s_0 = g$ and $F \circ s_1 = f$. Hence we have a homomorphism between the short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\Lambda}^*_{\widehat{\mathcal{F}}_1}(M^n \times \mathbb{R}) & \xrightarrow{\alpha'} & \Lambda^*(F) & \xrightarrow{\beta'} & \Lambda^*_{\mathcal{F}_2}(N^m) & \longrightarrow & 0 \\
 & & \big\downarrow s_1^* & & \big\downarrow id \oplus s_1^* & & \big\downarrow id & & \\
0 & \longrightarrow & \widetilde{\Lambda}^*_{\mathcal{F}_1}(M^n) & \longrightarrow & \Lambda^*(f) & \longrightarrow & \Lambda^*_{\mathcal{F}_2}(N^m) & \longrightarrow & 0
\end{array} .$$

This homomorphism defines a homomorphism between the corresponding long cohomologic sequences

$$\begin{array}{ccccccccccc}
\cdots \to & H^q_{\mathcal{F}_2}(N^m) & \to & H^q_{\widehat{\mathcal{F}}_1}(M^n \times \mathbb{R}) & \to & H^{q+1}(F) & \to & H^{q+1}_{\mathcal{F}_2}(N^m) & \to & H^{q+1}_{\widehat{\mathcal{F}}_1}(M^n \times \mathbb{R}) & \to \cdots \\
 & \downarrow id & & \downarrow s_1^* & & \downarrow \gamma & & \downarrow id & & \downarrow s_1^* & \\
\cdots \to & H^q_{\mathcal{F}_2}(N^m) & \to & H^q_{\mathcal{F}_1}(M^n) & \to & H^{q+1}(f) & \to & H^{q+1}_{\mathcal{F}_2}(N^m) & \to & H^{q+1}_{\mathcal{F}_1}(M^n) & \to \cdots
\end{array} ,$$

where γ is the mapping induced by $id \oplus s_1^*$. Since s_1^* is an isomorphism (Lemma 2.3), by virtue of the lemma on five homomorphisms we conclude that γ is also an isomorphism, i.e., $H^*(f) \approx H^*(F)$. By a similar reasoning we can conclude that $H^*(g) \approx H^*(F)$. $\square$

If $(M^n, \mathcal{F}_1)$ is a subfoliation of the foliation $(W^m, \mathcal{F}_2)$, i.e., the embedding $M^n \overset{j}{\hookrightarrow} W^m$ is simultaneously a leaf-to-leaf transforming mapping, then the cohomology algebra $H^*(j)$ will be said to be the algebra of relative vertical cohomologies. Denote it by $H^*_{\mathcal{F}_2, \mathcal{F}_1}(W; M)$.

Now sequence (2.6) can be rewritten as

$$\cdots \to H^{q-1}_{\mathcal{F}_1}(M) \xrightarrow{\alpha^*} H^q_{\mathcal{F}_2,\mathcal{F}_1}(W;M) \xrightarrow{\beta^*} H^q_{\mathcal{F}_2}(W) \xrightarrow{j^*}$$
$$\xrightarrow{j^*} H^q_{\mathcal{F}_1}(M) \xrightarrow{\alpha^*} \cdots . \tag{2.7}$$

Note that if we forget the structure of the foliation, then, as is known, the embedding $M^n \overset{j}{\hookrightarrow} W^m$ defines a long exact cohomological sequence of the pair (W^m, M^n) in de Rham's theory. One can easily verify that the homomorphism i^* from Proposition 2.2 defines a morphism between the long exact cohomological sequence of the pair (W^m, M^n) in de Rham's theory and sequence (2.7).

2.4. The Generalized Mayer–Vietoris Principle for Vertical Cohomologies. A Combinatorial Definition of Vertical Cohomologies. Let $(M^n, \mathcal{F})$ be a smooth foliation of dimension k; let $U = \{u_\alpha\}_{\alpha\in A}$ be an open countable covering of the manifold M^n. Similarly to Čech-de Rham's theory, we define a double complex which will be used to calculate vertical cohomologies of the foliation $(M^n, \mathcal{F})$.

Denote by $u_{\alpha_0\cdots\alpha_p}$ the intersection of open sets $u_0, \dots, u_p$ and by $\coprod$ the disjunctive union. Then we have a sequence of open sets

$$M^n \longleftarrow \coprod_{\alpha_0} u_{\alpha_0} \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} \coprod_{\alpha_0<\alpha_1} u_{\alpha_0\alpha_1} \overset{\partial_0}{\underset{\partial_2}{\overset{\partial_1}{\Lleftarrow}}} \coprod_{\alpha_0<\alpha_1<\alpha_2} u_{\alpha_0\alpha_1\alpha_2} \Lleftarrow \cdots ,$$

where ∂_i is the embedding $\partial_i(u_{\alpha_0\cdots\alpha_p}) = u_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p}$. This sequence of embeddings induces a sequence of restriction mappings of vertical forms

$$\Lambda^*_{\mathcal{F}}(M^n) \xrightarrow{r^*} \prod_{\alpha_0} \Lambda^*_{\mathcal{F}}(u_{\alpha_0}) \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \prod_{\alpha_0<\alpha_1} \Lambda^*_{\mathcal{F}}(u_{\alpha_0\alpha_1}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\Rrightarrow}}}$$
$$\overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\Rrightarrow}}} \prod_{\alpha_0<\alpha_1<\alpha_2} \Lambda^*_{\mathcal{F}}(u_{\alpha_0\alpha_1\alpha_2}) \Rrightarrow \cdots ,$$

where δ_i is induced by the imbedding ∂_i, i.e., $\delta_i = (\partial_i)^*$.

Let us define the difference operator δ by the following rule: if $\omega_{\alpha_0\cdots\alpha_p} \in \Lambda^q_{\mathcal{F}}(u_{\alpha_0\cdots\alpha_p})$ denotes the components of the element $\omega \in \prod\limits_{\alpha_0<\cdots<\alpha_p} \Lambda^q_{\mathcal{F}}(u_{\alpha_0\cdots\alpha_p})$, then

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+1}}, \tag{2.8}$$

where $\omega_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+1}} \equiv \delta_i(\omega_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{p+1}})$.

By a standard reasoning one can verify that $\delta^2 = 0$.

Consider now the double complex

$$K^{p,q} \equiv C^p(U, \Lambda^q_{\mathcal{F}}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Lambda^q_{\mathcal{F}}(u_{\alpha_0 \cdots \alpha_p}). \tag{2.9}$$

whose horizontal mappings are the operators δ^* and whose vertical mappings are the operators $d^*_{\mathcal{F}}$. As is known, this double complex can be reduced to an ordinary differential complex (K^*, D^*):

$$K^n = \bigoplus_{p+q=n} K^{p,q}, \quad D^n = \delta^p + (-1)^p d^q_{\mathcal{F}} \quad \text{on} \quad K^{p,q}.$$

Lemma 2.9. *The sequence*

$$\cdots \longrightarrow \Lambda^*_{\mathcal{F}}(M^n) \xrightarrow{r^*} K^{0,*} \xrightarrow{\delta^0} K^{1,*} \xrightarrow{\delta^1} K^{2,*} \xrightarrow{\delta^2} \cdots$$

is exact.

Proof. Let $q \geq 0$ be an integer number. Clearly, $\Lambda^q_{\mathcal{F}}(M^n)$ is the kernel of the first operator δ^0, since an element from $\prod_{\alpha_0} \Lambda^q(u_{\alpha_0})$ is a global form on M^n if and only if its components are consistent at the intersections.

Let $\{\theta_\alpha\}_{\alpha \in A}$ be the partitioning of unity subordinate to the covering $U = \{u_\alpha\}_{\alpha \in A}$. If $\omega \in K^{p,q}$ is the cocycle of the operator δ^p, then we can assign to it $p-1$ cochains $K\omega$ by the formula $(K\omega)_{\alpha_0 \cdots \alpha_{p-1}} = \sum_\alpha \theta_\alpha \omega_{\alpha\alpha_0 \cdots \alpha_{p-1}}$ (it is assumed here that $\omega_{\ldots,\alpha,\ldots,\beta,\ldots} = -\omega_{\ldots,\beta,\ldots,\alpha,\ldots}$, if $\alpha > \beta$; clearly, this is consistent with the operation δ, i.e., $(\delta\omega)_{\ldots,\beta,\ldots,\alpha,\ldots} = -(\delta\omega)_{\ldots,\alpha,\ldots,\beta,\ldots}$). In that case

$$(\delta^{p-1} K\omega)_{\alpha_0 \cdots \alpha_p} = \sum_i (-1)^i (K\omega)_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_p} = \sum_{i,\alpha} (-1)^i \theta_\alpha \omega_{\alpha\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_p} =$$

$$= \sum_\alpha \theta_\alpha [\omega_{\alpha_0 \cdots, \alpha_p} - (\delta\omega)_{\alpha\alpha_0 \cdots \alpha_p}] = \sum_\alpha \theta_\alpha \omega_{\alpha_0 \cdots \alpha_p} = \omega_{\alpha_0 \cdots \alpha_p}.$$

Hence $\delta^{p-1}(K\omega) = \omega$. □

Theorem 2.10. *The cohomologies of the double complex $K^{p,q}$, i.e., the cohomologies of the complex (K^*, D^*), are isomorphic to the vertical cohomologies $H^*_{\mathcal{F}}(M^n)$. This isomorphism is obtained by the mapping of the restriction r^*.*

Proof. Since $D^* r^* = (\delta^0 + d^*_{\mathcal{F}}) r^* = \delta^0 r^* + d^*_{\mathcal{F}} r^* = d^*_{\mathcal{F}} r^* = r^{*+1} d^*_{\mathcal{F}}$, r^* is a cochain mapping, it induces the mapping in cohomologies r_*.

Let $\varphi \in K^m$ be the cocycle, i.e., $D^m \varphi = 0$. We can represent φ as a sum $\varphi = \varphi^{0,m} + \varphi^{1,m-1} + \cdots + \varphi^{p,m-p}$, where $\varphi^{i,j} \in K^{ij}$ and $\varphi^{p,m-p} \neq 0$. Then $\delta^p \varphi^{p,m-p} = 0$. Hence by Lemma 2.9 we find that there exists an element $\varphi' \in K^{p-1,m-p}$ such that $\delta^{p-1} \varphi' = \varphi^{p,m-p}$. Then the element $\varphi - D^{m-1}\varphi'$ is obviously cohomologic to φ and has no component in $K^{p,m-p}$. After repeating this procedure several times, we obtain an element $\widetilde{\varphi} \in K^{0,m}$ which is cohomologic to φ. For this element we have $d_{\mathcal{F}}^m \widetilde{\varphi} = 0$, $\delta^0 \widetilde{\varphi} = 0$, i.e., $\widetilde{\varphi}$ defines the global vertical closed form on M^n which by means of r^m transforms to $\widetilde{\varphi}$. This shows that the mapping r_*^m is epimorphic. Let us now show that the mapping r_*^m is monomorphic.

Let $\varphi \in \Lambda_{\mathcal{F}}^m(M^n)$ and $d_{\mathcal{F}}^m \varphi = 0$. Then if $r^m(\varphi) = D^{m-1}\varphi'$, $\varphi' \in K^{m-1}$, $\varphi' = \varphi'^{0,m-1} + \cdots + \varphi'^{p,m-p-1}$, where $\varphi'^{p,m-p-1} \neq 0$, then $\delta^p \varphi'^{p,m-p-1} = 0$ (because $D^{m-1}\varphi'$ has only one component in $K^{0,m}$).

Proceeding as above, we can find an element φ'' such that $\varphi'' \in K^{0,m-1}$ and $\varphi' - \varphi'' \in D^{m-2}(K^{m-2})$. Then $r^m(\varphi) = d_{\mathcal{F}}^{m-1}\varphi''$ and $\delta^0 \varphi'' = 0$. Therefore φ'' defines a global form $\widetilde{\varphi}''$ on M^n such that $d_{\mathcal{F}}^{m-1}\widetilde{\varphi}'' = \varphi$. □

By analogy with de Rham's theory Theorem 2.10 can be called the generalized Mayer–Vietoris principle.

From the lower row of the double complex $K^{*,*}$ let us choose a subcomplex, namely, a kernel of the differential $d_{\mathcal{F}}^0$. We get a sequence

$$C_{\mathcal{F}}^0(U) \xrightarrow{\delta^0} C_{\mathcal{F}}^1(U) \xrightarrow{\delta^1} C_{\mathcal{F}}^2(U) \xrightarrow{\delta^2} \cdots , \tag{2.10}$$

where $C_{\mathcal{F}}^i(U) \equiv \ker d_{\mathcal{F}}^0 \subset \prod\limits_{\alpha_0 < \cdots < \alpha_i} \Lambda_{\mathcal{F}}^0(u_{\alpha_0 \cdots \alpha_i})$, and δ^* is the difference operator defined above. The cohomologies of complex (2.10) will be called the Čech cohomologies of the foliation $(M^n, \mathcal{F})$ for the covering U. They are a purely combinatorial object and will be denoted by $H_{\mathcal{F}}^*(U)$. Let the covering $U = \{u_\alpha\}_{\alpha \in A}$ consist of foliated open sets such that all finite nonempty intersections are contractible. Any manifold is known to have such a covering. Then, in view of the fact that Poincaré's lemma ([I]) is valid for vertical cohomologies, we obtain

Lemma 2.11. *The sequence*

$$0 \to C_{\mathcal{F}}^p(U) \xrightarrow{j^p} \prod_{\alpha_0 < \cdots < \alpha_p} \Lambda_{\mathcal{F}}^0(u_{\alpha_0 \cdots \alpha_p}) \xrightarrow{d_{\mathcal{F}}^0} \prod_{\alpha_0 < \cdots < \alpha_p} \Lambda^1(u_{\alpha_0 \cdots \alpha_p}) \xrightarrow{d_{\mathcal{F}}^1} \cdots$$

is exact, $p \geq 0$, where j^p is the embedding.

By Lemma 2.11 and the same reasoning as we used in proving Theorem 2.10 we can prove

Theorem 2.12. *The cochain mapping j^* defines an isomorphism between the Čech cohomologies $H_{\mathcal{F}}^*(U)$ and cohomologies of the double complex $K^{*,*}$. Hence the Čech cohomologies and vertical ones are isomorphic:*

$$H_{\mathcal{F}}^*(U) \approx H_{\mathcal{F}}^*(M^n).$$

Corollary 2.13. *A zero-dimensional vertical cohomology $H^0_{\mathcal{F}}(M^n)$ is isomorphic to a group of smooth functions on the foliated manifold M^n, which are constant on the leaves.*

§ 3. Completely Integrable Hamiltonian Systems and Vertical Cohomologies

3.1. Topology of Constant Energy Surfaces of Completely Integrable Hamiltonian Systems. In this subsection we shall briefly recall the basic facts from the topological theory of integrable Hamiltonian systems ([5]).

Let $v = \operatorname{sgrad} H$ be a Hamiltonian system on the symplectic manifold (M^{2n}, ω), where ω is a symplectic 2-form on M^n, and let v be integrable. Thus there exist n independent (almost everywhere) smooth integrals $f_1 = H$, $f_2, \dots, f_n$ in involution, i.e., $\{f_i, f_j\} = 0$, $i, j = 1, n$, where $\{\ ,\ \}$ is the Poisson bracket. Let $F : M^{2n} \to \mathbb{R}^n$ be the moment mapping which corresponds to these integrals, i.e., $F(x) = (f_1(x), \dots, f_n(x))$, $x \in M^{2n}$. Let N be the set of critical points of the moment mapping, and $\Sigma = F(N)$ be the set of all critical values which is called the bifurcation diagram.

Clearly, we have two cases: (a) $\dim \Sigma < n - 1$ and (b) $\dim \Sigma = n - 1$. In the case (a) the set Σ does not separate the space $\mathbb{R}^n$ and therefore all nonsingular leaves $B_a = F^{-1}(a)$ are diffeomorphic to one another (it is well known that if they are compact, then they are diffeomorphic to the tori T^n, and if they are noncompact, then they are diffeomorphic to the cylinders $T^k \times \mathbb{R}^{n-k}$). The case (b) is more difficult. Below we shall consider a theorem from [5].

Suppose that the restriction $f = f_1|_{X^{n+1}}$ to a joint compact nonsingular surface of the level of the rest of the $n-1$ integrals $X^{n+1} = \{x \in M^{2n} | f_i(x) = c_i,\ i = \overline{2, n}\}$ is a Bott function, i.e., all critical points of this restriction are organized into nondegenerate critical submanifolds (a critical submanifold $L^k \subset X^{n+1}$ is nondegenerate if the restriction of the function f to every normal plane P^{n+1-k} has a nondegenerate Morse singularity at the point $P^{n+1-k} \cap L^k$).

Let c_1 be a critical value of the function f on the surface X^{n+1}, and let $c = (c_1, \dots, c_n)$, i.e., $c = (c_1, c_2, \dots, c_n) \in \Sigma$. Let $B_c = F^{-1}(c)$ be a critical fiber of the moment mapping. Thus $B_c = \{f_1 = c_1\}$ is a critical level surface of f_1 on X^{n+1}.

Theorem 3.1 (A. T. Fomenko). *Each connected compact component B^0_c of the critical fiber B_c is homeomorphic to a set which is one of the following four types:* (1) *a torus T^n;* (2) *the nonorientable manifolds K^n_0 and K^n_1;* (3) *a torus T^{n-1}; or* (4) *a cell complex $T^n_1 \cup T^n_2$ obtained by removing $n-1$-dimensional tori T^{n-1}_1 from T^n_1 and T^{n-1}_2 from T^n_2, which realize nonzero generators of the homology groups $H_{n-1}(T^n_1, \mathbb{Z})$ and $H_{n-1}(T^n_2, \mathbb{Z})$,* *and glueing T_1^n and T_2^n together by identifying only the tori T_1^{n-1} and T_2^{n-1} (by means of a diffeomorphism). In cases* (1)–(3) *the critical fibers consist entirely of critical points of the function f on which a maximum or a minimum is attained. In case* (4) *the critical points of f in the critical fiber B_c^0 forms a torus T^{n-1} (the result of glueing together the tori T_1^{n-1} and T_2^{n-1}) which is a "saddle" for the function f.*

The manifolds K_1^n and K_2^n from Theorem 3.1 are the factor-sets of the torus T^n generated by the action of some group on T^n without fixed points ([5]).

It is important to investigate a particular case of the integrable Hamiltonian system $v = \operatorname{sgrad} H$ on the four-dimensional symplectic manifold (M^4, ω). For mechanical and physical reasons, it is useful to study the integrability effect on an individual isoenergetic surface $Q^3 \subset M^4$ given by the equation $H(x) = h$, where h is a regular value of the Hamiltonian H. The restriction of the system v on Q^3 will be denoted by the same symbol v. In what follows Q^3 will be assumed to be the closed manifold. We also assume that an additional integral f is the Bott function on Q^3. Then to the critical fibers from Theorem 3.1 there correspond the following manifolds:

(1) the torus T^2;

(2) Klein's bottle K^2;

(3) the circle S^1;

(4) a piecewise smooth two-dimensional polyhedron with a singularity of the type of "a fourfold line" (a transversal intersection of two planes).

The Hamiltonian H is called the nonresonance one on Q^3 if the set of Liouville tori with irrational windings is everywhere dense in Q^3.

We say that the Hamiltonian system v is integrable in the Bott sense if among Bott functions there is an additional first integral of v.

Definition 3.2 ([6]). Two integrable, in the Bott sense, nonresonance Hamiltonian systems v_1, v_2 on the oriented manifolds Q_1^3, Q_2^3 are said to be topologically equivalent if there exists an orientation preserving diffeomorphism $g : Q_1^3 \to Q_2^3$ which transforms the Liouville tori of the system v_1 to those of the system v_2 (critical tori are included in the number of Liouville tori), and the isolated critical circles to the isolated critical circles with the same orientation which is by the field v.

The partitioning of the manifold Q^3 into Liouville tori and critical level surfaces of the Bott integral f is called the Liouville foliation on Q^3 (for a given integrable nonresonance Hamiltonian system v). Obviously, the diffeomorphism g, appearing in Definition 3.2, is a leaf-to-leaf mapping between Liouville foliated manifolds.

Note that the Liouville foliation defined above does not depend on a choice of the Bott integral f.

An analogous definition can be introduced for the multidimensional case as well. Two completely integrable, in Bott sense, nonresonance Hamiltonian systems v_1 and v_2 on the nonsingular level surfaces X_1^{n+1} and X_2^{n+1} are said to be topologically equivalent if there exists a diffeomorphism $h : X_1^{n+1} \to X_2^{n+1}$ which transforms the Liouville tori of the system v_1 to those of the system v_2 (critical tori T^n are included in the number of Liouville tori), and critical fibers B_c of the system v_1 to those of the system v_2.

Note that the Liouville foliation is not a foliation in the sense of §2.

Let $v = \operatorname{sgrad} H$ be a completely integrable, in the Bott sense, nonresonance Hamiltonian system on the isoenergetic surface Q^3. As we already know, this system defines the Liouville foliation on Q^3. Denote it by $\mathcal{F}$.

If from Q^3 we remove all critical (i.e., maximal, minimal and "saddle"-type) circles S^1, then on the resulting open manifold $Q^{'3}$ the foliation $\mathcal{F}$ induces a true foliation $\mathcal{F}'$. Thus we have defined the vertical cohomology groups $H^i_{\mathcal{F}'}(Q^{'3})$, $i = 0, 1, 2$. Similarly, one can derive the vertical cohomology groups $H^i_{\mathcal{F}'}(X^{'n+1})$ in the multidimensional case, $i = 0, 1, \ldots, m$.

If now v_1 and v_2 are assumed to be the topologically equivalent systems on the manifolds Q_1^3 and Q_2^3, respectively, then the diffeomorphism $g : Q_1^3 \to Q_2^3$ which preserves the Liouville foliation obviously induces the leaf-to-leaf transforming diffeomorphism $g' : Q_1^{'3} \to Q_2^{'3}$. Therefore the following theorem is valid.

Theorem 3.3. *The cohomology groups $H^i_{\mathcal{F}'}(Q^{'3})$ are a topological invariant of completely integrable, in the Bott sense, nonresonance Hamiltonian systems $v = \operatorname{sgrad} H$ on the isoenergetic surface Q^3.*

Note that in the multi-dimensional case the groups $H^i_{\mathcal{F}'}(X^{'n+1})$, $i = \overline{0, n}$, are also topological invariants.

Thus to determine a topological invariant we had to deal with the open manifold $Q^{'3}$. To avoid this, we shall slightly modify the definition of cohomology groups.

The isoenergetic surface Q^3 can be considered as the set of Liouville tori, circles S^1 (which are maximal, minimal and "saddle"-type) and rings of the type $S^1 \times (0, 1)$ (which are obtained from a hyperbolic critical fiber of type (4) (Theorem 3.1) by removing hyberbolic critical circles of the integral f). We denote this partitioning of the manifold Q^3 by $\mathcal{P}$. By $S(Q^3)$ we denote a Lie algebra of smooth vector fields on Q^3 and by $\overline{S}(Q^3)$ a Lie subalgebra of the algebra $S(Q^3)$ consisting of smooth vector fields on Q^3 tangent to the manifolds of the partitioning $\mathcal{P}$. The algebra $C^\infty(Q^3)$ is, obviously, a module over $\overline{S}(Q^3)$ with respect to the product $x \cdot g = xg$, $x \in \overline{S}(Q^3)$, $g \in C^\infty(Q^3)$.

By a k-dimensional cochain of the algebra $\overline{S}(Q^3)$ with coefficients in $C^\infty(Q^3)$ we shall mean an element of the space $A^k(\overline{S}(Q^3))$ which is a skew-symmetric k-linear functional on $\overline{S}(Q^3)$ with values in $C^\infty(Q^3)$. The differential $d^k : A^k(\overline{S}(Q^3)) \to A^{k+1}(\overline{S}(Q^3))$ is defined by the formula

$$d^k\alpha(X_1,\dots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i-1}X_i\alpha(X_1,\dots,\widehat{X}_i,X_{i+1},\dots,X_{k+1}) + \\ +\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}),$$

where $\alpha \in A^k(\overline{S}(Q^3))$; $X_1,\dots,X_{k+1} \in \overline{S}(Q^3)$.

It is easy to verify that $d^k \circ d^{k-1} = 0$. Thus $(A^*(\overline{S}(Q^3)), d^*)$ is a cochain complex. We denote the cohomologies of the complex $(A^*(\overline{S}(Q^3)), d^*)$ by $H^*_{\mathcal{P}}(Q^3)$. Obviously, $H^i_{\mathcal{P}}(Q^3) = 0$, $i > 2$.

We have also defined the cohomology groups $H^*_{\mathcal{P}}(X^{n+1})$ for the multidimensional case. Note that here $H^i_{\mathcal{P}}(X^{n+1}) = 0$, $i > n$.

Obviously, $A^*(\overline{S}(Q^3))$ is a graduated ring, and the differential d is the antidifferentiation. Hence the cohomologies $H^*_{\mathcal{P}}(Q^3)$ also acquire a ring structure.

If now v_1 and v_2 are, as above, the topologically equivalent systems on the manifolds Q^3_1 and Q^3_2, respectively, then the diffeomorphism $g : Q^3_1 \to Q^3_2$ defines an isomorphism between the differential complexes $(A^*(\overline{S}(Q^3_1)), d^*)$ and $(A^k(\overline{S}(Q^3_2)), d^*)$. Therefore we have

Theorem 3.4. *Let $v = \operatorname{sgrad} H$ be an integrable, in the Bott sense, non-resonance Hamiltonian system on the isoenergetic surface Q^3, and let $\mathcal{P}$ be the partitioning of the manifold Q^3 into Liouville tori, critical circles of the additional Bott integral f, and rings of the type $S^1 \times (0,1)$. Then the cohomology groups $H^*_{\mathcal{P}}(Q^3)$ are topological invariants of the system v, i.e., to the topologically equivalent systems there correspond the isomorphic groups.*

An analogous theorem holds in the multidimensional case as well. Here the partitioning $\mathcal{P}$ consists of Liouville tori T^n, $(n-1)$-dimensional tori T^{n-1}, and rings of the type $T^{n-1} \times (0,1)$.

Since $A^0(\overline{S}(X^{n+1})) = C^\infty(X^{n+1})$, we have

$$H^0_{\mathcal{P}}(X^{n+1}) = \ker(d^0 : C^\infty(X^{n+1}) \to A^1(\overline{S}(X^{n+1})) = \\ = \{g \in C^\infty(X^{n+1}) \,|\, Xg = 0,\ \ \forall X \in \overline{S}(X^{n+1}\} = \operatorname{Inv}_{\overline{S}} C^\infty(X^{n+1}),$$

where $\operatorname{Inv}_{\overline{S}} C^\infty(X^{n+1})$ denotes the set of smooth functions on X^{n+1} whose restrictions are constant functions on the elements of the partitioning $\mathcal{P}$.

Denote by G the factor set of the space X^{n+1} with respect to Liouville tori T^n and the connected components of critical fibers of the integral f. If we introduce the factor topology on G, then $H^0_{\mathcal{P}}(X^{n+1})$ will coincide with the set of continuous functions on G whose liftings to X^{n+1} by the natural projection $X^{n+1} \to G$ are smooth functions.

Remark 3.5. For the four-dimensional case, a complete topological invariant was introduced in [6]. This is the so-called labelled molecule consisting of a graph with edges to which are attached rational numbers from $[0,1)$ or ∞. Note that in the four-dimensional case the above-mentioned G coincides with the graph-molecule from [6]. Therefore the zero-dimensional groups $H^0_{\mathcal{P}}(X^{n+1})$ already pick up "nonzero" information on the topological equivalence of integrable Hamiltonian systems.

§ 4. The Case of a Spherical Pendulum

For a spherical pendulum the phase space is a cotangent bundle to the two-dimensional sphere T^*S^2, where $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Using the Riemannian metric on S^2, we can identify T^*S^2 with the tangent bundle TS^2 and define the Hamiltonian of the system by the energy function

$$E(x,v) = \frac{1}{2}\langle v, v\rangle + x_3, \quad x \in S^2, \quad v \in T_xS^2. \tag{4.1}$$

A kinetic moment with respect to the x_3-axis has the form

$$I(x,v) = x_1v_2 - v_1x_2. \tag{4.2}$$

The critical points of the moment mapping $F = (E, I) : TS^2 \to \mathbb{R}^2$ are the points $x = \pm(0,0,1)$, $v = 0$ and $v = \alpha(-x_2, x_1, 0)$, $1 + \alpha^2 x_3 = 0$, $x_3 \neq \pm 1$. The corresponding singular values of the moment mapping are

$$F = (\pm 1, 0) \ \text{ and } \ F = \Big(\frac{1}{2}\alpha^2 - \frac{3}{2}\alpha^{-2}, \alpha - \alpha^{-3}\Big), \ \text{ where } \ |\alpha| > 1. \tag{4.3}$$

When α tends to ± 1, the point of the curve defined by formula (4.3) tends to $(-1,0)$. For $I \neq 0$, i.e., for $x \neq \pm(0,0,1)$, we can introduce the polar coordinates

$$x_1 = \sin\varphi\cos\theta, \quad x_2 = \sin\varphi\sin\theta, \quad x_3 = \cos\varphi,$$

where $\theta \in [0.2\pi]$, $\varphi \in]0,\pi[$.

The functions E and I in terms of these coordinates are written as

$$E = \frac{1}{2}\dot{\varphi}^2 + V_I(\varphi), \quad I = (\sin^2\varphi)\dot{\theta}, \tag{4.4}$$

where $V_I(\varphi) = \frac{1}{2}(\sin^2\varphi)\dot{\theta}^2 + \cos\varphi$ is the effective potential, and $(\varphi, \theta, \dot{\varphi}, \dot{\theta})$ define the local coordinate system on TS^2. The image F is given by the relation $I = \alpha - \alpha^{-3}$, $E \geq \frac{1}{2}\alpha^2 - \frac{3}{2}\alpha^{-2}$, $|\alpha| \geq 1$. For regular values of the mapping F we have $E > \frac{1}{2}\alpha^2 - \frac{3}{2}\alpha^{-2}$; the point $(0,1)$ should be discarded (see Fig. 1).

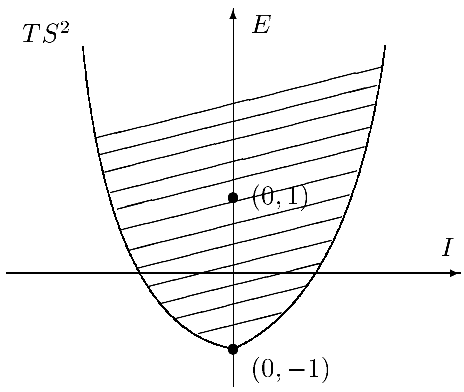

Fig. 1

Let us consider the function I on the isoenergetic surface $Q^3 = \{E = \frac{1}{2}\}$. If $(\varphi, \theta, \dot\varphi, \dot\theta)$ is a critical point of the function I on Q^3, then at that point $\operatorname{grad} E$ and $\operatorname{grad} I$ are collinear. Hence we obtain $\varphi_0 = \arccos(\frac{1-\sqrt{13}}{6})$, $\dot\theta_1 = \sqrt{\frac{\sqrt{13}+1}{2}}$, $\dot\theta_2 = -\sqrt{\frac{\sqrt{13}+1}{2}}$. Thus gives us two isolated critical circles $S_1 = \{(\varphi_0, \theta, \dot\varphi = 0, \dot\theta_1) | \theta \in [0, 2\pi]\}$ and $S_2 = \{(\varphi_0, \theta, \dot\varphi = 0, \dot\theta_2) | \theta \in [0, 2\pi]\}$. Using the method of constant Lagrange multipliers, we can conclude that S_1 is the maximal critical circle, and S_2 is the minimal one.

By Morse's lemma it follows that the compact isoenergetic surface corresponding to the values of energy E close to -1 is the three-dimensional sphere S^3. Therefore Q^3 is also diffeomorphic to S^3, since the points ± 1 are critical values of the integral E. The isoenergetic surface Q^3 can be represented as two solid tori $S^1_1 \times D^2$ and $S^1_2 \times D^2$ glued together along the boundary tori by means of a diffeomorphism which transforms the parallel to the meridian, and vice versa (here D^2 is a two-dimensional disk, and the central circles of the solid tori $S^1_1 \times \{0\}$ and $S^1_2 \times \{0\}$ coincide with the critical circles of the integral I); the tori $S^1_i \times S_r$, $r \in (0, 1]$, $i = 1, 2$, where $S_r = \{x \in D^2 \,|\, \|x\| = r\}$, are the usual Liouville tori of the Hamiltonian system $v = s \operatorname{grad} E$. In that case, using the notation from [6], the labelled molecule has the form

$$A \bullet\!\underset{r=0}{\longrightarrow}\!\bullet\, A,$$

and the factor set G coincides with the segment.

Now consider the function I on the isoenergetic surface $Q^3 = \{E = \alpha > 1\}$. A direct calculation shows that, as in the preceding case, for the integral I we have two isolated critical circles $S_1 = \{(\varphi_0, \theta, 0, \dot\theta_1) \,|\, \theta \in [0, 2\pi]\}$ and $S_2 = \{(\varphi_0, \theta, 0, \dot\theta_2) \,|\, \theta \in [0, 2\pi]\}$, where $\varphi_0 = \arccos \frac{\alpha - \sqrt{\alpha^2+12}}{3}$ and $\dot\theta_1 = -\sqrt{\frac{3}{\sqrt{\alpha^2+12}-\alpha}}$, $\dot\theta_2 = \sqrt{\frac{3}{\sqrt{\alpha^2+12}-\alpha}}$.

In terms of the coordinates $(\varphi, \theta, \dot\varphi, \dot\theta)$ the Euclidean metric is written as $ds^2 = d\varphi^2 + \sin^2\varphi d\theta^2$ so that the the equation $E = \alpha$ takes the form

$$||\xi||^2 + 2\cos\varphi = 2\alpha,$$

where $\xi = (\dot\varphi, \dot\theta)$ is the tangent vector at the point $(\varphi, \theta) \in S^2$, and $||\xi||$ denotes the norm of ξ. Hence it follows that the norm of the vector ξ is a function of φ, and $||\xi|| \neq 0$. Therefore Q^3 is diffeomorphic to the space T_1S^2, where $T_1S^2 = \{(a, \xi) \,|\, x \in S^2,\ ||\xi|| = 1\}$. As is wellknown, T_1S^2 is diffeomorphic to the three-dimensional projective space $\mathbb{R}P^3$ so that $Q^3 \approx \mathbb{R}P^3$. As above, Q^3 can be represented as two solid tori $S^1_1 \times D^2$ and $S^1_2 \times D^2$ glued together along the boundary tori by means of the diffeomorphism h whose corresponding induced mapping $h_* : H^1(T^2; \mathbb{Z}) \to H^1(T^2; \mathbb{Z})$ is given by the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Thus for the integrable Hamiltonian system $v = \operatorname{sgrad} E$ on Q^3 the labelled molecule has the form

$$A \bullet\!\underset{r=\frac{1}{2}}{\rule[0.5ex]{3em}{0.4pt}}\!\bullet A,$$

and the factor set again coincides with the segment. This example shows that vertical cohomologies are not a complete topological invariant of integrable Hamiltonian systems.

Acknowledgments

The author expresses his gratitude to Dr. Z. Giunashvili for helpful discussions.

This research was partly carried out under Grant No. RJV200 from the Intenational Science Foundation.

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(Received 1.03.1996)

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